J. Austral. Math. Soc. (Series A) 42 (1987), 296–311

BASIC REPRESENTATIONS OF SOME AFFINE LIE ALGEBRAS AND GENERALIZED EULER IDENTITIES

KAILASH C. MISRA

(Received 5 October 1985; revised 9 March 1986)

Communicated by R. Lidl

Abstract

We consider certain affine Kac-Moody Lie algebras. We give a Lie theoretic interpretation of the generalized Euler identities by showing that they are associated with certain filtrations of the basic representations of these algebras. In the case when the algebras have prime rank, we also give algebraic proofs of the corresponding identities.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 17 B 65, 05 A 19.

1. Introduction

In this paper we show that certain power series identities (viz. the generalized Euler identities (cf. [1])) are associated with natural filtrations of the basic modules for the affine Kac-Moody Lie algebras $A_n^{(1)}$ (cf. [2]). This association arises in the following way. If $\mathfrak{g}$ is any Lie algebra, V is a $\mathfrak{g}$-module, $S \subseteq \mathfrak{g}$ generates $\mathfrak{g}$ (as an algebra) and $T \subseteq V$ is a generating set for V as a $\mathfrak{g}$-module (i.e. $U(\mathfrak{g}) \cdot T = V$), then we may define

$$(1.1) \qquad V_{[i]} = \operatorname{span}\{ s_1 s_2 \cdots s_r \cdot t \mid s_1, \dots, s_r \in S, t \in T, r \leqslant i \}.$$

Then

$$(0) = V_{[-1]} \subseteq V_{[0]} \subseteq V_{[1]} \subseteq \cdots$$

This research was partially supported by NSF Grant DMS 85-01514.

is a filtration of V. If V and each $V_{[i]}$ has a grading

(1.2) $$V = \sum_{l \geqslant 0} V_{-l}, \qquad V_{[i]} = \sum_{l \geqslant 0} V_{[i],-l}$$

(by nonpositive integers) with each $\dim(V_{-l}) < \infty$, then we may define the corresponding characters

(1.3) $$\chi(V) = \sum_{l \geqslant 0} (\dim V_{-l}) q^{l}$$

and

$$\chi(V_{[i]}) = \sum_{l \geqslant 0} (\dim V_{[i],-l}) q^{l}.$$

It is immediate that

(1.4) $$\chi(V) = \sum_{i \geqslant 0} \chi(V_{[i]}/V_{[i-1]}).$$

For a particular choice of $\mathfrak{g}, V, S, T$ and a grading of V, it may be possible to explicitly determine $\chi(V)$ and each $\chi(V_{[i]}/V_{[i-1]})$, in which case (1.4) will become a power series identity. Certain interesting identities are known to arise in this way (for example, see [4–9]).

Here, for a suitable choice of the generating sets, we define a filtration of the basic modules of the affine Lie algebra $A_n^{(1)}$. We use this to give Lie algebraic interpretations of a set of identities called the generalized Euler identities. In the case when $A_n^{(1)}$ has prime rank we also give a Lie theoretic proof of these identities. But, as is well known, these identities have very simple combinatorial proofs. However, it is not the proof of these identities but rather the Lie algebraic approach which justifies the publication of this paper. We are grateful to R. L. Wilson for his valuable suggestions and to G. Benkart for her advice and support. We would like to thank the referee whose suggestions improved an earlier version of this paper.

2. The setting

Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbf{C})$ be the complex simple Lie algebra of $(n+1) \times (n+1)$ trace zero matrices with complex entries. Consider the basis

$$\left\{ \beta_i, x_{\pm(\beta_1 + \beta_{i+1} + \cdots + \beta_j)} \,|\, 1 \leqslant i \leqslant j \leqslant n \right\}$$

of $\mathfrak{g}$, where

$$\beta_i = E_{ii} - E_{i+1,i+1}, \qquad i = 1, 2, \ldots, n;$$
$$x(\beta_i + \beta_{i+1} + \cdots + \beta_j) = E_{il,j+1}; \quad x_{-(\beta_i + \beta_{i+1} + \cdots + \beta_j)} = E_{j+1,1}$$

for $1 \leqslant i \leqslant j \leqslant n$, with E_{ij} denoting the $(n+1) \times (n+1)$ matrix unit; i.e. E_{ij} is the $(n+1) \times (n+1)$ matrix with 1 as the ij-th entry and zero everywhere else. Let $\mathfrak{h}$ denote the Cartan subalgebra (CSA) of $\mathfrak{g}$ with basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$. Consider the bilinear form

$$\langle x, y \rangle = \operatorname{tr}(xy), \quad \text{for all } x, y \in \mathfrak{g},$$

on $\mathfrak{g}$. Let Φ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}$ is nondegenerate and hence we may identify Φ with a subset of $\mathfrak{h}$. Then

$$\Phi = \left\{ \pm(\beta_i + \beta_{i+1} + \cdots + \beta_j) \,|\, 1 \leqslant i \leqslant j \leqslant n \right\}.$$

Note that for all $\beta \in \Phi$, $x_\beta \in \mathfrak{g}$ is a root vector. Also note that $\langle \beta, \beta \rangle = 2$, $\langle \beta, x_{\pm\beta} \rangle = 0$, and $\langle x_\beta, x_{-\beta} \rangle = 1$ for all $\beta \in \Phi$.

Consider the automorphism ν: $\mathfrak{g} \to \mathfrak{g}$ defined by $\nu(x) = e^{-1}xe$, for all $x \in \mathfrak{g}$, where $e = E_{12} + E_{23} + \cdots + E_{n,n+1} + E_{n+1,1}$. Note that ν is an automorphism of order $n+1$ stabilizing $\mathfrak{h}$, and so ν acts on Φ. Under this action Φ splits into n distinct orbits each of cardinality $n+1$. Choose the following orbit representatives:

$$\gamma_1 = \beta_1, \gamma_2 = \beta_1 + \beta_2, \ldots, \gamma_n = \beta_1 + \beta_2 + \cdots + \beta_n.$$

Let $\mathbf{Z}_{n+1}$ denote the additive group of integers modulo $n+1$. Then, for all $\beta \in \Phi$, $p \in \mathbf{Z}_{n+1}$, we have

$$\nu^p x_\beta = x_{\nu^p \beta}. \tag{2.1}$$

Let ω be a primitive $(n+1)$st root of unity. For $p \in \mathbf{Z}_{n+1}$ define

$$\mathfrak{g}_{(p)} = \{ x \in \mathfrak{g} \,|\, \nu x = \omega^p x \}.$$

Then

$$\mathfrak{g} = \bigoplus_{p \in \mathbf{Z}_{n+1}} \mathfrak{g}_{(p)}$$

is a $\mathbf{Z}_{n+1}$-gradation of $\mathfrak{g}$. For $x \in \mathfrak{g}$, write $x = \sum_{p \in \mathbf{Z}_{n+1}} x_{(p)}$, where $x_{(p)} \in \mathfrak{g}_{(p)}$. Then note that for all $p \in \mathbf{Z}_{n+1}$, $x \in \mathfrak{g}$, we have

$$x_{(p)} = \frac{1}{n+1} \sum_{i=0}^{n} \omega^{-ip} \nu^i(x). \tag{2.2}$$

In particular, for all $\beta \in \Phi$, $\beta_{(0)} = 0$. If $\mathfrak{b}$ is any ν-invariant subalgebra of $\mathfrak{g}$, define

$$\mathfrak{b}_{(p)} = \mathfrak{b} \cap \mathfrak{g}_{(p)}, \quad \text{for all } p \in \mathbf{Z}_{n+1}.$$

Then

$$\mathfrak{b} = \bigoplus_{p \in \mathbf{Z}_{n+1}} \mathfrak{b}_{(p)}$$

is a $\mathbf{Z}_{n+1}$-gradation of $\mathfrak{b}$. Then observe that

$$\mathfrak{h}_{(0)} = (0).$$

Further observe that

$$\mathfrak{t} = \mathfrak{g}_{(0)} = \bigoplus_{i=1}^{n} \mathbf{C}(x_{\gamma_i})_{(0)}$$

is a CSA of $\mathfrak{g}$. Consider the matrix $P = (p_{ij})$, where $p_{ij} = \omega^{ij}$, $0 \leqslant i,\ j \leqslant n$. Note that $P^{-1} = [1/(n+1)\omega^{-ij}]$. Consider the automorphism θ: $\mathfrak{g} \to \mathfrak{g}$, defined by $\theta(x) = PxP^{-1}$, for all $x \in \mathfrak{g}$. Now observe that θ maps $\mathfrak{h}$ to $\mathfrak{t}$ and maps the root vectors with respect to $\mathfrak{h}$ to root vectors with respect to $\mathfrak{t}$. For $j = 1, 2, \ldots, n$, let

$$(2.3)\qquad \begin{cases} H_j = \dfrac{1}{n+1} \displaystyle\sum_{i=1}^{n} \sum_{p \in \mathbf{Z}_{n+1}} \omega^{-ij}(\omega^i - 1)\nu^p x_{\gamma_i}, \\ E_j = \dfrac{1}{n+1}\left[-\displaystyle\sum_{i=1}^{n} \omega^{-i}\gamma_i + \sum_{i=1}^{n} \sum_{p \in \mathbf{Z}_{n+1}} \omega^{-ij-p}\nu^p x_{\gamma_i}\right], \\ F_j = \dfrac{1}{n+1}\left[-\displaystyle\sum_{i=1}^{n} \omega^{i}\gamma_i + \sum_{i=1}^{n} \sum_{p \in \mathbf{Z}_{n+1}} \omega^{-ij+i+p}\nu^p x_{\gamma_i}\right]. \end{cases}$$

Then $\{E_j, F_j, H_j \mid 1 \leqslant j \leqslant n\}$ is a set of canonical generators of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Also note that, with respect to the CSA $\mathfrak{t}$ of $\mathfrak{g}$, the vector

$$(2.4)\qquad E_0 = \frac{1}{n+1}\left[-\sum_{i=1}^{n} \omega^{-i}\gamma_i + \sum_{i=1}^{n} \sum_{p \in \mathbf{Z}_{n+1}} \omega^{-p}\nu^p x_{\gamma_i}\right]$$

is a lowest root vector, the vector

$$(2.5)\qquad F_0 = \frac{1}{n+1}\left[-\sum_{i=1}^{n} \omega^{i}\gamma_i + \sum_{i=1}^{n} \sum_{p \in \mathbf{Z}_{n+1}} \omega^{p+i}\nu^p x_{\gamma_i}\right]$$

is a highest root vector, and

$$H_0 = [E_0, F_0] = \frac{1}{n+1} \sum_{i=1}^{n} \sum_{p \in \mathbf{Z}_{n+1}} (\omega^i - 1)\nu^p x_{\gamma_i}.$$

Let $E = \sum_{j=0}^{n} E_j$. Then $\mathfrak{h} = C_{\mathfrak{g}}(E)$, the centralizer of E in $\mathfrak{g}$. Thus $\mathfrak{h}$ is a CSA in apposition to $\mathfrak{t}$ (cf. [5]). Consider the Lie algebra

$$(2.6)\qquad \hat{\mathfrak{g}} = \coprod_{i \in \mathbf{Z}} \mathfrak{g}_{(\bar{i})} \otimes t^i \oplus \mathbf{C}c, \qquad (\bar{i} = i (\operatorname{mod} n+1)),$$

where c is central, and where

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + \frac{1}{n+1} i\delta_{i+j,0}\langle x, y\rangle c,$$

for all $x \in \mathfrak{g}_{(\bar{i})}$ and $y \in \mathfrak{g}_{(\bar{j})}$. In $\hat{\mathfrak{g}}$, set

$$(2.7) \qquad e_j = E_j \otimes t, \quad f_j = F_j \otimes t^{-1} \quad \text{and} \quad h_j = H_j \otimes 1 + \frac{1}{n+1}c$$

for $j = 0, 1, 2, \ldots, n$. Then $\{e_j, f_j, h_j \mid 0 \leqslant j \leqslant n\}$ is a system of canonical generators for $\hat{\mathfrak{g}}$, viewed as the (twisted) affine Lie algebra $A_n^{(1)}$. Observe that $c = \sum_{j=0}^n h_j$. Let d be the derivation of $\hat{\mathfrak{g}}$ defined by the conditions

$$(2.8) \qquad d(c) = 0, \qquad d(x \otimes t^i) = i(x \otimes t^i), \qquad i \in \mathbf{Z},\ x \in \mathfrak{g}_{(\bar{i})},$$

so that

$$d(h_j) = 0, \qquad d(e_j) = e_j \quad \text{and} \quad d(f_j) = -f_j.$$

Consider the semidirect product Lie algebra

$$(2.9) \qquad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes \mathbf{C}d.$$

For $i \in \mathbf{Z}$, set

$$(2.10) \qquad \tilde{\mathfrak{g}}_i = \{x \in \tilde{\mathfrak{g}} \mid [d, x] = ix\}.$$

Note that for $i \neq 0$, $\tilde{\mathfrak{g}}_i = \mathfrak{g}_{(\bar{i})} \otimes t^i$ and $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{(0)} \oplus \mathbf{C}c \oplus \mathbf{C}d$. Furthermore,

$$(2.11) \qquad \tilde{\mathfrak{g}} = \coprod_{i \in \mathbf{Z}} \tilde{\mathfrak{g}}_i$$

is a $\mathbf{Z}$-gradation of $\mathfrak{g}$. In a similar way define

$$(2.12) \qquad \tilde{\mathfrak{h}} = \coprod_{i \in \mathbf{Z}} \mathfrak{h}_{(\bar{i})} \otimes t^i \otimes \mathbf{C}c \oplus \mathbf{C}d.$$

Observe that $\tilde{\mathfrak{h}}_0 = \mathbf{C}c \oplus \mathbf{C}d$, since $\mathfrak{h}_{(0)} = (0)$. Then the derived algebra

$$(2.13) \qquad \tilde{\mathfrak{h}}' = [\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}] = \coprod_{i \in \mathbf{Z}} \mathfrak{h}_{(\bar{i})} \otimes t^i \oplus \mathbf{C}c$$

is an infinite dimensional Heisenberg subalgebra of $\mathfrak{g}$. For $i \in \mathbf{N}$, ($\mathbf{N}$ denotes the set of nonnegative integers), $i \not\equiv 0 \pmod{n+1}$, choose

$$(2.14) \qquad u_i = \operatorname{diag}(1, \omega^{-i}, \omega^{-2i}, \ldots, \omega^{-ni}), \quad v_i = \operatorname{diag}(1, \omega^i, \omega^{2i}, \ldots, \omega^{ni}).$$

Note that $u_i \in \mathfrak{h}_{(\bar{i})}$, $v_i \in \mathfrak{h}_{(-\bar{i})}$, and for $j = 1, 2, \ldots, n$, $\langle \nu_j, u_i \rangle = (1 - \omega^{-ij})$ and $\langle \gamma_j, v_i \rangle = (1 - \omega^{ij})$. Define

$$(2.15) \qquad p_i = u_i \otimes t^i, \qquad q_i = v_i \otimes t^{-i},$$

for $i \not\equiv 0 \pmod{n+1}$ and $i \in \mathbf{N}$. Then $[p_i, q_j] = ic\delta_{i,j}$, and

$$(2.16) \qquad \{p_i, q_i, c \mid i \not\equiv 0 \pmod{n+1}, i \in \mathbf{N}\}$$

forms a basis of the Heisenberg subalgebra $\tilde{\mathfrak{h}}'$. Furthermore,

$$(2.17) \qquad \{p_i, q_i, c, d, (x_{\gamma_k})_{(\bar{j})} \otimes t^j \mid i \in \mathbf{N}, i \not\equiv 0 \pmod{n+1}, j \in \mathbf{Z}, \quad k = 1, 2, \ldots, n\}$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

is a basis of $\tilde{\mathfrak{g}}$. Let ζ be an indeterminate. For any $\beta \in \Phi$ (as in [5]), define the elements

(2.18)
$$\beta(\zeta) = \sum_{\substack{i \in \mathbf{Z} \\ i \equiv 0\,(\mathrm{mod}(n+1))}} \left(\beta_{(i)} \otimes t^i\right)\zeta^i$$

and

(2.19)
$$X(\beta,\zeta) = \sum_{i \in \mathbf{Z}} \left((x_\beta)_{(i)} \otimes t^i\right)\zeta^i$$

in $\tilde{\mathfrak{g}}(\zeta)$. Let

(2.20)
$$X(\beta,\zeta) = \sum_{i \in \mathbf{Z}} X_i(\beta)\zeta^i,$$

where $X_i(\beta) = ((x_\beta)_{(i)} \otimes t^i)$ is the homogeneous component of degree i of $X(\beta,\zeta)$. Define

(2.21)
$$\delta(\zeta) = \sum_{i \in \mathbf{Z}} \zeta^i \quad \text{and} \quad (D\delta)(\zeta) = \sum_{i \in \mathbf{Z}} i\zeta^i.$$

The following lemma now follows from Theorem 2.4(3) in [5].

LEMMA 2.1 *Let ζ_1, ζ_2 be two commuting indeterminates.*
(1) *For $1 \leqslant j < n$, we have*

$$\left[X(\gamma_1,\zeta_1), X(\gamma_j,\zeta_2)\right] = -\frac{1}{n+1}X(\gamma_{j+1},\zeta_2)\delta(\omega^{-j}\zeta_1/\zeta_2) + \frac{1}{n+1}X(\gamma_{j+1},\omega^{-1}\zeta_2)\delta(\omega\zeta_1/\zeta_2).$$

(2) *For $1 \leqslant j \leqslant n$, we have*

$$\left[X(\gamma_{n+1-j},\zeta_1), X(\gamma_j,\zeta_2)\right] = -\frac{1}{n+1}\gamma_j(\gamma_{j+1},\zeta_2)\delta(\omega^{-j}\zeta_1/\zeta_2) + \frac{1}{(n+1)^2}c(D\delta)(\omega^{-j}\zeta_1/\zeta_2).$$

Consider the abelian subalgebra

$$\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbf{C}c \oplus \mathbf{C}d$$

of $\tilde{\mathfrak{g}}$. There is a triangular decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{n}}^+$$

where

$$\tilde{\mathfrak{n}}^- = \bigoplus_{i>0} \tilde{\mathfrak{g}}_{-i} \quad \text{and} \quad \tilde{\mathfrak{n}}^+ = \bigoplus_{i>0} \tilde{\mathfrak{g}}_i.$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

For a dominant integral weight $\lambda \in \mathfrak{k}^*$ (i.e. $\lambda(h_i) \in \mathbf{N}$, and we assume $\lambda(d) = 0$) define a $(\mathfrak{k} + \tilde{\mathfrak{n}}^+)$-module structure on $\mathbf{C}$ by setting

$$\tilde{\mathfrak{n}}^+ \cdot 1 = (0), \qquad h \cdot 1 = \lambda(h)1, \quad \text{for all } h \in \mathfrak{k}.$$

Then the induced module

$$(2.22) \qquad M(\lambda) = U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{k}+\tilde{\mathfrak{n}}^+)} \mathbf{C}$$

(where U denotes the universal enveloping algebra functor) is the Verma module with highest weight λ and highest weight vector $1 \otimes 1$. Note that

$$M(\lambda) \cong U(\tilde{\mathfrak{n}}^-) \cdot (1 \otimes 1)$$

is a vector space isomorphism. Let

$$(2.23) \qquad W(\lambda) = \sum_{i=0}^{n} U(\tilde{\mathfrak{n}}^-) f_i^{\lambda(h_i)+1} \cdot (1 \otimes 1).$$

Then $W(\lambda)$ is the unique maximal proper $\tilde{\mathfrak{g}}$-submodule of $M(\lambda)$, and

$$(2.24) \qquad L(\lambda) = M(\lambda)/W(\lambda)$$

is the unique (up to isomorphism) irreducible highest weight module with highest weight λ. If v_0 is a highest weight vector in $L(\lambda)$, then

$$(2.25) \qquad f_i^{\lambda(h_i)+1} \cdot v_0 = 0.$$

The $\mathbf{Z}$-gradation on $\tilde{\mathfrak{g}}$ induces a $\mathbf{Z}$-gradation on $M(\lambda)$ and also on $W(\lambda)$. Hence the standard module $L(\lambda)$ has a direct sum decomposition

$$L(\lambda) = \bigoplus_{j \geqslant 0} L(\lambda)_{-j},$$

where, for each j, $\dim(L(\lambda)_{-j}) < \infty$. For an indeterminante q, define the (principally specialized) character by

$$(2.26) \qquad \chi(L(\lambda)) = \sum_{j \geqslant 0} \left(\dim\left(L(\lambda)_{-j}\right)\right) q^j.$$

Then $\chi(L(\lambda))$ has a well-known product expansion (cf. [7, Formula 1.1]). Consider the linear functionals $\lambda_j \in \mathfrak{k}^*$, defined by $\lambda_j(h_i) = \delta_{i,j}$ and $\lambda_j(d) = 0$, $j = 0, 1, 2, \ldots, n$. The irreducible highest weight modules $L(\lambda_j)$ are called the basic $\tilde{\mathfrak{g}}$-modules with highest weight λ_j. It can easily be checked that

$$(2.27) \qquad \chi\left(L(\lambda_j)\right) = \prod_{\substack{k \geqslant 1 \\ k \not\equiv 0 \ (\mathrm{mod}(n+1))}} (1 - q^k)^{-1}.$$

PROOF. In this proof, t will range through all positive integers $\not\equiv 0 \pmod{n+1}$, while s will range through all positive integers. Observe that, by Theorem 3.1, we have

$$\begin{aligned} X(\gamma_1, \zeta_i)\cdot 1 &= a_1 \exp\Big(\sum_t (1-\omega^{-t}) L(y_t)\zeta_i^{-t}/t\Big) \\ &\quad \cdot \exp\Big(-\sum_t (1-\omega^t)\frac{\partial}{\partial y_t}\zeta_i^t\Big)\cdot 1 \\ &= a_1 \exp\Big(\sum_t \nu_i^t z_t\Big). \end{aligned}$$

Now using the Campbell-Baker-Hausdorff formula (i.e. $\exp(a)\exp(b) = \exp(b)\exp(a)\exp([a,b])$ if $[a,b]$ commutes with a and b), we have

$$\begin{aligned} &X(\gamma_1,\zeta_k)X(\gamma_1,\zeta_{k-1})\cdots X(\gamma_1,\zeta_1)\cdot 1 \\ &\quad = (a_1)^k \exp\Big(\sum_t (1-\omega^{-t})L(y_t)\zeta_k^{-t}/t\Big)\exp\Big(-\sum_t(1-\omega^t)\frac{\partial}{\partial y_t}\zeta_k^t\Big) \\ &\qquad \cdot \exp\Big(\sum_t (1-\omega^{-t})L(y_t)\zeta_{k-1}^{-t}/t\Big)\exp\Big(-\sum_t(1-\omega^t)\frac{\partial}{\partial y_t}\zeta_{k-1}^t\Big)\cdots \\ &\qquad \cdot \exp\Big(\sum_t (1-\omega^{-t})L(y_t)\zeta_1^{-t}/t\Big)\exp\Big(-\sum_t(1-\omega^t)\frac{\partial}{\partial y_t}\zeta_1^t\Big)\cdot 1 \\ &\quad = (a_1)^k \prod_{1\leqslant u<v\leqslant k}\exp\Big(\sum_s \frac{1}{s}(\omega^s+\omega^{-s}l-2)\Big(\frac{\nu_u}{\nu_v}\Big)^s\Big) \\ &\qquad \cdot \exp\Big(\sum_t \sum_{1\leqslant i\leqslant k}\nu_i^t z_t\Big) \\ &\quad = (a_1)^k \prod_{1\leqslant u<v\leqslant k}\Big(1-\frac{\nu_u}{\nu_v}\Big)^2\Big/\Big(\Big(1-\omega\frac{\nu_u}{\nu_v}\Big)\Big(1-\omega^{-1}\frac{\nu_u}{\nu_v}\Big)\Big) \\ &\qquad \cdot \exp\Big(\sum_t \sum_{1\leqslant i\leqslant k}\nu_i^t z_t\Big), \end{aligned}$$

since $\log(1-\zeta) = -\Sigma_s(\zeta^s/s)$.

REMARK. For the above proof, we could also have quoted Proposition 3.4 of [5].

For $k > 0$, let us set

$$(3.1)\qquad P_k(\nu_1,\nu_2,\ldots,\nu_k) = (a_1)^k \prod_{1\leqslant u<v\leqslant k}\frac{(1-\nu_u/\nu_v)^2}{(1-\omega\nu_u/\nu_v)(1-\omega^{-1}\nu_u/\nu_v)},$$

$$(3.2)\qquad f_k(\nu_1,\nu_2,\ldots,\nu_k) = \exp\Bigg(\sum_{\substack{t>0\\ t\not\equiv 0\,(\mathrm{mod}(n+1))}}\ \sum_{1\leqslant i\leqslant k}\nu_i^t z_t\Bigg)$$

and

$$g_k(\nu_1,\nu_2,\ldots,\nu_k) = P_k(\nu_1,\nu_2,\ldots,\nu_k) f_k(\nu_1,\nu_2,\ldots,\nu_k). \tag{3.3}$$

For $\alpha \in \mathbb{N}^K$, where $K = \{1,2,\ldots,k\}$ (i.e., α: $K \to \mathbb{N}$), denote $\nu_1^{\alpha(1)}\nu_2^{\alpha(2)} \cdots \nu_k^{\alpha(k)}$ by ν^α and let $|\alpha| = \sum_{i=1}^k \alpha(i)$. Observe that

$$f_k(\nu_1,\nu_2,\ldots,\nu_k) = \sum_{\substack{\alpha \\ \alpha(i)\geqslant 0 \\ 1\leqslant i \leqslant k}} \nu^\alpha f_{k,\alpha} \tag{3.4}$$

and

$$g_k(\nu_1,\nu_2,\ldots,\nu_k) = \sum_{\substack{\alpha \\ \alpha(1)\geqslant 0 \\ \alpha(1)+\alpha(2)\geqslant 0 \\ \alpha(1)+\cdots+\alpha(k)\geqslant 0}} \nu^\alpha g_{k,\alpha}, \tag{3.5}$$

where $f_{k,\alpha} \in \mathbb{C}[y]$, where $g_{k,\alpha} \in \mathbb{C}[y]$, and where both have degree $-|\alpha|$. Define $V_{[k]}$ to be the span of all the elements $x_1x_2 \cdots x_r \cdot 1$, $0 \leqslant r \leqslant k$, where each x_i is one of the $X_j(\gamma_1)$, $j \in \mathbf{Z}$. Then by Lemma 2.1, we have

$$0 = V_{[-1]} \subseteq V_{[0]} \subseteq V_{[1]} \subseteq \cdots \subseteq V, \tag{3.6}$$

and

$$V = \bigcup_{k \geqslant 0} V_{[k]}. \tag{3.7}$$

Furthermore, by Proposition 3.2 and equation (3.5), we have

$$V_{[k]} = \operatorname{span}\{ g_{r,\alpha} \mid 0 \leqslant r \leqslant k, \alpha \in \mathbb{N}^k \}. \tag{3.8}$$

LEMMA 3.3. *For all $k \geqslant 0$, we have*

$$V_{[k]} = \operatorname{span}\{ f_{r,\alpha} \mid 0 \leqslant r \leqslant k, \alpha \in \mathbb{N}^K \}.$$

PROOF. Let $W_{[k]} = \operatorname{span}\{ f_{r,\alpha} \mid 0 \leqslant r \leqslant k, \alpha \in \mathbb{N}^K \}$. Observe that the identity

$$g_r(\nu_1,\nu_2,\ldots,\nu_r) = \sum_{\substack{\alpha \\ \alpha(1)\geqslant 0 \\ \alpha(1)+\alpha(2)\geqslant 0 \\ \cdots \\ \alpha(1)+\alpha(2)+\cdots+\alpha(r)\geqslant 0}} \nu^\alpha g_{r,\alpha}$$

may be viewed as a formal power series in the indeterminates ν_1/ν_2, $\nu_2/\nu_3, \ldots, \nu_{r-1}/\nu_r$ and ν_r, since

$$\nu_1^{\alpha(1)}\nu_2^{\alpha(2)} \cdots \nu_r^{\alpha(r)} = \left(\frac{\nu_1}{\nu_2}\right)^{\alpha(1)} \left(\frac{\nu_2}{\nu_3}\right)^{\alpha(1)+\alpha(2)} \cdots \left(\frac{\nu_{r-1}}{\nu_r}\right)^{\alpha(1)+\cdots+\alpha(r-1)} \nu_r^{\alpha(1)+\cdots+\alpha(r)}.$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

The generalized Euler identities state that (cf. [1])

$$\prod_{\substack{k\geqslant 1\\ k\not\equiv 0\ (\mathrm{mod}(n+1))}} (1-q^k)^{-1} = 1 + \sum_{m,\,l\geqslant 1} c_{m,l}q^l, \tag{2.28}$$

where

(2.29) $c_{m,l}$ = the number of partitions of l into m parts in which each integer appears as a part at most n times.

3. The spanning theorem

Let $\mathbf{C}[y]$ denote the polynomial algebra $\mathbf{C}[y_m \mid m>0, m\not\equiv 0\ (\mathrm{mod}(n+1))]$ on infinitely many variables y_m, $m\not\equiv 0\ (\mathrm{mod}(n+1))$, $m\in\mathbf{N}$, and give $\mathbf{C}[y]$ the structure of a graded algebra by setting $\deg(y_m) = -m$, for all $m>0$, $m\not\equiv 0\ (\mathrm{mod}(n+1))$.

Now define on $\mathbf{C}$ a structure of $(\tilde{\mathfrak{h}}_0 + \tilde{\mathfrak{h}}^+)$-module by setting

$$(\tilde{\mathfrak{h}}^+ + \mathbf{C}d)\cdot 1 = (0), \qquad c\cdot 1 = 1, \tag{3.1}$$

and consider the induced $\tilde{\mathfrak{h}}$-module

$$V = U(\tilde{\mathfrak{h}}) \otimes_{U(\tilde{\mathfrak{h}}_0+\tilde{\mathfrak{h}}^+)} \mathbf{C}.$$

Then as vector spaces

$$V \cong U(\tilde{\mathfrak{h}}^-) = S(\tilde{\mathfrak{h}}^-) \cong \mathbf{C}[y],$$

where S is the symmetric algebra functor. Also the (principal) gradation on V with respect to the degree operator d coincides with the gradation on $\mathbf{C}[y]$. Now we can identify $V = \mathbf{C}[y]$, with the elements p_i acting as the operators $i\partial/\partial y_i$, the elements q_i acting as the multiplication operators $L(y_i)$, c acting as the identity operator and d acting as the zero operator. The following theorem is the well-known vertex operator construction of the fundamental representation of $\tilde{\mathfrak{g}}$ due to Kac, Kazhdan, Lepowsky and Wilson [3].

THEOREM 3.1 (*cf.* [5]). *Fix* $j = 0,1,2,\ldots,n$. *Let* ζ *be an indeterminate and denote by*

$$X'_m(\gamma_k), \qquad k = 1,2,\ldots,n, \quad m\in\mathbf{Z},$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

the coefficients of the Laurent series

$$X'(\gamma_k,\zeta) = a_k \exp\Big(\sum_i (1-\omega^{-ik})L(y_i)\zeta^{-i}/i\Big)$$
$$\cdot \exp\Big(-\sum_i (1-\omega^{ik})\frac{\partial}{\partial y_i}\zeta^{i}\Big),$$

where i ranges over all positive integers $\not\equiv 0 \pmod{n+1}$, and where $a_k = n\omega^{kj}/((\omega^k-1)(n+1)^2)$. Then the map $X_m(\gamma_k) \to X'_m(\gamma_k)$, $m \in \mathbf{Z}$, $k = 1,2,\ldots,n$, extends the representation of $\tilde{\mathfrak{h}}$ on $V = \mathbf{C}[y]$ to a representation of $\tilde{\mathfrak{g}}$ on V. Moreover, for

$$a_k = \lambda_j\big((X_{\gamma_k})_{(0)}\big),$$

and for $k = 1,2,\ldots,n$, the module V is the basic $\tilde{\mathfrak{g}}$-module $L(\lambda_j)$ with highest weight $\lambda_j \in \mathfrak{t}^*$, and $1 \in V$ is a highest weight vector.

Note that for an indeterminate ζ, we have used and will continue to use the standard formal power series notation for the log series, the exponential series, and the binomial series.

Let $\zeta_1, \zeta_2, \zeta_3, \ldots$ be a (possibly finite) set of commuting formal indeterminates, and let W be any vector space. Denote by $W\{\zeta_1,\zeta_2,\ldots\}$ the vector space of formal Laurent series in $\zeta_1, \zeta_2, \ldots$ with coefficients in W, i.e. the space of all (possibly infinite) formal sums

$$\sum_{i_1, i_2 \cdots \in \mathbf{Z}} v_{i_1 i_2} \cdots \zeta_1^{i_1}\zeta_2^{i_2}\cdots$$

with each $v_{i_1 i_2 \ldots} \in W$. Observe that

$$X(\gamma_l,\zeta_k)X(\gamma_l,\zeta_{k-1})\cdots X(\gamma_l,\zeta_1) \in (\operatorname{End} V)\{\zeta_1,\zeta_2,\ldots,\zeta_k\}.$$

PROPOSITION 3.2. *Let $\zeta_1, \zeta_2, \ldots, \zeta_k$ be a set of commuting formal indeterminates and let $V = L(\lambda_j)$. Then in $V\{\zeta_1,\zeta_2,\ldots,\zeta_k\}$, we have*

$$X(\gamma_1,\zeta_k)X(\gamma_1,\zeta_{k-1})\cdots X(\gamma_1,\zeta_1)\cdot 1$$
$$= (a_1)^k \prod_{1\leqslant u<v\leqslant k} \frac{(1-\nu_u/\nu_v)^2}{(1-\omega\nu_u/\nu_v)(1-\omega^{-1}\nu_u/\nu_v)} \exp\Big(\sum_t \sum_{1\leqslant i\leqslant k} \nu_i^t z_t\Big),$$

where $\nu_i = \zeta_i^{-1}$, where $z_t = (1-\omega^{-t})y_t/t$, so that $\deg z_t = -t$, where $a_1 = n\omega^j/((n+1)^2(\omega-1))$, and where t runs through all positive integers $\not\equiv 0$ $(\bmod(n+1))$.

Also any formal power series in $\nu_1, \nu_2, \ldots, \nu_r$ can be expressed as a formal power series in $\nu_1/\nu_2, \nu_2/\nu_3, \ldots, \nu_{r-1}/\nu_r, \nu_r$, since

$$\nu_i^{\alpha(i)} = \left(\frac{\nu_i}{\nu_{i+1}}\right)^{\alpha(i)}\left(\frac{\nu_{i+1}}{\nu_{i+2}}\right)^{\alpha(i)} \cdots \left(\frac{\nu_{r-1}}{\nu_r}\right)^{\alpha(i)} \nu_r^{\alpha(i)}.$$

Furthermore, $P_r(\nu_1, \nu_2, \ldots, \nu_r)$ is a formal power series in the indeterminates $\nu_1/\nu_2, \nu_2/\nu_3, \ldots, \nu_{r-1}/\nu_r$ with constant coefficients. Therefore, as formal power series, the coefficients of $g_r(\nu_1, \nu_2, \ldots, \nu_r)$ are linear combinations of coefficients of $f_r(\nu_1, \nu_2, \ldots, \nu_r)$. Hence it follows that $V_{[k]} \subseteq W_{[k]}$. Now the lemma follows, since, as a formal power series, $P_r(\nu_1, \nu_2, \ldots, \nu_r)$ is invertible.

For $l \geqslant 0$, define

(3.9) $$V_{[k],-l} = \operatorname{span}\{ f_{r,\alpha} \mid 0 \leqslant r \leqslant k, |\alpha| = l \}.$$

Let $A(k,l) = \{\alpha \in \mathbf{N}^K \mid |\alpha| = l, 0 \leqslant \alpha(1) \leqslant \cdots \leqslant \alpha(k)\}$. Since f_r is symmetric, it follows from (3.9) that

(3.10) $$V_{[k],-l} = \operatorname{span}\{ f_{r,\alpha} \mid 0 \leqslant r \leqslant k, \alpha \in A(k,l)\}.$$

Also note that

(3.11) $$V_{[k],-l}/V_{[k-1],-l} = \operatorname{span}\{ f_{k,\alpha} + V_{[k-1],-l} \mid \alpha \in A(k,l)\}.$$

LEMMA 3.4. *Modulo* $V_{[k-1]}$,

(1) $$f_k(0, \nu_2, \ldots, \nu_k) \equiv 0, \quad \textit{for } k \geqslant 1, \quad \textit{and}$$

(2) $$f_k(\omega^n \nu_{n+1}, \omega^{n-1}\nu_{n+1}, \ldots, \omega\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \ldots, \nu_k) \equiv 0,$$

for $k \geqslant n+1$. *Furthermore*, f_k *is a symmetric function of* $\nu_1, \nu_2, \ldots, \nu_k$.

PROOF. To prove (2), observe that, for $t \not\equiv 0 \pmod n$, we have

$$\omega^{nt} + \omega^{(n-1)t} + \cdots + \omega^t + 1 = 0.$$

The rest is clear.

Let

$$B = \{\alpha \in A(k,l) \mid \alpha(1) > 0, \alpha(i+n) > \alpha(i) \text{ for } i = 1, 2, \ldots, (k-n)\}.$$

THEOREM 3.5. *For* $k, l \geqslant 0$,

$$\dim(V_{[k],-l}/V_{[k-1],-l}) \leqslant c_{k,l},$$

where $c_{k,l}$ *is defined in* (2.29).

PROOF. Since $|B| = c_{k,l}$, we will prove the theorem if we prove that

$$\{ f_{k,\alpha} + V_{[k-1],-l} \mid \alpha \in \beta\} \text{ spans } V_{[k],-l}/V_{[k-1],-l}.$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

Now order $A(k,l)$ by the lexicographic ordering. Then to prove the theorem it is enough to show that if $\alpha \in A(k,l)$ and $\alpha \notin B$, then $f_{k,\alpha}$ is congruent modulo $V_{[k-1],-l}$ to a linear combination of $f_{k,\beta}$ with $\beta \in A(k,l)$ and $\beta < \alpha$.

Lemma 3.4(1) implies that if $\alpha(1) = 0$, then $f_{k,\alpha} \in V_{[k-1],-l}$. Let $f_{k,-l} = \sum_{|\alpha|=l} \nu^{\alpha} f_{k,\alpha}$, let $\bar{\alpha} = (\alpha(n+2),\ldots,\alpha(k))$, and let $\bar{\nu} = (\nu_{n+2},\ldots,\nu_k)$. Then

$$f_{k,-l} = \sum_{|\alpha|=l} \nu_1^{\alpha(1)} \cdots \nu_{n+1}^{\alpha(n+1)} \bar{\nu}^{\bar{\alpha}} f_{k,\alpha}.$$

By Lemma 3.4(2), we have

$$\begin{aligned} 0 &\equiv f_{k,-l}\left(\omega^n \nu_{n+1},\ldots,\nu_{n+1},\nu_{n+2},\ldots,\nu_k\right) \\ &= \sum_{\substack{\alpha(1),\ldots,\alpha(n+1),\bar{\alpha} \\ |\alpha|=l}} \omega^{n\alpha(1)+(n-1)\alpha(2)+\cdots+\alpha(n)} \nu_{n+1}^{\alpha(1)+\cdots+\alpha(n+1)} \cdot \bar{\nu}^{\bar{\alpha}} f_{k,\alpha} \\ &= \sum_{\alpha} \nu_{n+1}^{l-|\bar{\alpha}|} \nu^{\bar{\alpha}} \left[\sum_{\substack{\alpha(1),\ldots,\alpha(n+1)\geqslant 0 \\ \alpha(1)+\cdots+\alpha(n+1)=l-|\bar{\alpha}|}} \omega^{n\alpha(1)+\cdots+\alpha(n)} \cdot f_{k,(\alpha(1),\ldots,\alpha(n+1),\bar{\alpha})} \right]. \end{aligned}$$

But since the monomials

$$\left\{ \nu_{n+1}^{l-|\bar{\alpha}|} \bar{\nu}^{\bar{\alpha}} \,\middle|\, \bar{\alpha} \colon \{(n+2),\ldots,k\} \to \mathbb{N} \right\}$$

are independent, it follows that

$$(*) \qquad \sum_{\substack{\alpha(1),\ldots,\alpha(n+1)\geqslant 0 \\ \alpha(1)+\cdots+\alpha(n+1)=l-|\bar{\alpha}|}} \omega^{n\alpha(1)+\cdots+\alpha(n)} f_{k,\alpha} \in V_{[k-1],-l}.$$

Now define the functions $\varepsilon_i \in \mathbb{N}^K$ by $\varepsilon_i(j) = \delta_{ij}$ for $1 \leqslant j \leqslant k$. Replacing α by $\beta + i_1(\varepsilon_1 - \varepsilon_2) + \cdots + i_n(\varepsilon_n - \varepsilon_{n+1})$, where β is any element of $\mathbb{N}^K$, and rearranging the order of summation, we obtain from $(*)$ that, modulo $V_{[k-1],-l}$)

$$\sum_{i_n=-\beta(1)\cdots-\beta(n)}^{\beta(n+1)} \omega^{i_n+\beta(1)+\cdots+\beta(n)} \sum_{i_{n-1}=-\beta(1)-\cdots-\beta(n-1)}^{i_n+\beta(n)} \omega^{i_{n-1}+\beta(1)+\cdots+\beta(n-1)} \cdots \sum_{i_1=-\beta(1)}^{i_2+\beta(2)} \omega^{i_1+\beta(1)} f_{k,\beta+i_1(\varepsilon_1-\varepsilon_2)+\cdots+i_n(\varepsilon_n-\varepsilon_{n+1})} \equiv 0.$$

By symmetry this is true for any consecutive n-tuple of $\beta(i)$'s. Hence we have

$$(**) \qquad \sum_{i_n=-\beta(j)\cdots-\beta(j+n-1)}^{\beta(j+1)} \omega^{i_n+\beta(j)+\cdots+\beta(j+n-1)} \cdot \sum_{i_{n-1}=-\beta(j)-\cdots-\beta(j+n-2)}^{i_n+\beta(j+n-1)} \omega^{i_{n-1}} + \beta(j) + \cdots + \beta(j+n-2) \cdots \sum_{i_1=-\beta(j)}^{i_2+\beta(j+1)} \omega^{i_1+\beta(j)} f_{k,\beta+i_1(\varepsilon_j-\varepsilon_{j+1})+\cdots+i_n(\varepsilon_{j+n+1}-\varepsilon_{j+n})} \equiv 0.$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

In the case when $\beta(j) = \beta(j+n)$ (i.e. $\beta(j) = \beta(j+1) = \cdots = \beta(j+n)$), if i_l, $l = 1, 2, \ldots, n$, are not all zero, then, for a suitable permutation σ of $\{1, 2, \ldots, k\}$, we have

$$\left(\beta + i_1(\varepsilon_j - \varepsilon_{j+1}) + \cdots + i_n(\varepsilon_{j+n-1} - \varepsilon_{j+n})\right) \circ \sigma \in A(k, l),$$

and it is less than β in the lexicographic ordering. Furthermore, in equation (**), $f_{k,\beta}$ occurs only once with nonzero coefficient (namely, when $i_1 = i_2 = \cdots = i_n = 0$). All other terms in (**) are multiples of some $f_{k,\gamma}$ with $\gamma < \beta$ (since f_k is symmetric and $\beta(j) = \beta(j+1) = \cdots = \beta(j+n)$). Hence the theorem follows.

4. The independence theorem

Now let $\mathfrak{A}[[z_t \mid t > 0, t \not\equiv 0 \pmod{n+1}]]$, denote the $\mathbf{Z}$-module consisting of all formal integral linear combinations of products of the elements $z_t^{(s)} = z_t^s/s!$. Then $\mathfrak{A}[z_t]]$ is an algebra over $\mathbf{Z}$. If a, b lie in the maximal ideal of nonunits of $\mathfrak{A}[[z_t]]$, then we define $a^{(s)} = a^s/s! \in \mathfrak{A}[[z_t]]$ for all $s \in \mathbf{Z}$, $s \geqslant 0$. Then $(a+b)^{(s)} = \sum_{i=0}^{s} a^{(i)} b^{(s-i)}$ and $\exp(a) = \sum_{i=0}^{\infty} a^{(i)}$ are elements of $\mathfrak{A}[[z_t]]$.

Let $p = n+1$ be a prime number. Let F be any field of characteristic p. Let

$$\mathfrak{A}_F[[z_t]] = \mathfrak{A}[[z_t]] \otimes_{\mathbf{C}} F.$$

Now for $k \geqslant 0$, we have

$$(4.1)\qquad f_k(\nu_1, \nu_2, \ldots, \nu_k) = \exp\left(\sum_t \sum_{i=1}^{k} \nu_i^t z_t\right) = \prod_{i=1}^{k} \exp\left(\sum_t \nu_i^t z_t\right)$$
$$= \prod_{i=1}^{k} \left(\sum_\gamma \nu_i^{\|\gamma\|} z^{(\gamma)}\right) \in \mathfrak{A}[[z_t]][[\nu_1, \ldots, \nu_k]],$$

where γ: $\{t \in \mathbf{Z} \mid t > 0, t \not\equiv 0 \pmod{n+1}\} \to \mathbf{N}$ is such that all but finitely many $\gamma(t)$ are zero. Furthermore, $\|\gamma\| = \sum_t t\gamma(t)$, and

$$z^{(\gamma)} = \left(z_1^{\gamma(1)} z_2^{\gamma(2)} \cdots z_t^{\gamma(t)} \cdots\right).$$

Let

$$f_k^*(\nu_1, \nu_2, \ldots, \nu_k) = f_k(\nu_1, \ldots, \nu_k) \otimes 1 \in \mathfrak{A}_F[[z_t]][[\nu_1, \ldots, \nu_k]],$$

and for $|\alpha| = l$, $l \geqslant 0$, let

$$(4.2)\qquad f_{k,\alpha}^* = f_{k,\alpha} \otimes 1.$$

Define, for $l \geqslant 0$,

$$(4.3)\qquad V_{[k],-l}^* = \operatorname{span}\left\{ f_{r,\alpha}^* \mid 0 \leqslant r \leqslant k, |\alpha| = l \right\}.$$

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

Then, since f_k is symmetric, we have

(4.4) $$V^*_{[k],-l} = \operatorname{span}\{ f^*_{r,\alpha} \mid 0 \leqslant r \leqslant k, \alpha \in A(k,l)\}.$$

Note that

(4.5) $$\dim_F\left(V^*_{[k],-l}/V^*_{[k-1],-l}\right) \leqslant \dim_{\mathbf{C}}\left(V_{[k],-l}/V_{[k-1],-l}\right).$$

Now we want to show that, modulo $V^*_{[k-1],-l}$, the set

(4.6) $$\{ f^*_{k,\alpha} \mid |\alpha| = l, 0 < \alpha(1) \leqslant \cdots \leqslant \alpha(k), \alpha(j+n) > \alpha(j)\}$$

is an independent set. But since each $z_t^{(p^{i_j})}$ is indecomposable, it is now clear from (4.1) that the coefficient of

(4.7) $$z^{(p^{i_1})}_{\alpha(1)/p^{i_1}} z^{(p^{i_2})}_{\alpha(2)/p^{i_2}} \cdots z^{(p^{i_k})}_{\alpha(k)/p^{i_k}} \otimes 1$$

(where $p^{i_j} \| \alpha(j)$) in $f^*_{k,\alpha}$ is 1, and the coefficent of this element in $f^*_{k,\beta}$ is zero for any $\beta \in \mathbf{N}^K$ with $\beta \neq \alpha$, and with $0 \leqslant \beta(1) \leqslant \cdots \leqslant \beta(k)$ and $\beta(j+n) > \beta(j)$. Note that the hypothesis that $\alpha(j+n) > \alpha(j)$ for all j implies that no subscript in (4.7) is repeated more than n times. Therefore the monomial occurring in (4.7) is nonzero. Hence from (4.5) we now have

(4.8) $$\dim_{\mathbf{C}}\left(V_{[k],-l}/V_{[k-1],-l}\right) \geqslant c_{k,l}.$$

The next theorem now follows from (4.8) and from Theorem 3.5.

Theorem 4.1. *For k, $l \geqslant 0$, and for $(n+1)$ prime, we have*

$$\dim\left(V_{[k],-l}/V_{[k-1],-l}\right) = c_{k,l}.$$

Now observe that

$$\chi(V) = \sum_{k \geqslant 0} \chi\left(V_{[k]}/V_{[k-1]}\right)$$
$$= \sum_{k \geqslant 0} \sum_{l \geqslant 0} \dim\left(V_{[k],-l}/V_{[k-1],-l}\right) q^l.$$

Therefore, when the rank is prime (i.e. when $(n+1)$ is prime) the generalized Euler identities (2.28) follow from (2.27) and Theorem 4.1.

Remark. The above argument to prove Theorem 4.1 does not hold if $(n+1)$ is not prime. However, because the generalized Euler identities are true for all n, Theorem 4.1 does hold for all n.

https://doi.org/10.1017/S1446788700028585 Published online by Cambridge University Press

References

[1] G. E. Andrews, *The Theory of Partitions* (G. C. Rota (ed.), *Encyclopedia of Mathematics and its Applications*, *Vol.* 2, Addison-Wesley, Reading, Mass., 1976).

[2] V. G. Kac, *Infinite Dimensional Lie Algebras* (Progress in Mathematics, Vol. 44, Birkhäuser, Boston, Mass., 1983).

[3] V. G. Kac, D. A. Kazhdam, J. Lepowsky and R. L. Wilson, 'Realization of the basic representations of the Euclidean Lie algebras', *Advances in Math.* **2** (1981), 83–112.

[4] J. Lepowsky and R. L. Wilson, 'A Lie Theoretic interpretation and proof of the Rogers-Ramanujan identities', *Advances in Math.* **45** (1982), 21–72.

[5] J. Lepowsky and R. L. Wilson, 'The structure of standard modules, I: Universal algebras and the Rogers-Ramanujan identities', *Invent. Math.* **77** (1984), 199–290.

[6] J. Lepowsky and R. L. Wilson, 'The structure of standard modules, II: The case $A_1^{(1)}$, principal gradation', *Invent. Math.* **79** (1985), 417–442.

[7] K. C. Misra, 'Structure of certain standard modules for $A_n^{(1)}$ and the Rogers-Ramanujan identities', *J. Algebra* **88** (1984), 196–227.

[8] K. C. Misra, 'Structure of some standard modules for $C_n^{(1)}$', *J. Algebra* **90** (1984), 385–409.

[9] K. C. Misra, 'Standard representations of some affine Lie algebras' (J. Lepowsky, S. Mandelstam and I. M. Singer (eds.), *Vertex Operators in Mathematics and Physics*, Publ. Math. Sci. Res. Inst., Vol. 3, Springer-Verlag, Berlin and New York, 1985), pp. 163–183.

Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27695-8205
U. S. A.