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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

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In this paper we are dealing with oscillatory and asymptotic behaviour of solutions of second order nonlinear difference equations of the form

$$\Delta(r_{n-1}\Delta x_{n-1})+F(n,x_n)=G(n,x_n,\Delta x_n), n\in N(n_0). \tag{E}$$

Some sufficient conditions for all solutions of (E) to be oscillatory are obtained. Asymptotic behaviour of nonoscillatory solutions of (E) is considered also.

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1. Introduction

Recently, there has been a lot of interest in the oscillation and nonoscillation of second order difference equations. See, for example, [1–6] and the references cited therein. In this paper, we consider the second order nonlinear difference equation of the form

$$\Delta(r_{n-1}\Delta x_{n-1})+F(n,x_n)=G(n,x_n,\Delta x_n), \tag{E}$$

where $n\in N(n_0)=\{n_0,n_0+1,n_0+2,\ldots\}$ (n_0 is a fixed non-negative integer) and Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$. Moreover, F and G are real-valued functions with $x\colon N(n_0)\to\mathbf{R}$, $r\colon N(n_0)\to(0,+\infty)$, $F\colon N(n_0)\times R\to R$ and $G\colon N(n_0)\times\mathbf{R}^2\to\mathbf{R}$.

The purpose of this paper is to establish some new results on the oscillatory and asymptotic behaviour of solutions of (E). Our results differ greatly from those in [1–6] and the known literature.

As is customary (see [3], [4] and [6]), a nontrivial solution $\{x_n\}$ of (E) is said to be oscillatory if for every $N>0$ there exists a $k\geqq N$ such that $x_kx_{k+1}\leqq 0$. Otherwise the solution is called nonoscillatory.

In this paper, we further assume that the following conditions hold:

(H) There exist sequences $\{f(n)\}$, $\{g(n)\}$ and ratio m of two odd integers such that for all sufficiently large n

$$\frac{F(n,u)}{u^m} \geqq f(n) \quad \text{for } u \neq 0,$$

and

$$\frac{G(n,u,v)}{u^m} \leqq g(n) \quad \text{for } u \neq 0.$$

2. Asymptotic behaviour of nonoscillatory solutions

In this section, we assume that

$$\sum_{k=n_0}^{\infty} [f(k)-g(k)] = \infty. \tag{1}$$

Theorem 1. *Let conditions* (*H*) *and* (1) *hold, then any nonoscillatory solution of* (*E*) *must belong to one of the following two types:*

$$A_c: x_n \to C \neq 0,\ n \to \infty,$$

$$A_0: x_n \to 0,\ n \to \infty.$$

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (E), then x_n is eventually positive or negative. Thus, from (*E*), we have

$$\begin{aligned}\Delta\left(\frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}^m}\right) &= \frac{r_n \Delta x_n}{x_n^m} - \frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}^m} \\ &= \frac{x_{n-1}^m r_n \Delta x_n - x_n^m r_{n-1}\Delta x_{n-1}}{x_n^m x_{n-1}^m} \\ &= \frac{\Delta(r_{n-1}\Delta x_{n-1})}{x_n^m} - \frac{\Delta x_{n-1} m \cdot r_{n-1}\Delta x_{n-1}}{(x_{n-1}x_n)^m} \\ &\leqq -[f(n)-g(n)] - \frac{\Delta x_{n-1}^m \cdot r_{n-1}\Delta x_{n-1}}{(x_{n-1}x_n)^m}. \end{aligned} \tag{2}$$

By the mean value theorem

$$\Delta x_{n-1}^m = m\xi_n^{m-1}\Delta x_{n-1}, \tag{3}$$

where $x_{n-1} < \xi_n < x_n$ or $x_n < \xi_n < x_{n-1}$. Thus from (2), (3) we have

$$\Delta\left(\frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}^m}\right) \leqq -[f(n)-g(n)] - \frac{m\xi_n^{m-1} \cdot r_{n-1}(\Delta x_{n-1})^2}{(x_{n-1}x_n)^m}$$

$$\leqq -[f(n)-g(n)]. \tag{4}$$

Summing (4) from n_0+1 to n, we get

$$\frac{r_n \Delta x_n}{x_n^m} \leqq \frac{r_{n_0} \Delta x_{n_0}}{x_{n_0}^m} - \sum_{k=n_0+1}^{n} [f(k)-g(k)]. \tag{5}$$

If x_n is eventually positive, then there exists $n_1 \in N(n_0)$ such that $x_n > 0$ for $n \in N(n_1)$, thus from (5) and (1) we have

$$\Delta x_n < 0 \quad \text{for } n \in N(n_1).$$

Hence x_n is monotone decreasing, and $\lim_{n\to\infty} x_n = C \geqq 0$, where C is a constant.

If x_n is eventually negative, then there exists $n_2 \in N(n_0)$ such that $x_n < 0$ for $n_2 \in N(n_0)$, thus from (5) and (1) we have

$$\Delta x_n > 0 \quad \textit{for } n \in N(n_2).$$

Hence x_n is monotone increasing, then $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} x_n = C \leqq 0$.

Thus any nonoscillatory solution of (E) must belong to the following two types: A_c or A_0. The proof of Theorem 1 is complete.

Theorem 2. *Let conditions* (H) *and* (1) *hold.*

(*i*) *If* $m=1$, *then a necessary condition for equation* (E) *to have a nonoscillatory solution* $\{x_n\}$ *which belongs to* A_c *is that*

$$\sum_{k=n_1+1}^{k} \frac{1}{r_k} \sum_{i=n_1+1}^{\infty} [f(i)-g(i)] < \infty, \tag{6}$$

where $n_1 \in N(n_0)$ *is sufficiently large.*

(*ii*) *If* $0<m<1$, *then a necessary condition for equation* (E) *to have a nonoscillatory solution* $\{x_n\}$ *which belongs to* A_0 *or* A_c *is also* (6).

Proof. (i) if $m=1$, let $\{x_n\}$ be a nonoscillatory solution of (E) which belongs to A_c. If $C>0$, then x_n is eventually positive. From the proof of Theorem 1, we have that Δx_n is eventually negative and from (1), there exists $n_1 \in N(n_0)$ such that $x_n > 0$, $\Delta x_n < 0$, and $\sum_{i=n_1+1}^{n} [f(i)-g(i)] > 0$ for $n \in N(n_1)$. Summing (4) from n_1+1 to n, it follows that

$$\frac{r_n \Delta x_n}{x_n} \leqq \frac{r_{n_1} \Delta x_{n_1}}{x_{n_1}} - \sum_{i=n_1+1}^{n} [f(i)-g(i)] \leqq - \sum_{i=n_1+1}^{n} [f(i)-g(i)],$$

this is,

$$\frac{\Delta x_n}{x_n} \leqq -\frac{1}{r_n} \sum_{i=n_1+1}^{n} [f(i)-g(i)]. \tag{7}$$

Let $q(t)=x_n+(t-n)\Delta x_n$, $n\leqq t\leqq n+1$. Then $q'(t)=\Delta x_n<0$, and $0<x_{n+1}\leqq q(t)\leqq x_n$ for $n<t<n+1$. Hence

$$\begin{aligned}\sum_{k=n_1+1}^{n}\frac{\Delta x_k}{x_k}&=\sum_{k=n_1+1}^{n}\int_k^{k+1}\frac{q'(t)}{x_k}dt\geqq\sum_{k=n_1+1}^{n}\int_k^{k+1}\frac{q'(t)}{q(t)}dt\\&=\sum_{k=n_1+1}^{n}[\log q(k+1)-\log q(k)]\\&=\sum_{k=n_1+1}^{n}[\log x_{k+1}-\log x_k]\\&=\log x_{n+1}-\log x_{n_1+1}.\end{aligned}\tag{8}$$

Thus from (7) and (8), we have

$$\sum_{k=n_1+1}^{n}\frac{1}{r_k}\sum_{i=n_1+1}^{k}[f(i)-g(i)]\\ \leqq\log x_{n_1+1}-\log x_{n+1},$$

from which letting $n\to\infty$ and noting $\lim_{n\to\infty}x_n=C>0$, we obtain (6).

(ii) If $0<m<1$, let $\{x_n\}$ be a solution of (E) which belongs to A_0 or A_c. As shown in the proof of case $m=1$, we can obtain

$$\frac{\Delta x_n}{x_n^m}\leqq-\frac{1}{r_n}\sum_{i=n_1+1}^{n}[f(i)-g(i)]\tag{9}$$

and

$$\sum_{k=n_1+1}^{n}\frac{\Delta x_k}{x_k^m}\leqq(1-m)[x_{n_1+1}^{1-m}-x_{n+1}^{1-m}].\tag{10}$$

From (9) and (10) we have

$$\sum_{k=n_1+1}^{n}\frac{1}{r_k}\sum_{i=n_1+1}^{k}[f(i)-g(i)]\leqq(1-m)(x_{n_1+1}^{1-m}-x_{n+1}^{1-m}),$$

from which letting $n\to\infty$, and noting $0<m<1$ and $\lim_{n\to\infty}x_n=0$ or $\lim_{n\to\infty}x_n=C>0$, we obtain (6), that is.

$$\sum_{k=n_1+1}^{\infty}\frac{1}{r_k}\sum_{i=n_1+1}^{k}[f(i)-g(i)]<\infty.$$

If $\{x_n\}$ is eventually negative, similarly we can show that (6) holds. Thus the proof Theorem 2 is complete.

3. Oscillation of solutions

Theorem 3. *Let conditions (H), (1) and the following condition hold,*

$$\sum_{k=n_1+1}^{\infty} \frac{1}{r_k} = \infty. \tag{11}$$

Then all solutions of (E) are oscillatory.

Proof. Suppose on the contrary that there exists a nonoscillatory solution $\{x_n\}$. Without loss of generality, we assume that x_n is eventually positive. From the proof of Theorem 1, we have that Δx_n is eventually negative and from (1), there exists $n_1 \in N(n_0)$ such that $x_n > 0$, $\Delta x_n < 0$ for $n \in N(n_1)$ and

$$\sum_{i=n_1+1}^{n} [f(i) - g(i)] \geqq 0 \quad \text{for } n \in N(n_1).$$

Summing (E) from $n_1 + 1$ to n, we have

$$\begin{aligned} r_n \Delta x_n &= r_{n_1} \Delta x_{n_1} - \sum_{i=n_1+1}^{n} [F(k, x_k) - G(k, x_k, \Delta x_k)] \\ &\leqq r_{n_1} \Delta x_{n_1} - \sum_{k=n_1+1}^{n} x_k^m [f(k) - g(k)] \\ &= r_{n_1} \Delta x_{n_1} - x_n^m \sum_{k=n_1+1}^{n} [f(k) - g(k)] + \sum_{k=n_1+1}^{n-1} \Delta x_k^m \sum_{i=n_1+1}^{k} [f(i) - g(i)] \\ &= r_{n_1} \Delta x_{n_1} - x_n^m \sum_{k=n_1+1}^{n} [f(k) - g(k)] + \sum_{k=n_1+1}^{n-1} (m\xi_k^{m-1} \Delta x_k) \sum_{i=n_1+1}^{k} [f(i) - g(i)] \end{aligned} \tag{12}$$

where $x_{k+1} < \xi_k < x_k$.

From $x_n > 0$, $\Delta x_n < 0$ for $n \in N(n_1)$ and (12), we have

$$r_n \Delta x_n \leqq r_{n_1} \Delta x_{n_1}.$$

Thus

$$\Delta x_n \leqq \frac{1}{r_n} r_{n_1} \Delta x_{n_1}. \tag{13}$$

Summing (13) from $n_1 + 1$ to $n - 1$, we get

$$x_n \leqq x_{n_1+1} + r_{n_1} \Delta x_{n_1} \sum_{k=n_1+1}^{n-1} \frac{1}{r_k} \tag{14}$$

from (14), letting $n\to\infty$ and using (11) and $\Delta x_{n_1}<0$, we have $x_n\to-\infty$, which contradicts $x_n>0$. Thus Theorem 3 is proved.

Theorem 4. *Let conditions* (H) *with* $m=1$, (11) *and the following conditions hold,*
(*i*) *There exists a sufficiently large* $n_1\in N(n_0)$ *such that for* $n\in N(n_1)$, $f(n)-g(n)\geqq 0$ *and*

$$\sum_{k=n_1+1}^{\infty}[f(k)-g(k)]<\infty. \tag{15}$$

(*ii*) *There exists positive sequence* $\{C_n\}$ *such that*

$$\sum_{k=n_1+1}^{\infty}C_k[f(k)-g(k)]=\infty \tag{16}$$

and

$$\sum_{k=n_1+1}^{\infty}\frac{(\Delta C_{k-1})^2}{C_k\left(\frac{1}{r_{k-1}}\sum_{i=k}^{\infty}[f(i)-g(i)]\right)}<\infty. \tag{17}$$

Then all solutions of (E) *are oscillatory.*

Proof. Suppose that there exists a nonoscillatory solution $\{x_n\}$. Without loss of generality, we assume that $x_n>0$ for $n\in N(n_1)$. Hence (4) holds. Now, we show that $\Delta x_n<0$ for sufficiently large n and that this will lead to a contradiction.

Case (*a*). If there exists $n_2\in N(n_1)$ such that $\Delta x_{n_2}=0$, then summing (4) from n_2+1 to n, we have

$$\frac{r_n\Delta x_n}{x_n}\leqq\frac{r_{n_2}\Delta x_{n_2}}{x_{n_2}}-\sum_{k=n_2+1}^{n}[f(k)-g(k)]$$

$$=-\sum_{k=n_2+1}^{n}[f(k)-g(k)].$$

Thus from (15), we have $\Delta x_n<0$ for $n\in N(n_2)$. Hence summing (E) from $n_3\in N(n_2)$ to n, we can obtain that

$$\lim_{n\to\infty}x_n=-\infty$$

which contracts $x_n>0$.

Case (*b*) If $\Delta x_n>0$ for $n\in N(n_1)$. Similarly to (4) we have

$$\Delta\left(\frac{r_{n-1}\Delta x_{n-1}}{x_{n-1}}\right) < -[f(n)-g(n)]. \tag{18}$$

Summing (18) from $n+1$, $n\in N(n_1)$, to N and letting $N\to\infty$, we have

$$0 \leqq \lim_{N\to\infty}\frac{r_N\Delta x_N}{x_N} \leqq \frac{r_n\Delta x_n}{x_n} - \sum_{k=n+1}^{\infty}[f(k)-g(k)].$$

Thus

$$\sum_{k=n+1}^{\infty}[f(k)-g(k)] \leqq \frac{r_n\Delta x_n}{x_n}.$$

From $\Delta x_n > 0$ for $n\in N(n_1)$, we have

$$\frac{1}{r_n}\sum_{k=n+1}^{\infty}[f(k)-g(k)] \leqq \frac{1}{x_{n_1}}\Delta x_n. \tag{19}$$

Hence

$$\begin{aligned}
\Delta\left(\frac{r_{n-1}C_{n-1}\Delta x_{n-1}}{x_{n-1}}\right) &= \frac{r_nC_n\Delta x_n}{x_n} - \frac{r_{n-1}C_{n-1}\Delta x_{n-1}}{x_{n-1}}\\
&= \frac{C_n(r_n\Delta x_n - r_{n-1}\Delta x_{n-1})}{x_n} + \frac{C_nr_{n-1}\Delta x_{n-1}}{x_n} - \frac{r_{n-1}C_{n-1}\Delta x_{n-1}}{x_{n-1}}\\
&= C_n\frac{G(n,x_n,\Delta x_n)-F(n,x_n)}{x_n} - \frac{C_nr_{n-1}(\Delta x_{n-1})^2}{x_nx_{n-1}} + \frac{\Delta C_{n-1}r_{n-1}\Delta x_{n-1}}{x_{n-1}}\\
&\leqq -C_n[f(n)-g(n)] - \frac{r_{n-1}x_n}{x_{n_1}}\left[\frac{\sqrt{C_n}\,\Delta x_{n-1}}{x_n} - \frac{\Delta C_{n-1}}{2\sqrt{C_n}}\right]^2 + \frac{r_{n-1}(\Delta C_{n-1})^2}{4C_n}\cdot\frac{x_n}{x_{n-1}}\\
&\leqq -C_n[f(n)-g(n)] + \frac{r_{n-1}(\Delta C_{n-1})^2}{4C_n}\cdot\frac{x_n}{x_{n-1}}\\
&\leqq -C_n[f(n)-g(n)] + \frac{r_{n-1}(\Delta C_{n-1})^2}{4C_n}\\
&= -C_n[f(n)-g(n)] + \frac{r_{n-1}\Delta x_{n-1}\cdot(\Delta C_{n-1})^2}{4C_n\cdot\Delta x_{n-1}}.
\end{aligned} \tag{20}$$

Summing the following inequality from n_1+1 to $n+1$,

$$r_{k-1}\Delta x_{k-1} \leqq -x_k[f(k)-g(k)],$$

we find that

$$r_{n-1}\,\Delta x_{n-1} \leqq r_{n_1}\,\Delta x_{n_1} - \sum_{k=n_1+1}^{n-1} x_k[f(k)-g(k)]$$
$$\leqq x_{n_1}\,\Delta x_{n_1} = M_0. \tag{21}$$

Using (21), (19), and (20) we have

$$\Delta\left(\frac{r_{n-1}C_{n-1}\,\Delta x_{n-1}}{x_{n-1}}\right)$$
$$\leqq -C_n[f(n)-g(n)] + \frac{M_0\cdot(\Delta C_{n-1})^2}{4x_{n_1}\cdot C_n\left(\frac{1}{r_{n-1}}\sum_{k=n}^{\infty}[f(k)-g(k)]\right)}$$
$$= -C_n[f(n)-g(n)] + M\cdot\frac{(\Delta C_{n-1})^2}{C_n\left(\frac{1}{r_{n-1}}\sum_{k=n}^{\infty}[f(k)-g(k)]\right)}, \tag{22}$$

where $M = M_0/4x_{n_1}$. Summing (22) from n_1+1 to n, we have

$$\frac{r_nC_n\,\Delta x_n}{x_n} \leqq \frac{r_{n_1}C_{n_1}\,\Delta x_{n_1}}{x_{n_1}} - \sum_{k=n_1+1}^{n} C_k[f(k)-g(k)]$$
$$+M\sum_{k=n_1+1}^{n}\frac{(\Delta C_{n-1})^2}{C_k\left(\frac{1}{r_{k-1}}\sum_{i=k}^{\infty}[f(i)-g(i)]\right)}.$$

Letting $n\to\infty$ and noting (16), (17), we get

$$\lim_{n\to\infty}\frac{r_nC_n\,\Delta x_n}{x_n} = -\infty.$$

Thus there exists $n_2 \in N(n_1)$ such that $\Delta x_n < 0$ for $n \in N(n_2)$, which contradicts $\Delta x_n > 0$ for $n \in N(n_1)$.

Thus from Cases (a) and (b) we can show that there exists $n_3 \in N(n_1)$ such that $\Delta x_{n_3} < 0$. Summing (4) from n_3+1 to n we have

$$\frac{r_n\,\Delta x_n}{x_n} \leqq \frac{r_{n_3}\,\Delta x_{n_3}}{x_{n_3}} - \sum_{k=n_3+1}^{n}[f(k)-g(k)].$$

Hence $\Delta x_n < 0$ for $n \in N(n_3)$. Similarly to the last part of the proof of Theorem 3 and from (11) we have $\lim_{n\to\infty} x_n = -\infty$, which contradicts $x_n > 0$. Theorem 4 is proved.

For the purpose of illustration we consider the following example.

Example. Consider the difference equation

$$\Delta\left(\frac{1}{2n^{1+\delta}}\Delta x_{n-1}\right)+\frac{1}{n^{1+\delta}}x_n+\frac{1}{4(n+1)^{1+\delta}}(\Delta x_n)^2=0,\ n\in N(n_0),\ n_0\geqq 1$$

where $0<\delta<1$. Let $C_n=n$, $f(n)=1/n^{1+\delta}$ and $g(n)=0$, $n\in N(n_0)$, then we find that conditions (H), (11), and (15)–(17) are satisfied. Thus from Theorem 4 all solutions of (E) are oscillatory. In fact, $\{x_n\}=\{(-1)^n\}$ is such a solution. We believe that the conclusion is not deducible from the oscillation criteria in [**3**, **4**, **6**] and the known literature.

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