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THE LATTICE OF EQUATIONAL CLASSES OF COMMUTATIVE SEMIGROUPS

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Introduction. There has been some interest lately in equational classes of commutative semigroups (see, for example, [**2**; **4**; **7**; **8**]). The atoms of the lattice of equational classes of commutative semigroups have been known for some time [**5**]. Perkins [**6**] has shown that each equational class of commutative semigroups is finitely based. Recently, Schwabauer [**7**; **8**] proved that the lattice is not modular, and described a distributive sublattice of the lattice.

The present paper describes a "skeleton" sublattice of the lattice, which is isomorphic to $A \times N^+$ with a unit adjoined, where A is the lattice of pairs (r, s) of non-negative integers with $r \leqq s$ and $s \geqq 1$, ordered component-wise, and N^+ is the natural numbers with division. Every other equational class "hangs between" two members of the skeleton in a certain way; the relationships between intervals of the form $[\mathfrak{K}_1, \mathfrak{K}_2]$ where $\mathfrak{K}_1$, $\mathfrak{K}_2$ are members of the skeleton are investigated. Finally, it is shown that Schwabauer's distributive sublattice is actually a maximal modular sublattice.

1. BASIC CONCEPTS

1.1. Equations and completeness. A *semigroup* is a pair (S, f) consisting of a set S and a binary operation f on S satisfying $f(f(a, b), c) = f(a, f(b, c))$ for all $a, b, c \in S$. (S, f) is called *commutative* if, for all $a, b \in S, f(a, b) = f(b, a)$. We deal exclusively with commutative semigroups and will write simply ab for $f(a, b)$ and S for (S, f).

The free commutative semigroup on countably many generators, $F(\omega)$, is the set of sequences $(u_n)_{n\in N}$ of non-negative integers, such that $u_n = 0$ for all but finitely many $n \in N$ and $\sum u_n \geqq 1$, with component-wise addition. For convenience we write (u_n) for $(u_n)_{n\in N}$ and, if $u_n = 0$ for all $n > m$, we sometimes write $(u_1, u_2, \ldots u_m)$ for $(u_n)_{n\in N}$.

A commutative semigroup equation is a pair $((u_n), (v_n))$ of elements of $F(\omega)$. A commutative semigroup S is said to satisfy the equation $((u_n), (v_n))$ if, for every family $(a_n)_{n\in N}$ of elements of S,

$$\prod \{a_i^{u_i} | u_i \neq 0\} = \prod \{a_i^{v_i} | v_i \neq 0\}.$$

A class $\mathfrak{K}$ of commutative semigroups is said to satisfy an equation e (a set Σ of equations) if every semigroup in $\mathfrak{K}$ satisfies e (satisfies every equation in Σ).

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For a set Σ of equations, we define a set $\Gamma\Sigma$ of equations as follows: $e \in \Gamma\Sigma$ if and only if there exists a finite sequence $e_1, e_2, \ldots e_m$ of equations such that $e_m = e$, and such that

(P): for each $i \leqq m$, one of the following holds.

(P1) $e_i \in \Sigma$ or $e_i = ((u_n), (u_n))$ for some $(u_n)_{n \in N} \in F(\omega)$.

(P2) There exists $j < i$ such that $e_j = ((u_n), (v_n))$ and $e_i = ((v_n), (u_n))$.

(P3) There exists $j < i$ and a permutation π of N such that $e_j = ((u_n), (v_n))$ and $e_i = ((u_{\pi(n)}), (v_{\pi(n)}))$.

(P4) There exists $j < i$ such that e_i is obtained from e_j by multiplication, i.e., $e_j = ((u_n), (v_n))$ and $e_i = ((u_n + w_n), (v_n + w_n))$ for some $(w_n)_{n \in N} \in F(\omega)$.

(P5) There exists $j < i$ such that e_i is obtained from e_j by substitution, i.e., $e_j = ((u_n), (v_n))$ and for some $p \in N$ and $(k_n)_{n \in N} \in F(\omega)$, $e_i = ((u_n + k_n u_p), (v_n + k_n v_p))$. (See note below.)

(P6) There exists $j < i$ such that e_i is obtained from e_j by identification of variables, i.e., $e_j = ((u_n), (v_n))$ and there exist p, q with $1 \leqq p < q$ such that

$$e_i = ((u_1, \ldots u_{p-1}, 0, u_{p+1}, \ldots u_{q-1}, u_q + u_p, u_{q+1}, \ldots), (v_1, \ldots v_{p-1}, 0, v_{p+1}, \ldots v_{q-1}, v_q + v_p, v_{q+1}, \ldots)).$$

(P7) There exist $j, k < i$ such that $e_j = ((u_n), (v_n))$, $e_k = ((v_n), (w_n))$ and $e_i = ((u_n), (w_n))$.

Note. (P5) does not yield what intuitively is the result of substituting some term $(h_n)_{n \in N}$ for the pth variable in e_j to obtain

$$((u_1 + h_1 u_p, \ldots u_{p-1} + h_{p-1} u_p, h_p u_p, u_{p+1} + h_{p+1} u_p, \ldots), (v_1 + h_1 v_p, \ldots v_{p-1} + h_{p-1} v_p, h_p v_p, v_{p+1} + h_{p+1} v_p, \ldots))$$

from $((u_n)_{n \in N}, (v_n)_{n \in N})$. However, these two operations are equivalent modulo (P6). For example, to obtain the above equation from $((u_n), (v_n))$ using (P5) and (P6): if $h_p \geqq 1$ then apply (P5) with $k_i = h_i$ for $i \neq p$ and $k_p = h_p - 1$. If $h_p = 0$ then we may assume, in view of (P3), that $h_q \geqq 1$ for some $q > p$ and then apply (P5) with $k_i = h_i$ for $i \neq q$, $k_q = h_q - 1$. The result will be

$$((u_1 + h_1 u_p, \ldots u_{p-1} + h_{p-1} u_p, u_p, u_{p+1} + h_{p+1} u_p, \ldots u_q + h_q u_p - u_p, \ldots), (v_1 + h_1 v_p, \ldots v_{p-1} + h_{p-1} v_p, v_p, v_{p+1} + h_{p+1} v_p, \ldots v_q + h_q v_p - v_p, \ldots)).$$

If (P6) is then applied to identify the pth variable with the qth, one obtains the desired result.

A set Σ of equations is called closed if $\Sigma = \Gamma\Sigma$. We also write $\Sigma \to e$ for $e \in \Gamma\Sigma$ and in the case Σ consists of exactly one equation f, we write $f \to e$.

Then $e \in \Gamma\Sigma$ if and only if every commutative semigroup that satisfies Σ also satisfies e; this is the completeness theorem for commutative semigroups.

LEMMA 1.1. *If $e = ((u_n)_{n \in N}, (v_n)_{n \in N})$, then for each $i \in N$,*

$$e \to ((v_1, \ldots, v_{i-1}, v_i + u_i, v_{i+1}, \ldots), (v_1, \ldots v_{i-1}, 2v_i, v_{i+1}, \ldots)).$$

Proof. By (P4),

$$e \to ((u_1, \ldots u_{i-1}, 2u_i, u_{i+1}, \ldots), (v_1, \ldots v_{i-1}, v_i + u_i, v_{i+1}, \ldots))$$

and by (P5),

$$e \to ((u_1, \ldots u_{i-1}, 2u_i, u_{i+1}, \ldots), (v_1, \ldots v_{i-1}, 2v_i, v_{i+1}, \ldots)).$$

The result follows from (P2) and (P7).

For a class $\mathfrak{K}$ of commutative semigroups, let $\mathfrak{K}^*$ be the set of all equations satisfied by every member of $\mathfrak{K}$; then $\mathfrak{K}^*$ is closed. For a set Σ of equations, let Σ^* be the class of all commutative semigroups satisfying Σ; then Σ^* is equational. For equational classes $\mathfrak{K}$, $\mathfrak{K}'$, $\mathfrak{K} \subseteq \mathfrak{K}'$ if and only if $\mathfrak{K}'^* \subseteq \mathfrak{K}^*$, and for closed sets Σ, Σ' of equations, $\Sigma \subseteq \Sigma'$ if and only if $\Sigma'^* \subseteq \Sigma^*$.

Let $\mathfrak{L}$ be the lattice of equational classes of commutative semigroups, and $\mathfrak{L}'$ the lattice of closed sets of equations; then $\mathfrak{L}$ is dually isomorphic to $\mathfrak{L}'$ by the mapping $\mathfrak{K} \rightsquigarrow \mathfrak{K}^*$. For $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathfrak{L}$,

$$\mathfrak{K}_1 \wedge_{\mathfrak{L}} \mathfrak{K}_2 = \mathfrak{K}_1 \cap \mathfrak{K}_2 = (\mathfrak{K}_1^* \vee_{\mathfrak{L}'} \mathfrak{K}_2^*)^*,$$

and

$$\mathfrak{K}_1 \vee_{\mathfrak{L}} \mathfrak{K}_2 = (\mathfrak{K}_1^* \wedge_{\mathfrak{L}'} \mathfrak{K}_2^*)^* = (\mathfrak{K}_1^* \cap \mathfrak{K}_2^*)^*.$$

1.2. The invariants D, V, L, U. The equation $((u_n), (v_n))$ is called non-trivial if $u_n \neq v_n$ for some $n \in N$. A set of equations is called non-trivial if it contains at least one non-trivial equation; an equational class $\mathfrak{K}$ is called non-trivial if $\mathfrak{K}^*$ is non-trivial.

For a non-trivial equation $e = ((u_n), (v_n))$, define

$D(e) = \text{g.c.d.}\{|u_n - v_n| \mid n \in N, u_n \neq v_n\}$

$V(e) = \min\{u_n, v_n \mid n \in N, u_n \neq v_n\}$

$L(e) = \min\{\max\{u_n \mid n \in N\}, \max\{v_n \mid n \in N\}\}$

$$U(e) = \begin{cases} \min\left\{\sum_{n\in N} u_n, \sum_{n \in N} v_n\right\} & \text{if } \sum_{n\in N} u_n \neq \sum_{n\in N} v_n \\ \sum_{n \in N} u_n + V(e) & \text{if } \sum_{n\in N} u_n = \sum_{n \in N} v_n. \end{cases}$$

Note that $D(e)$, $L(e)$, and $U(e) \geqq 1$ and that $V(e) \leqq L(e) \leqq U(e)$.

For example, if $e = ((0, 1), (1, 0))$, then $D(e) = 1$, $V(e) = 0$, $L(e) = 1$ and $U(e) = 1 + 0 = 1$. A semigroup S satisfies e if and only if S has at most one element.

If $e = ((1, 0), (1, p))$ then $D(e) = p$, $V(e) = 0$, and $L(e) = U(e) = 1$. A commutative semigroup S satisfies e if and only if for all $s, t \in S$, $s = st^p$, i.e., if and only if S is an abelian group satisfying $s^p = 1$ for all $s \in S$.

For a non-trivial set Σ of equations, we define

$D(\Sigma) = \text{g.c.d.}\{D(e) \mid e \in \Sigma, e \text{ non-trivial}\}$

$V(\Sigma) = \min\{V(e) \mid e \in \Sigma, e \text{ non-trivial}\}$

$L(\Sigma) = \min\{L(e) \mid e \in \Sigma, e \text{ non-trivial}\}$

$U(\Sigma) = \min\{U(e) \mid e \in \Sigma, e \text{ non-trivial}\}.$

For each pair (r, n) of natural numbers, define

$$f_{r,n} : I^+ \to \{0, 1, \ldots r + n - 1\}$$

as follows:

$$f_{r,n}(k) = \begin{cases} k & \text{if } k \leqq r \\ r + [k - r]_n & \text{if } k > r, \end{cases}$$

where $[m]_n$ is the least non-negative residue of m modulo n, and I^+ is the set of non-negative integers.

Let $F_{r,n} = \{1, 2, \ldots r + n - 1\}$ and for $i_1, i_2 \in F_{r,n}$ define $i_1 i_2 = f_{r,n}(i_1 + i_2)$. Then $F_{r,n}$ with this operation is a commutative semigroup satisfying $((r), (r + n))$.

Let $F_{r,n}{}^+$ be $F_{r,n}$ with a unit adjoined, i.e., $F_{r,n}{}^+ = F_{r,n} \cup \{u\}$ where $ux = xu = x$ for all $x \in F_{r,n}{}^+$.

LEMMA 1.2. *$F_{r,n}$ satisfies a non-trivial equation e if and only if $U(e) \geqq r$ and $n|D(e)$.*

Proof. Let $e = ((u_i)_{i\in N}, (v_i)_{i\in N})$ be a non-trivial equation with $r \leqq U(e)$ and $n|D(e)$.

Case 1. $\sum u_i \neq \sum v_i$. Then $\sum u_i, \sum v_i \geqq r$. If $(k_i)_{i\in N}$ is a family in $F_{r,n}$ then

$$\prod \{k_i^{u_i}|u_i \neq 0\} = \prod \{1^{k_i u_i}|u_i \neq 0\} = f_{r,n}(\textstyle\sum \{k_i u_i|u_i \neq 0),$$

and

$$\prod \{k_i^{v_i}|v_i \neq 0\} = f_{r,n}(\textstyle\sum \{k_i v_i|v_i \neq 0\}).$$

But $\sum \{k_i u_i|u_i \neq 0\} \geqq \sum \{u_i\,|u_i \neq 0\} = \sum u_i \geqq r$; similarly

$$\textstyle\sum \{k_i v_i|v_i \neq 0\} \geqq r.$$

Moreover, since $n|D(e)$, it follows that $n|u_i - v_i$ for all i; thus, $\sum k_i u_i \equiv \sum k_i v_i$ (modulo n). This implies that $f_{r,n}(\sum k_i u_i) = f_{r,n}(\sum k_i v_i)$; thus,

$$\prod \{k_i^{u_i}|u_i \neq 0\} = \prod \{k_i^{v_i}|v_i \neq 0\}.$$

Case 2. $\sum u_i = \sum v_i$. Then $U(e) = \sum u_i + V(e) \geqq r$. Let $(k_i)_{i\in N}$ be a family in $F_{r,n}$. If $k_i = 1$ for all i with $u_i \neq v_i$ then $\sum k_i u_i = \sum k_i v_i$. If $k_i > 1$ for some i with $u_i \neq v_i$ then, since $u_i \neq v_i$ implies that $u_i, v_i \geqq V(e)$, it follows that $\sum k_i u_i \geqq \sum u_i + V(e) \geqq r$, $\sum k_i v_i \geqq \sum v_i + V(e) \geqq r$. Thus we again have $f_{r,n}(\sum k_i u_i) = f_{r,n}(\sum k_i v_i)$; hence

$$\prod \{k_i^{u_i}|u_i \neq 0\} = \prod \{k_i^{v_i}|v_i \neq 0\}.$$

It follows that $F_{r,n}$ satisfies e.

For the converse, assume that $e = ((u_i)_{i\in N}, (v_i)_{i\in N})$ is a non-trivial equation with $U(e) < r$. If $\sum u_i \neq \sum v_i$, then $r > \min\{\sum u_i, \sum v_i\}$, $(1)_{i\in N}$ is a family in $F_{r,n}$ and

$$\prod \{1^{u_i}|u_i \neq 0\} = f_{r,n}(\textstyle\sum u_i) \neq f_{r,n}(\sum v_i) = \prod \{1^{v_i}|v_i \neq 0\}.$$

Thus, in this case $F_{r,n}$ does not satisfy e. If $\sum u_i = \sum v_i$, then we may assume without loss of generality that $V(e) = u_1$ (and then

$$v_1 > u_1 \quad \text{and} \quad \sum u_i + u_1 < r).$$

Thus, by Lemma 1.1, $e \rightarrow ((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$. $(1)_{i \in N}$ is a family in $F_{r,n}$ and

$$\prod \{1^{w_i} | w_1 = 2v_1, w_i = v_i \text{ for } i \geqq 2\} = f_{r,n}(\sum v_i + v_1) \neq f_{r,n}(\sum v_i + u_1)$$
$$= \prod \{1^{x_i} | x_1 = u_1 + v_1, x_i = v_i \text{ for } i \geqq 2\}.$$

Thus $F_{r,n}$ does not satisfy $((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$ and hence does not satisfy e.

If $U(e) \geqq r$ but $n \nmid D(e)$ then we may assume without loss of generality that $u_1 \neq v_1$ and $n \nmid |u_1 - v_1|$. As above, $e \rightarrow ((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$. But since $\sum v_i + v_1 \not\equiv \sum v_i + u_1$ (modulo n), it follows that $F_{r,n}$ does not satisfy $((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$ and hence does not satisfy e.

This completes the proof.

LEMMA 1.3. *$F_{r,n}{}^+$ satisfies a non-trivial equation e if and only if $V(e) \geqq r$ and $n|D(e)$.*

Proof. $F_{r,n}{}^+$ satisfies $((r), (r + n))$. $((r), (r + n)) \rightarrow ((r), (r + kn))$ for all $k \geqq 1$. If $e = ((u_i)_{i \in N}, (v_i)_{i \in N})$ and if $V(e) \geqq r$ and $n|D(e)$, then $u_k \neq v_k$ implies that $u_k, v_k \geqq r$ and $n|u_k - v_k$; thus, $((r), (r + n)) \rightarrow ((u_k), (v_k))$. Thus $((r), (r + n)) \rightarrow ((u_i)_{i \in N}, (v_i)_{i \in N}) = e$. It follows that if $V(e) \geqq r$ and $n|D(e)$ then $F_{r,n}{}^+$ satisfies e.

Conversely, if $F_{r,n}{}^+$ satisfies $e = ((u_i)_{i \in N}, (v_i)_{i \in N})$, then since $F_{r,n}$ is a sub-semigroup of $F_{r,n}{}^+$, it follows that $F_{r,n}$ satisfies e and thus $n|D(e)$. We may assume without loss of generality that $V(e) = u_1$ (and then $u_1 < v_1$). Let $a_1 = 1 \in F_{r,n}{}^+$ and for $i \geqq 2$, let $a_i = u \in F_{r,n}{}^+$. Then $(a_i)_{i \in N}$ is a family in $F_{r,n}{}^+$ and

$$\prod \{a_i{}^{u_i} | u_i \neq 0\} = \begin{cases} f_{r,n}(u_1) & \text{if} \quad u_1 \neq 0 \\ u & \text{if} \quad u_1 = 0 \end{cases}$$

and

$$\prod \{a_i{}^{v_i} | v_i \neq 0\} = f_{r,n}(v_1).$$

Since $F_{r,n}{}^+$ satisfies e, it follows that $u_1 = 0$ and $f_{r,n}(u_1) = f_{r,n}(v_1)$. But $u_1 \neq v_1$: thus, $u_1, v_1 \geqq r$. This means that $V(e) \geqq r$.

THEOREM 1.1. *If $\Sigma \rightarrow e$ and e is non-trivial, then $U(\Sigma) \leqq U(e)$, $V(\Sigma) \leqq V(e)$, $L(\Sigma) \leqq L(e)$ and $D(\Sigma)|D(e)$.*

Proof. Assume $\Sigma \rightarrow e$. Since $F_{U(\Sigma), D(\Sigma)}$ satisfies Σ, it also satisfies e; thus, $U(\Sigma) \leqq U(e)$ and $D(\Sigma)|D(e)$. Moreover, if $V(\Sigma) > 0$, then $F_{V(\Sigma), D(\Sigma)}{}^+$ satisfies Σ, and hence also e, and thus $V(\Sigma) \leqq V(e)$.

To show that $L(\Sigma) \leqq L(e)$ it is enough to show that if $e_1, \ldots e_m$ is a sequence of equations satisfying (P) and $L(e_i) \geqq L(\Sigma)$ for all $i < m$, then $L(e_m) \geqq L(\Sigma)$. Let $e_i = (\alpha_i, \beta_i)$ where $\alpha_i, \beta_i \in F(\omega)$. Then $L(e_i) \geqq L(\Sigma)$ means that there exists an entry $\geqq L(\Sigma)$ in each of α_i and β_i. But if this holds for all $i < m$, then it is clear that whichever of (P1) to (P7) e_m satisfies, there will be an entry $\geqq L(\Sigma)$ in each of α_m and β_m, i.e., $L(e_m) \geqq L(\Sigma)$.

COROLLARY 1. *If* $\Sigma \rightarrow \Sigma'$, *then* $U(\Sigma) \leqq U(\Sigma')$, $V(\Sigma) \leqq V(\Sigma')$, $L(\Sigma) \leqq L(\Sigma')$ *and* $D(\Sigma)|D(\Sigma')$.

COROLLARY 2. D, V, L, U *as operators on sets of equations are invariant under* Γ, *i.e., for any non-trivial set* Σ *of equations,* $D(\Sigma) = D(\Gamma\Sigma)$, $V(\Sigma) = V(\Gamma\Sigma)$, $L(\Sigma) = L(\Gamma\Sigma)$ *and* $U(\Sigma) = U(\Gamma\Sigma)$.

For a non-trivial equational class $\mathfrak{K}$, define $D(\mathfrak{K}) = D(\mathfrak{K}^*)$, $V(\mathfrak{K}) = V(\mathfrak{K}^*)$, $L(\mathfrak{K}) = L(\mathfrak{K}^*)$ and $U(\mathfrak{K}) = U(\mathfrak{K}^*)$. Since for two equational classes $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$ if and only if $\mathfrak{K}_1^* \rightarrow \mathfrak{K}_2^*$, it follows that if $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$ then $U(\mathfrak{K}_1) \leqq U(\mathfrak{K}_2)$, $V(\mathfrak{K}_1) \leqq V(\mathfrak{K}_2)$, $L(\mathfrak{K}_1) \leqq L(\mathfrak{K}_2)$ and $D(\mathfrak{K}_1)|D(\mathfrak{K}_2)$.

2. THE SKELETON SUBLATTICE CONSISTING OF THE CLASSES $\Omega_{r,s,n}$.

2.1. Definition of the skeleton. For non-negative integers r, s, n with $r \leqq s$ and $n \geqq 1$, let $\Omega_{r,s,n} = \{((r, s), (r + n, s)), ((s), (s + n))\}^*$. Then $U(\Omega_{r,s,n}) = s = L(\Omega_{r,s,n})$, $V(\Omega_{r,s,n}) = r$ and $D(\Omega_{r,s,n}) = n$.

Note that since $((0, s), (n, s)) \rightarrow ((s), (s + n))$ by (P6), $\Omega_{0,s,n} = \{((0, s), (n, s))\}^*$. Since $((r), (r + n)) \rightarrow ((r, r), (r + n, r))$ by (P4), $\Omega_{r,r,n} = \{((r), (r + n))\}^*$.

$\Omega_{0,1,p}$ is the class of all commutative groups G satisfying $x^p = 1$ for all $x \in G$. $\Omega_{0,1,1} = \{((0, 1), (1, 1))\}^*$ and since $((0, 1), (1, 1)) \rightarrow ((0, 1), (1, 0))$ it follows that $\Omega_{0,1,1}$ is the zero of the lattice $\mathfrak{L}$.

Clearly, in view of (P4), if $r \leqq t$ and $s \leqq u$ then $\Omega_{r,s,n} \subseteq \Omega_{t,u,n}$. If in addition $n|m$, then a simple induction argument yields $\Omega_{r,s,n} \subseteq \Omega_{t,u,m}$. On the other hand, by the remark at the end of Chapter 1, if $\Omega_{r,s,n} \subseteq \Omega_{t,u,m}$ then $r \leqq t$, $s \leqq u$ and $n|m$. Thus $\Omega_{r,s,n} \subseteq \Omega_{t,u,m}$ if and only if $r \leqq t$, $s \leqq n$ and $n|m$.

2.2. The set $\Omega_{r,s,n}^*$ of equations holding in $\Omega_{r,s,n}$.

THEOREM 2.1. *For a non-trivial equation* e, $e \in \Omega_{r,s,n}^*$ *if and only if* $r \leqq V(e)$, $s \leqq L(e)$ *and* $n|D(e)$.

Proof. The "only if" part is a direct consequence of the results of the last section of Chapter 1.

For the converse, let $e = ((u_i)_{i \in N}, (v_i)_{i \in N}))$ and assume that $r \leqq V(e)$, $s \leqq L(e)$ and $n|D(e)$. It follows directly from the definition of V, D, and L that there exist j, k with $u_j, v_k \geqq s$ and that if $u_i \neq v_i$, then $n|u_i - v_i$ and $u_i, v_i \geqq r$. We may assume without loss of generality that $u_1 \geqq s$. But then

$$((r, s), (r + n, s)) \rightarrow ((u_1, u_2 \ldots), (u_1, v_2, v_3, \ldots)).$$

If $v_1 \geqq s$ then

$$((s), (s + n)) \rightarrow ((u_1), (v_1)) \rightarrow ((u_1, v_2, \ldots), (v_1, v_2, \ldots)).$$

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If $v_1 < s$ then $v_j \geqq s$ for some $j \geqq 2$ and then

$$((r, s), (r + n, s)) \to ((u_1, v_j), (v_1, v_j)) \to ((u_1, v_2, \ldots), (v_1, v_2, \ldots)).$$

Thus $\Omega_{r,s,n}{}^* \to e$, i.e., $e \in \Omega_{r,s,n}{}^*$.

COROLLARY 1. *For an equational class* $\mathfrak{K}$, $\Omega_{r,s,n} \subseteq \mathfrak{K}$ *if and only if* $r \leqq V(\mathfrak{K})$, $s \leqq L(\mathfrak{K})$ *and* $n|D(\mathfrak{K})$.

COROLLARY 2. $\Omega_{r,s,n} \vee \Omega_{t,u,m} = \Omega_{v,w,p}$, *where* $v = \max\{r, t\}$, $w = \max\{s, u\}$, $p = \text{l.c.m.}\{n, m\}$.

Proof. Since $\Omega_{r,s,n} \subseteq \Omega_{v,w,p}$ and $\Omega_{t,u,m} \subseteq \Omega_{v,w,p}$, it follows that

$$\Omega_{r,s,n} \vee \Omega_{t,u,m} \subseteq \Omega_{v,w,p}.$$

Thus it is enough to show that $\Omega_{v,w,p} \subseteq \Omega_{r,s,n} \vee \Omega_{t,u,m}$, i.e., that

$$\Omega_{r,s,n}{}^* \cap \Omega_{t,u,m}{}^* \subseteq \Omega_{v,w,p}{}^*.$$

But e non-trivial and $e \in \Omega_{r,s,n}{}^* \cap \Omega_{t,u,m}{}^*$ imply by the theorem that $V(e) \geqq r$, $L(e) \geqq s$, $n|D(e)$ and $V(e) \geqq t$, $L(e) \geqq u$ and $m|D(e)$; thus, $V(e) \geqq v$, $L(e) \geqq w$ and $p|D(e)$. It follows from the theorem that $e \in \Omega_{v,w,p}{}^*$ and this completes the proof.

Since every non-trivial equational class is contained in some $\Omega_{r,s,n}$, it follows from Corollary 2 that the class of all commutative semigroups is not the join of two smaller classes. This was also proved in [**2**].

THEOREM 2.2. $\Omega_{r,s,n} \wedge \Omega_{t,u,m} = \Omega_{v,w,d}$, *where* $v = \min\{r, t\}$, $w = \min\{s, u\}$ *and* $d = \text{g.c.d.}\{n, m\}$.

Proof. Since $\Omega_{r,s,n} \supseteq \Omega_{v,w,d}$, and $\Omega_{t,u,m} \supseteq \Omega_{v,w,d}$, it follows that

$$\Omega_{r,s,n} \wedge \Omega_{t,u,m} \supseteq \Omega_{v,w,d}.$$

To show the reverse inclusion, it is enough to show that

$$\{((v, w), (v + d, w)), ((w), (w + d))\} \subseteq (\Omega_{r,s,n} \wedge \Omega_{t,u,m})^* = \Omega_{r,s,n}{}^* \vee \Omega_{t,u,m}{}^*.$$

Assume that $s \leqq u$. Then there exist natural numbers p, q such that $pn = qm + d$ and $pn \geqq u$. By Theorem 2.1,

$$((s), (s + 2pn)) = ((s), (s + pn + qm + d)) \in \Omega_{r,s,n}{}^*$$
$$((s + pn + qm + d), (s + pn + d)) \in \Omega_{t,u,m}{}^*$$
$$((s + pn + d, s + d)) \in \Omega_{r,s,n}{}^*.$$

Thus $((s), (s + d)) \in \Omega_{r,s,n}{}^* \vee \Omega_{t,u,m}{}^*$. The case $u < s$ follows by symmetry; thus, $((w), (w + d)) \in \Omega_{r,s,n}{}^* \vee \Omega_{t,u,m}{}^*$.

Now assume that $r \leqq t$. Then $v = r$. There exist natural numbers h, k such that $w + kd \geqq s$, $r + hn \geqq w$. Then:

$$((r, w), (r, w + kd)) \in \Omega_{r,s,n}{}^* \vee \Omega_{t,u,m}{}^*$$
$$((r, w + kd), (r + hn, w + kd)) \in \Omega_{r,s,n}{}^*$$
$$((r + hn, w + kd), (r + hn + d, w + kd)) \in \Omega_{r,s,n}{}^* \vee \Omega_{t,u,m}{}^*$$
$$((r + hn + d, w + kd), (r + d, w + kd)) \in \Omega_{r,s,n}{}^*$$
$$((r + d, w + kd), (r + d, w)) \in \Omega_{r,s,n}{}^* \vee \Omega_{t,u,m}{}^*.$$

Thus, $((v, w), (v + d, w)) \in (\Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*) \vee \Omega_{t,u,m}^* = \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$. The case $t < r$ follows by symmetry. This completes the proof.

Let A be the lattice of pairs (r, s) of non-negative integers such that $r \leqq s$ and $s \geqq 1$, ordered component-wise, i.e., $(r, s) \leqq (t, u)$ if and only if $r \leqq t$ and $s \leqq u$. Let N^+ be the lattice of natural numbers ordered by division. Then, by the above theorems, the map given by $(r, s, n) \rightsquigarrow \Omega_{r,s,n}$ is a lattice isomorphism of $A \times N^+$ onto a sublattice of $\mathfrak{L}$.

2.3. Equations implying $\Omega_{r,s,n}^*$.

THEOREM 2.3. *For a non-trivial equation e, $e \rightarrow \Omega_{r,s,n}^*$ if and only if $V(e) \leqq r$, $U(e) \leqq s$ and $D(e)|n$.*

Proof. It follows from the results in the last section of Chapter 1 that if $e \rightarrow \Omega_{r,s,n}^*$ then $V(e) \leqq r = V(\Omega_{r,s,n})$, $U(e) \leqq s = U(\Omega_{r,s,n})$ and $D(e)|n = D(\Omega_{r,s,n})$.

For the converse, let $e = ((u_i)_{i\in N}, (v_i)_{i\in N})$ and assume that $V(e) \leqq r$, $U(e) \leqq s$ and $D(e)|n$. For each $i \in N$, by Lemma 1.1,

$$e \rightarrow ((v_1, \ldots v_{i-1}, v_i + u_i, v_{i+1}, \ldots), (v_1, \ldots v_{i-1}, 2v_i, v_{i+1}, \ldots)).$$

Let $w_i = \sum_{j\in N} v_j + \min\{u_i, v_i\}$ and let $d_i = |u_i - v_i|$. Then for each $i \in N$, $e \rightarrow ((w_i), (w_i + d_i))$. Thus for each $i \in N$ with $u_i \neq v_i$, $e^* \subseteq \Omega_{w_i,w_i,d_i}$. By Theorem 2.2, $e^* \subseteq \Omega_{w,w,d}$ where $w = \min\{w_i | u_i \neq v_i\}$, and

$$d = \text{g.c.d.}\{d_i | d_i \neq 0\} = D(e).$$

If $\sum_{i\in N} u_i = \sum_{i\in N} v_i$ then $U(e) = w_j$ for some $j \in N$ and thus $e^* \subseteq \Omega_{U(e),U(e),D(e)}$. If $\sum u_i \neq \sum v_i$, then

$$e \rightarrow ((\textstyle\sum_{i\in N} u_i), (\sum_{i\in N} v_i)) \rightarrow ((U(e)), (U(e) + h))$$

where $h = |\sum u_i - \sum v_i|$ is divisible by $D(e)$. But then $e^* \subseteq \Omega_{U(e),U(e),D(e)}$.

Now assume without loss of generality that $V(e) = u_1$. Then

$$e \rightarrow ((u_1, \textstyle\sum_{i\geqq 2} u_i), (v_1, \sum_{i\geqq 2} v_i))$$
$$\rightarrow ((u_1, U(e) + \textstyle\sum_{i\geqq 2} u_i), (v_1, U(e) + \sum_{i\geqq 2} v_i)).$$

Since $D(e)|\sum_{i\geqq 2} u_i - \sum_{i\geqq 2} v_i$ and since $e \rightarrow ((U(e)), (U(e) + D(e)))$, it follows that $e \rightarrow ((u_1, U(e)), (v_1, U(e)))$. Thus $e^* \subseteq \Omega_{V(e),U(e),h}$, where $h = v_1 - u_1$ is divisible by $D(e)$. This, together with $e^* \subseteq \Omega_{U(e),U(e),D(e)}$ yields $e^* \subseteq \Omega_{V(e),U(e),D(e)}$. Since $V(e) \leqq r$, $U(e) \leqq s$ and $D(e)|n$, it follows that $e^* \subseteq \Omega_{r,s,n}$.

COROLLARY. *For an equational class $\mathfrak{K}$, $\mathfrak{K} \subseteq \Omega_{r,s,n}$ if and only if $V(\mathfrak{K}) \leqq r$, $U(\mathfrak{K}) \leqq s$ and $D(\mathfrak{K})|n$.*

Proof. The "only if" part follows from the remark at the end of Chapter 1; the converse follows from the fact that if $\mathfrak{K}$ is a non-trivial equational class, then there exist equations $e_1, e_2, e_3 \in \mathfrak{K}^*$ such that $V(e_1) = V(\mathfrak{K})$, $U(e_2) = U(\mathfrak{K})$ and $D(e_3) = D(\mathfrak{K})$.

LEMMA 2.1. *For a non-trivial equation e, if $L(e) \leqq t$ then there exists $k \in N$ such that*

$$e \rightarrow (\underbrace{(t, t, \ldots t),}_{k} \ (t + D(e)\underbrace{, t, t, \ldots t)}_{k-1}).$$

Proof. Let e be a non-trivial equation with $L(e) \leqq t$ and $D(e) = d$. We may assume without loss of generality that $e = ((u_1, \ldots u_n), (v_1, \ldots v_n))$ where $u_i \leqq t$ for all $i \leqq n$. If $u_j < v_j$ for some $j \leqq n$ then

$$e \rightarrow ((t, t, \ldots t), (v_1 + t - u_1, \ldots v_n + t - u_n))$$

where $v_j + t - u_j > t$. If $u_i \geqq v_i$ for all $i \leqq n$, then

$$e \rightarrow ((u_1 + t - v_1, \ldots u_n + t - v_n), (t, \ldots t))$$

where, since e is non-trivial, $u_i + t - v_i > t$ for some $i \leqq n$. Thus, in either case, there exist $w_2, \ldots w_n$ and $s \geqq 1$ such that

$$e \rightarrow (\underbrace{(t, \ldots t)}_{n}, (t + s, w_2, \ldots w_n)).$$

Choose h so that $t + hs \geqq U(e)$ and let $k = h(n - 1) + 1$. For each m with $0 \leqq m \leqq h$, let

$$\alpha_m = (t + ms, \underbrace{w_2, \ldots w_n, \ldots w_2, \ldots w_n}_{m(n-1)}, \underbrace{t, \ldots t}_{k - m(n-1) - 1}).$$

By (P4),

$$e \rightarrow ((t + ms, \underbrace{t, \ldots t),}_{n-1} \ (t + ms + s, w_2, \ldots w_n))$$

for each $m \geqq 0$. Thus, again by (P4), $e \rightarrow (\alpha_m, \alpha_{m+1})$ for each m with $0 \leqq m < h$. By (P7), it follows that

$$e \rightarrow (\alpha_0, \alpha_h) = (\underbrace{(t, \ldots t),}_{k} \ \underbrace{(t + hs, w_2, \ldots w_n, \ldots w_n)}_{k})$$

and thus by (P4),

$$e \rightarrow (\underbrace{(t + d, t, \ldots t),}_{k} \ \underbrace{(t + hs + d, w_2, \ldots w_n, \ldots w_2, \ldots w_n)}_{k}).$$

But $t + hs \geqq U(e)$; thus $e \rightarrow ((t + hs), (t + hs + d))$. It follows that

$$e \rightarrow (\underbrace{(t, \ldots t),}_{k} \ \underbrace{(t + d, t, \ldots t)}_{k}).$$

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COROLLARY. *For non-trivial equational classes* $\mathfrak{K}_1, \mathfrak{K}_2, L(\mathfrak{K}_1 \vee \mathfrak{K}_2) = \max\{L(\mathfrak{K}_1), L(\mathfrak{K}_2)\}$.

Proof. Assume without loss of generality that $L(\mathfrak{K}_1) = t \geqq L(\mathfrak{K}_2)$. Then there exist $e_i \in \mathfrak{K}_i^*$ for $i = 1, 2$ such that $L(e_i) \leqq t$. By the lemma there exists $k \in N$ such that

$$e_i \to ((\underbrace{t, t, \ldots t}_{k}), (t + d, \underbrace{t, \ldots t}_{k-1}))$$

for $i = 1, 2$ where d is the least common multiple of $D(\mathfrak{K}_1)$ and $D(\mathfrak{K}_2)$. Thus

$$((\underbrace{t, \ldots t}_{k}), (t + d, \underbrace{t, \ldots t}_{k-1})) \in (\mathfrak{K}_1 \vee \mathfrak{K}_2)^*.$$

It follows that $L(\mathfrak{K}_1 \vee \mathfrak{K}_2) \leqq t$. On the other hand $L(\mathfrak{K}_1 \vee \mathfrak{K}_2) \geqq L(\mathfrak{K}_1) = t$; thus $L(\mathfrak{K}_1 \vee \mathfrak{K}_2) = \max\{L(\mathfrak{K}_1), L(\mathfrak{K}_2)\}$.

Summarizing the results of this section and the preceding one, we have that for a non-trivial equational class $\mathfrak{K}$, $\Omega_{V(\mathfrak{K}),L(\mathfrak{K}),D(\mathfrak{K})} \subseteq \mathfrak{K} \subseteq \Omega_{V(\mathfrak{K}),U(\mathfrak{K}),D(\mathfrak{K})}$. Moreover, these choices of the Ω's are the best possible in the following sense: if $\Omega_{r,s,n} \subseteq \mathfrak{K}$ then $\Omega_{r,s,n} \subseteq \Omega_{V(\mathfrak{K}),L(\mathfrak{K}),D(\mathfrak{K})}$ and if $\mathfrak{K} \subseteq \Omega_{r,s,n}$ then

$$\Omega_{V(\mathfrak{K}),U(\mathfrak{K}),D(\mathfrak{K})} \subseteq \Omega_{r,s,n}.$$

Thus if $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$, then $V(\mathfrak{K}) = r$, $L(\mathfrak{K}) \geqq s$, $U(\mathfrak{K}) \leqq t$ and $D(\mathfrak{K}) = n$.

THEOREM 2.4. *D is a lattice homomorphism from $\mathfrak{L} - \{E\}$ to N^+ and V, L and U are lattice homomorphisms from $\mathfrak{L} - \{E\}$ to the non-negative integers with their usual order, where E is the class of all commutative semigroups.*

Proof. For non-trivial equational classes $\mathfrak{K}_1, \mathfrak{K}_2$,

$$U(\mathfrak{K}_1 \wedge \mathfrak{K}_2) = U(\mathfrak{K}_1^* \cup \mathfrak{K}_2^*) = \min\{U(\mathfrak{K}_1), U(\mathfrak{K}_2)\}.$$

The rest of the proof follows from the above remarks, and the corollary to the last lemma.

3. HANGING THE MEAT ON THE BONES

3.1. The intervals $[\Omega_{r,s,n}, \Omega_{t,u,m}]$. Since for each equational class $\mathfrak{K}$ there exist $r, s, t, n \in N$ with $\Omega_{r,s,n} \subseteq \mathfrak{K} \subseteq \Omega_{r,t,n}$, it follows that the interval $[\Omega_{r,s,n}, \Omega_{r,s,pn}]$ is a jump for p prime and that $[\Omega_{r,s,n}, \Omega_{r+1,s,n}]$ is a jump for all $r \geqq 0$. Thus $[\Omega_{r,s,1}, \Omega_{r,s,m}]$ consists of exactly the classes $\Omega_{r,s,n}$ where $n|m$ and $[\Omega_{0,s,n}, \Omega_{s,s,n}]$ consists of exactly the classes $\Omega_{r,s,n}$ where $r \leqq s$. Moreover, if $\mathfrak{K} \subset \Omega_{r,r,n}$, then either $\mathfrak{K} \subseteq \Omega_{r-1,r,n}$ or $\mathfrak{K} \subseteq \Omega_{r,r,m}$ for some $m < n$. Thus $\Omega_{1,1,1}$, the class of all semilattices, is an atom in $\mathfrak{L}$ and for p prime, $\Omega_{0,1,p}$, the class of all abelian groups G satisfying $x^p = 1$ for all $x \in G$, is an atom in $\mathfrak{L}$.

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$\{((1, 1, 0), (0, 0, 2))\}^*$, the class of all semigroups with constant multiplication, is also an atom in $\mathfrak{L}$. Moreover, it is an easy consequence of the above remarks that this exhausts the set of atoms in $\mathfrak{L}$, a result proved in [**5**].

It remains only to investigate intervals of the form $[\Omega_{r,s,n}, \Omega_{r,t,n}]$ where $s < t$. It is easy to see that every such interval is infinite: for each $p \geqq t$, let $\mathfrak{K}_p = \Omega_{r,s,n} \vee \{e_{t,p}\}^*$ where

$$e_{t,p} = ((\underbrace{1, 1, \ldots 1}_{p}, 0), (\underbrace{0, \ldots 0}_{p}, t)).$$

Then $\mathfrak{K}_p \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and if $p \leqq q$, then $\mathfrak{K}_p \subseteq \mathfrak{K}_q$. Moreover, if $p > r + s$, then

$$f_p = ((r, s, \underbrace{1, 1, \ldots 1}_{p-r-s}), (r + n, s, \underbrace{1, 1, \ldots 1}_{p-r-s})) \in \mathfrak{K}_p^*$$

but $f_p \notin \mathfrak{K}_{p+1}^*$ since $e_{t,p} \not\rightarrow f_p$. Thus $\{\mathfrak{K}_p | p \geqq t, p > r + s\}$ is an infinite chain in $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.

The following lemma will be useful in the rest of this chapter.

LEMMA 3.1. *If $\Omega_{r,t,n} \subseteq \mathfrak{K}$ for some t then $e \in (\mathfrak{K} \wedge \Omega_{r,s,n})^*$ if and only if there exist $\tau_1, \tau_2, \tau_3, \tau_4 \in F(\omega)$ such that $e = (\tau_1, \tau_4)$ and (τ_1, τ_2), $(\tau_3, \tau_4) \in \mathfrak{K}^*$, $(\tau_2, \tau_3) \in \Omega_{r,s,n}^*$.*

Proof. The "if" part is trivial. On the other hand, if $e \in (\mathfrak{K} \wedge \Omega_{r,s,n})^* = \mathfrak{K}^* \vee \Omega_{r,s,n}^*$, then, since for arbitrary congruence relations θ_1, θ_2,

$$\theta_1 \vee \theta_2 = \bigcup \{\underbrace{\theta_1 \circ \theta_2 \circ \theta_1 \ldots . \theta_1}_{n} | n \geqq 1, n \text{ odd}\}$$

and since $\mathfrak{K}^*$ and $\Omega_{r,s,n}^*$ are congruence relations on $F(\omega)$, it follows that there exists a finite sequence $\tau_1, \tau_2, \ldots \tau_{2p} \in F(\omega)$ such that $e = (\tau_1, \tau_{2p})$ and

$$(\tau_i, \tau_{i+1}) \in \begin{cases} \mathfrak{K}^* & \text{for } i \text{ odd} \\ \Omega_{r,s,n}^* & \text{for } i \text{ even.} \end{cases}$$

We may assume without loss of generality that (τ_i, τ_{i+1}) is non-trivial for all $i \neq 1$ or $2p - 1$ and that $p \geqq 2$. But then, by Theorem 2.1, $L((\tau_i, \tau_{i+1})) \geqq s$ for even i, i.e., for all i with $2 \leqq i \leqq 2p - 1$, τ_i has an entry $\geqq s$. But then for all odd i with $3 \leqq i \leqq 2p - 3$, $V((\tau_i, \tau_{i+1})) \geqq r$, $L((\tau_i, \tau_{i+1})) \geqq s$ and $n | D((\tau_i, \tau_{i+1}))$; thus, $(\tau_i, \tau_{i+1}) \in \Omega_{r,s,n}^*$. It follows that $(\tau_2, \tau_{2p-1}) \in \Omega_{r,s,n}^*$. Thus we may take $\tau_1, \tau_2, \tau_{2p-1}, \tau_{2p}$ for the four elements of $F(\omega)$ in the theorem statement.

THEOREM 3.1. *If $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and if $u \leqq r$ and $m | n$, then*

$$\mathfrak{K} = (\mathfrak{K} \wedge \Omega_{u,t,m}) \vee \Omega_{r,s,n}.$$

Proof. Since $\mathfrak{R} \wedge \Omega_{u,t,m} \subseteq \mathfrak{R}$ and $\Omega_{r,s,n} \subseteq \mathfrak{R}$ it follows that

$$(\mathfrak{R} \wedge \Omega_{u,t,m}) \vee \Omega_{r,s,n} \subseteq \mathfrak{R}.$$

Thus it is enough to show that if $e \in ((\mathfrak{R} \wedge \Omega_{u,t,m}) \vee \Omega_{r,s,n})^*$, then $e \in \mathfrak{R}^*$.

Assume that $e \in ((\mathfrak{R} \wedge \Omega_{u,t,m}) \vee \Omega_{r,s,n})^* = (\mathfrak{R} \wedge \Omega_{u,t,m})^* \cap \Omega_{r,s,n}{}^*$. Then by Lemma 3.1, there exist $\tau_1, \tau_2, \tau_3, \tau_4 \in F(\omega)$ such that $(\tau_1, \tau_2), (\tau_3, \tau_4) \in \mathfrak{R}^*$, $(\tau_2, \tau_3) \in \Omega_{u,t,m}{}^*$ and $e = (\tau_1, \tau_4) \in \Omega_{r,s,n}{}^*$. But $\mathfrak{R}^* \subseteq \Omega_{r,s,n}{}^*$; thus,

$$\{(\tau_1, \tau_4), (\tau_1, \tau_2), (\tau_3, \tau_4)\} \subseteq \Omega_{r,s,n}{}^*.$$

Since $\{(\tau_1, \tau_4), (\tau_1, \tau_2), (\tau_3, \tau_4)\} \rightarrow (\tau_2, \tau_3)$, it follows that $(\tau_2, \tau_3) \in \Omega_{r,s,n}{}^*$. Since $(\tau_2, \tau_3) \in \Omega_{u,t,m}{}^*$, we have that $(\tau_2, \tau_3) \in \Omega_{r,s,n}{}^* \cap \Omega_{u,t,m}{}^* = \Omega_{r,t,n}{}^*$. But $\Omega_{r,t,n}{}^* \subseteq \mathfrak{R}^*$; thus, it follows that $e \in \mathfrak{R}^*$.

COROLLARY. *If $\mathfrak{R} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and $m|n$, then $\mathfrak{R} = (\mathfrak{R} \wedge \Omega_{r,t,m}) \vee \Omega_{0,1,n}$.*

Proof. By the theorem, $\mathfrak{R} = (\mathfrak{R} \wedge \Omega_{r,t,m}) \vee \Omega_{r,s,n}$. But $\Omega_{r,s,m} \subseteq \mathfrak{R} \wedge \Omega_{r,t,m}$ and $\Omega_{r,s,m} \vee \Omega_{0,1,n} = \Omega_{r,s,n}$; this yields the desired result.

LEMMA 3.2. *If*

$$V(\mathfrak{R}_1) \leqq V(\mathfrak{R}_2), q \leqq V(\mathfrak{R}_1), p \leqq V(\mathfrak{R}_2), q \leqq p, \max\{p, V(\mathfrak{R}_1)\} \geqq V(\mathfrak{R}_2)$$

and if $n|m = D(\mathfrak{R}_1) = D(\mathfrak{R}_2)$, then

$$(\mathfrak{R}_1 \wedge \Omega_{q,u,n}) \vee (\mathfrak{R}_2 \wedge \Omega_{p,u,n}) = (\mathfrak{R}_1 \vee \mathfrak{R}_2) \wedge \Omega_{p,u,n},$$

where $u \geqq U(\mathfrak{R}_1), U(\mathfrak{R}_2)$.

Proof. Clearly $(\mathfrak{R}_1 \vee \mathfrak{R}_2) \wedge \Omega_{p,u,n} \supseteq (\mathfrak{R}_1 \wedge \Omega_{q,u,n}) \vee (\mathfrak{R}_2 \wedge \Omega_{p,u,n})$. On the other hand, if $e \in (\mathfrak{R}_1{}^* \vee \Omega_{q,u,n}{}^*) \cap (\mathfrak{R}_2{}^* \vee \Omega_{p,u,n}{}^*)$, then, by Lemma 3.1, there exist $\tau_i \in F(\omega)$ for $1 \leqq i \leqq 6$ such that $e = (\tau_1, \tau_6)$ and (τ_1, τ_2), $(\tau_3, \tau_6) \in \mathfrak{R}_1{}^*$, (τ_1, τ_4), $(\tau_5, \tau_6) \in \mathfrak{R}_2{}^*$, $(\tau_2, \tau_3) \in \Omega_{q,u,n}{}^*$ and $(\tau_4, \tau_5) \in \Omega_{p,u,n}{}^*$. Let $r = V(\mathfrak{R}_1)$. If both (τ_2, τ_3) and (τ_4, τ_5) are non-trivial then τ_2, τ_3, τ_4 and τ_5 all contain an entry $\geqq u$. But $(\tau_1, \tau_2) \in \mathfrak{R}_1{}^* \subseteq \Omega_{r,r,m}{}^*$ and

$$(\tau_1, \tau_4) \in \mathfrak{R}_2{}^* \subseteq \Omega_{r,r,m}{}^* \quad \text{implies that} \quad (\tau_2, \tau_4) \in \Omega_{r,r,m}{}^*.$$

It follows that $(\tau_2, \tau_4) \in \Omega_{r,u,m}{}^* \subseteq \mathfrak{R}_1{}^*$. Thus $(\tau_1, \tau_4) \in \mathfrak{R}_1{}^*$. Similarly, $(\tau_5, \tau_6) \in \mathfrak{R}_1{}^*$. Thus $e \in (\mathfrak{R}_1{}^* \cap \mathfrak{R}_2{}^*) \vee \Omega_{p,u,n}{}^*$. If (τ_2, τ_3) is trivial, then $e = (\tau_1, \tau_6) \in \mathfrak{R}_1{}^*$. Thus (τ_1, τ_4), (τ_5, τ_6) and $(\tau_1, \tau_6) \in \Omega_{r,r,m}{}^*$. Since $\{(\tau_1, \tau_4), (\tau_5, \tau_6), (\tau_1, \tau_6)\} \rightarrow (\tau_4, \tau_5)$, we have $(\tau_4, \tau_5) \in \Omega_{r,r,m}{}^*$. Thus $(\tau_4, \tau_5) \in \Omega_{r,r,m}{}^* \cap \Omega_{p,u,n}{}^* \subseteq \mathfrak{R}_2{}^*$. It follows that $e \in \mathfrak{R}_1{}^* \cap \mathfrak{R}_2{}^*$. Similarly, if (τ_4, τ_5) is trivial then $e \in \mathfrak{R}_1{}^* \cap \mathfrak{R}_2{}^*$. Thus in any case,

$$e \in (\mathfrak{R}_1{}^* \cap \mathfrak{R}_2{}^*) \vee \Omega_{p,u,n}{}^*.$$

This completes the proof.

3.2. The sublattice $\mathfrak{L}_n$ with constant D. For $n \in N$, let

$$\mathfrak{L}_n = \{\mathfrak{R} \in \mathfrak{L} | D(\mathfrak{R}) = n\}$$

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and for each non-negative integer k, let $\mathfrak{L}_{n,k} = \{\mathfrak{K} \in \mathfrak{L}_n | V(\mathfrak{K}) = k\}$. Then the $\mathfrak{L}_{n,k}$'s are pairwise disjoint, and $\mathfrak{L}_n = \bigcup_{k \geqq 0} \mathfrak{L}_{n,k}$.

For $p \leqq q$, define a mapping $\delta_{p,q,n} : \mathfrak{L}_{n,q} \to \mathfrak{L}_{n,p}$ as follows: for $\mathfrak{K} \in \mathfrak{L}_{n,q}$ with $U(\mathfrak{K}) = u$, $\delta_{p,q,n}(\mathfrak{K}) = \mathfrak{K} \wedge \Omega_{p,u,n}$. If $u \leqq s$, then, since $\mathfrak{K} \subseteq \Omega_{q,u,n}$,

$$\mathfrak{K} \wedge \Omega_{p,s,n} = \mathfrak{K} \wedge \Omega_{q,u,n} \wedge \Omega_{p,s,n} = \mathfrak{K} \wedge \Omega_{p,u,n} = \delta_{p,q,n}(\mathfrak{K}).$$

Thus $\delta_{p,q,n}$ is a meet homomorphism. It follows from Lemma 3.2 that $\delta_{p,q,n}$ is a join homomorphism. By Theorem 3.1, if $\mathfrak{K} \in \mathfrak{L}_{n,q}$ and $p < q$ then

$$\delta_{p,q,n}(\mathfrak{K}) \vee \Omega_{q,q,n} = \mathfrak{K};$$

thus, $\delta_{p,q,n}$ is one-to-one.

Thus, for $p < q$, $\delta_{p,q,n}$ is a lattice monomorphism of $\mathfrak{L}_{n,q}$ into $\mathfrak{L}_{n,p}$ with the property that $\delta_{p,q,n}(\mathfrak{K}) \vee \Omega_{q,q,n} = \mathfrak{K}$. Clearly, if $p < q < r$ then

$$\delta_{p,r,n} = \delta_{p,q,n} \circ \delta_{q,r,n}.$$

THEOREM 3.2. *The mapping $\mathfrak{K} \rightsquigarrow (\delta_{0,V(\mathfrak{K}),n}(\mathfrak{K}), V(\mathfrak{K}))$ is an embedding of $\mathfrak{L}_n$ as a meet subsemilattice into $\mathfrak{L}_{n,0} \times I^+$, where I^+ is the lattice of non-negative integers with their usual order.*

Proof. Since $V(\mathfrak{K}_1 \wedge \mathfrak{K}_2) = \min\{V(\mathfrak{K}_1), V(\mathfrak{K}_2)\}$ and since the $\delta_{0,p,n}$'s are one-to-one, it is enough to show that if $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathfrak{L}_n$ then

$$\delta_{0,V(\mathfrak{K}_1),n}(\mathfrak{K}_1) \wedge \delta_{0,V(\mathfrak{K}_2),n}(\mathfrak{K}_2) = \delta_{0,V(\mathfrak{K}_1 \wedge \mathfrak{K}_2),n}(\mathfrak{K}_1 \wedge \mathfrak{K}_2).$$

Assume that $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathfrak{L}_n$ and let $u = \max\{U(\mathfrak{K}_1), U(\mathfrak{K}_2)\}$. Then

$$\begin{aligned}\delta_{0,V(\mathfrak{K}_1),n}(\mathfrak{K}_1) \wedge \delta_{0,V(\mathfrak{K}_2),n}(\mathfrak{K}_2) &= (\mathfrak{K}_1 \wedge \Omega_{0,u,n}) \wedge (\mathfrak{K}_2 \wedge \Omega_{0,u,n}) \\ &= (\mathfrak{K}_1 \wedge \Omega_{0,u,n}) \wedge (\mathfrak{K}_2 \wedge \Omega_{0,u,n}) \\ &= (\mathfrak{K}_1 \wedge \mathfrak{K}_2) \wedge \Omega_{0,u,n} \\ &= \delta_{0,V(\mathfrak{K}_1 \wedge \mathfrak{K}_2),n}(\mathfrak{K}_1 \wedge \mathfrak{K}_2)\end{aligned}$$

and this completes the proof.

It will be shown in Section 4 that this embedding is not a lattice embedding, i.e., that it does not preserve joins.

3.3. A mapping between intervals of the lattice. If r, s, t, u, n are non-negative integers such that $r < s \leqq t < u$ and $n \geqq 1$, then, since

$$\Omega_{s,t,n} \wedge \Omega_{r,u,n} = \Omega_{r,t,n} \quad \text{and} \quad \Omega_{s,u,n} \wedge \Omega_{r,u,n} = \Omega_{r,u,n},$$

it follows that the restriction of $\delta_{r,s,n}$ to $[\Omega_{s,t,n}, \Omega_{s,u,n}]$ is a lattice monomorphism mapping into $[\Omega_{r,t,n}, \Omega_{r,u,n}]$. Let $\phi_{r,s,t,u,n} : [\Omega_{s,t,n}, \Omega_{s,u,n}] \to [\Omega_{r,t,n}, \Omega_{r,u,n}]$ be the restriction of $\delta_{r,s,n}$. We will investigate which of the $\phi_{r,s,t,u,n}$'s are actually isomorphisms, i.e., for which values of r, s, t, u, n the image of $\phi_{r,s,t,u,n}$ is the whole interval $[\Omega_{r,t,n}, \Omega_{r,u,n}]$.

LEMMA 3.3. *$\phi_{0,1,t,u,n}$ maps onto $[\Omega_{0,t,n}, \Omega_{0,u,n}]$ for all $t, u, n \geqq 1$.*

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Proof. Let $\mathfrak{K} \in [\Omega_{0,t,n}, \Omega_{0,u,n}]$. It is enough to show that for each non-trivial $e \in \mathfrak{K}^*$, there exists $\Sigma_e \subseteq \Omega_{1,t,n}{}^*$ such that $\{e\} \cup \Omega_{0,u,n}{}^* \leftrightarrow \Sigma_e \cup \Omega_{0,u,n}{}^*$, for then $\mathfrak{K} = \phi_{0,1,t,u,n}((\bigcup_{e\in\mathfrak{K}^*}\Sigma_e)^* \wedge \Omega_{1,u,n})$.

Let $e \in \mathfrak{K}^*$ be non-trivial. Then $L(e) \geqq t$ and $n|D(e)$. If $V(e) \geqq 1$, then we can take $\Sigma_e = \{e\}$. If $V(e) = 0$, then we may assume without loss of generality that $e = ((u_i)_{i\in N}, (v_i)_{i\in N})$ where $u_1 = 0, v_1 > 0$ (and then $n|v_1$) and $u_2 > 0$. Then let

$$\Sigma_e = \{((v_i)_{i\in N}, (2v_1, v_2, \ldots)), ((u_2, u_3, \ldots), (u_2 + v_1, u_3, \ldots))\}.$$

Clearly, $\Sigma_e \subseteq \Omega_{1,t,n}{}^*$ and $e \rightarrow \Sigma_e$. It remains only to show that $\Sigma_e \cup \Omega_{0,u,n}{}^* \rightarrow e$. But

$$((v_i)_{i\in N}, (uv_1, v_2, v_3, \ldots)) \in \Gamma\Sigma_e$$
$$((uv_1, v_2, v_3, \ldots), (0, u_2 + uv_1, u_3, \ldots)) \in \Omega_{0,u,n}{}^*$$
$$((0, u_2 + uv_1, u_3, \ldots), (0, u_2, u_3, \ldots)) \in \Gamma\Sigma_e.$$

Thus, $\Sigma_e \cup \Omega_{0,u,n}{}^* \rightarrow e$. This completes the proof.

COROLLARY. *$\delta_{0,1,n}$ is an isomorphism of $\mathfrak{L}_{n,1}$ onto $\mathfrak{L}_{n,0}$ for each $n \in N$.*

LEMMA 3.4. *If $r > 0$ and $r + n < u$ then $\phi_{r,s,t,u,n}$ does not map onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$.*

Proof. Let $e = ((r, r + n, t), (r + n, r, t))$ and let $\mathfrak{K} = e^* \wedge \Omega_{r,u,n}$. Then $\mathfrak{K} \in [\Omega_{r,t,n}, \Omega_{r,u,n}]$. If $\mathfrak{K} = \phi_{r,s,t,u,n}(\mathfrak{K}')$ for some $\mathfrak{K}' \in [\Omega_{s,t,n}, \Omega_{s,u,n}]$, then by Theorem 3.1, $\mathfrak{K} \vee \Omega_{s,t,n} = (\mathfrak{K}' \wedge \Omega_{r,u,n}) \vee \Omega_{s,t,n} = \mathfrak{K}'$ and this implies that $\mathfrak{K} = \phi_{r,s,t,u,n}(\mathfrak{K} \vee \Omega_{s,t,n}) = (\mathfrak{K} \vee \Omega_{s,t,n}) \wedge \Omega_{r,u,n}$. Thus to prove the lemma, it is enough to show that $\mathfrak{K} \neq (\mathfrak{K} \vee \Omega_{s,t,n}) \wedge \Omega_{r,u,n}$.

Since $e \in \mathfrak{K}^*$, it is enough to show that $e \notin ((\mathfrak{K} \vee \Omega_{s,t,n}) \wedge \Omega_{r,u,n})^* = (\mathfrak{K}^* \cap \Omega_{s,t,n}{}^*) \vee \Omega_{r,u,n}{}^*$. Suppose that $e \in (\mathfrak{K}^* \cap \Omega_{s,t,n}{}^*) \vee \Omega_{r,u,n}{}^*$. Then there exist $\tau_1, \tau_2 \in F(\omega)$ such that

$$((r, r + n, t), \tau_1), (\tau_2, (r + n, r, t)) \in \mathfrak{K}^* \cap \Omega_{s,t,n}{}^*$$

and $(\tau_1, \tau_2) \in \Omega_{r,u,n}{}^*$.

Now $((r, r + n, t), \tau_1) \in \mathfrak{K}^* = \Gamma e \vee \Omega_{r,u,n}{}^*$ implies that there exist $\tau_3, \tau_4 \in F(\omega)$ such that $((r, r + n, t), \tau_3) \in \Gamma e$, $(\tau_3, \tau_4) \in \Omega_{r,u,n}{}^*$ and $(\tau_4, \tau_1) \in \Gamma e$. But $((r, r + n, t), \tau_3) \in \Gamma e$ implies that $\tau_3 = (r, r + n, t)$ or $(r + n, r, t)$ in the case $r + n \neq t$ and that

$$\tau_3 = (r, r + n, r + n), (r + n, r + n, r) \quad \text{or} \quad (r + n, r, r + n)$$

in the case $r + n = t$. In any case, since $r + n < u$ and $(\tau_3, \tau_4) \in \Omega_{r,u,n}{}^*$, it follows that $\tau_3 = \tau_4$. Thus $((r, r + n, t), \tau_1) \in \Gamma e$. A repetition of this argument yields $\tau_1 = \tau_2$. Thus $((r, r + n, t), (r + n, r, t)) \in \Omega_{s,t,n}{}^*$. But this is a contradiction, since $r < s$. This completes the proof.

3.4. Restriction of the mapping to Schwabauer classes. An equational class is called a Schwabauer class, or S-class, if it can be defined by equations of the form $((u_i)_{i\in N}, (v_i)_{i\in N})$ where $u_i \leqq v_i$ for all $i \in N$. Clearly, all the $\Omega_{r,s,n}$'s are S-classes. The set of all S-classes forms a distributive sublattice γ of the lattice of equational classes of commutative semigroups (see [**8**]); this will be proved in Section 7.

LEMMA 3.5. *If* $r + n \geqq t$, *then* $[\Omega_{r,s,n}, \Omega_{r,t,n}] \subseteq \gamma$.

Proof. Let $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ where $r + n \geqq t$. To prove that $\mathfrak{K}$ is an S-class, it is enough to prove that every $e \in \mathfrak{K}^*$ with $e \notin \Omega_{r,t,n}{}^*$ is equivalent to an equation of the form $((u_i)_{i\in N}, (v_i)_{i\in N})$, where $u_i \leqq v_i$ for all $i \in N$.

Assume that $e \in \mathfrak{K}^*$ and that $e \notin \Omega_{r,t,n}{}^*$. Since $\mathfrak{K}^* \subseteq \Omega_{r,s,n}{}^*$,

$$e = ((u_i)_{i\in N}, (v_i)_{i\in N})$$

where $u_i \neq v_i$ implies that $u_i, v_i \geqq r$ and $n|(u_i - v_i)$. Since $e \notin \Omega_{r,t,n}{}^*$ we may assume without loss of generality that $u_i < t$ for all $i \in N$. But then, if $u_i > v_i$ for some i, it follows that $u_i = v_i + kn$ where $k \geqq 1$ and $v_i \geqq r$. But then $u_i \geqq v_i + n \geqq r + n \geqq t$ and this is a contradiction. Thus $u_i \leqq v_i$ for all $i \in N$.

For $t < u$, let $T_{t,u} = \{(u_i)_{i\in N} \in F(\omega)|u_i < u \text{ for all } i, u_1 \geqq t\}$. For $T \subseteq T_{t,u}$ and $n \geqq 1$, let $T(n) = \{((u_i)_{i\in N}, (u_1 + n, u_2, u_3, \ldots))|(u_i)_{i\in N} \in T\}$. Then $T(n) \subseteq \Omega_{t,t,n}{}^*$, since if $(u_i)_{i\in N} \in T_{t,u}$, then $u_1 \geqq t$; thus,

$$((t), (t + n)) \rightarrow ((u_1), (u_1 + n)) \rightarrow ((u_i)_{i\in N}, (u_1 + n, u_2, u_3, \ldots)).$$

LEMMA 3.6. $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ *is an S-class if and only if there exists* $T \subseteq T_{s,t}$ *such that* $\mathfrak{K} = T(n)^* \wedge \Omega_{r,t,n}$.

Proof. Clearly if $\mathfrak{K} = T(n)^* \wedge \Omega_{r,t,n}$ for some $T \subseteq T_{s,t}$, then $\mathfrak{K}$ is an S-class.

On the other hand, if $\mathfrak{K}$ is an S-class there exists a set Σ of equations of the form $((u_i)_{i\in N}, (v_i)_{i\in N})$ where $u_i \leqq v_i$ for all $i \in N$ such that $\mathfrak{K} = \Sigma^*$. It is enough to show that for each $e \in \Sigma$ with $e \notin \Omega_{r,t,n}{}^*$ there exists $\bar{e} \in T_{s,t}(n)$ such that $\{e\} \cup \Omega_{r,t,n}{}^* \leftrightarrow \{\bar{e}\} \cup \Omega_{r,t,n}{}^*$; for then

$$\mathfrak{K} = \{\bar{e}|e \in \Sigma - \Omega_{r,t,n}{}^*\}^* \wedge \Omega_{r,t,n}.$$

Let $e = ((u_i)_{i\in N}, (v_i)_{i\in N}) \in \Sigma - \Omega_{r,t,n}{}^*$. Since $\Sigma \subseteq \mathfrak{K}^* \subseteq \Omega_{r,s,n}{}^*$, there exists $j \in N$ with $u_j \geqq s$ and if $u_i < v_i$, then $n|u_i - v_i$. Since $e \notin \Omega_{r,t,n}{}^*$, $u_i < t$ for all $i \in N$. Thus $e = ((u_i)_{i\in N}, (u_i + k_i n)_{i\in N})$, where we may assume without loss of generality that $u_1 \geqq s$. Let $\bar{e} = ((u_i)_{i\in N}, (u_1 + n, u_2, u_3, \ldots))$. Then $\bar{e} \in T_{s,t}(n)$. Now $k_j \geqq 1$ for some $j \in N$. Choose $k \in N$ so that $u_j + kk_j \geqq t$. Then

$$((u_i)_{i\in N}, (u_i + kk_i n)_{i\in N}) \in \Gamma e$$

$$((u_i + kk_i n)_{i\in N}, (u_1 + n + kk_1 n, u_2 + kk_2 n, u_3 + kk_3 n, \ldots)) \in \Omega_{r,t,n}{}^*$$

$$((u_1 + n + kk_1 n, u_2 + kk_2 n, u_3 + kk_3 n, \ldots), (u_1 + n, u_2, u_3, \ldots)) \in \Gamma e.$$

Thus $\{e\} \cup \Omega_{r,t,n}{}^* \rightarrow \bar{e}$.

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On the other hand, if $h \in N$ is chosen so that $u_1 + hn \geqq t$, then

$$((u_i)_{i \in N}, (u_1 + hn, u_2, u_3, \ldots)) \in \Gamma\bar{e}$$
$$((u_1 + hn, u_2, u_3, \ldots), (u_1 + hn + k_1 n, u_2 + k_2 n, u_3 + k_3 n, \ldots)) \in \Omega_{r,t,n}^*$$
$$((u_1 + hn + k_1 n, u_2 + k_2 n, u_3 + k_3 n, \ldots), (u_i + k_i n)_{i \in N}) \in \Gamma\bar{e}.$$

Thus $\{\bar{e}\} \cup \Omega_{r,t,n}^* \to e$. This completes the proof.

COROLLARY 1. *$\phi_{r,s,t,u,n}$ restricted to $[\Omega_{s,t,n}, \Omega_{s,u,n}] \cap \gamma$ is an isomorphism of $[\Omega_{s,t,n}, \Omega_{s,u,n}] \cap \gamma$ onto $[\Omega_{r,t,n}, \Omega_{r,u,n}] \cap \gamma$.*

Proof. It is an immediate consequence of the definition of $\phi_{r,s,t,u,n}$ that it maps S-classes to S-classes. We already know that $\phi_{r,s,t,u,n}$ is a lattice monomorphism; thus, it is enough to show that for every S-class in $[\Omega_{r,t,n}, \Omega_{r,u,n}]$ there is an S-class in $[\Omega_{s,t,n}, \Omega_{s,u,n}]$ that maps onto it under $\phi_{r,s,t,u,n}$.

Let $\mathfrak{R} \in [\Omega_{r,t,n}, \Omega_{r,u,n}] \cap \gamma$. By the lemma, there exists $T \subseteq T_{t,u}$ such that $\mathfrak{R} = T(n)^* \wedge \Omega_{r,u,n}$. But then $T(n)^* \wedge \Omega_{s,u,n} \in [\Omega_{s,t,n}, \Omega_{s,u,n}] \cap \gamma$ and $\phi_{r,s,t,u,n}(T(n)^* \wedge \Omega_{s,u,n}) = T(n)^* \wedge \Omega_{s,u,n} \wedge \Omega_{r,u,n} = \mathfrak{R}$. This completes the proof.

In view of Lemmas 3.4, 3.5, and the last corollary, $\phi_{r,s,t,u,n}$ maps onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$ for $r > 0$ if and only if $r + n \geqq u$. For $1 < s \leqq t < u$, since $\phi_{0,s,t,u,n} = \phi_{0,1,t,u,n} \circ \phi_{1,s,t,u,n}$ and since $\phi_{0,1,t,u,n}$ is an isomorphism by Lemma 3.3, it follows that $\phi_{0,s,t,u,n}$ maps onto $[\Omega_{0,t,n}, \Omega_{0,u,n}]$ if and only if $n + 1 \geqq u$.

Thus $\phi_{r,s,t,u,n}$ maps onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$ if and only if $r = 0$ and $s = 1$, or $r = 0$ and $n + 1 \geqq u$, or $r > 0$ and $r + n \geqq u$.

From this we see that the embedding of $\mathfrak{L}_n$ into $\mathfrak{L}_{n,0} \times I^+$ in Section 2 does not preserve joins. Let $n \geqq 1$ and let $p > n$; then $\phi_{0,p,p,p+1,n}$ does not map onto $[\Omega_{0,p,n}, \Omega_{0,p+1,n}]$. Let $\mathfrak{R} \in [\Omega_{0,p,n}, \Omega_{0,p+1,n}]$ such that $\mathfrak{R}$ is not in the image of $\phi_{0,p,p,p+1,n}$. Let $\mathfrak{R}' = \Omega_{p,p,n}$. If the above-mentioned embedding preserved joins, then we would have

$$\delta_{0,V(\mathfrak{R}),n}(\mathfrak{R}) \vee \delta_{0,V(\mathfrak{R}'),n}(\mathfrak{R}') = \delta_{0,V(\mathfrak{R} \vee \mathfrak{R}'),n}(\mathfrak{R} \vee \mathfrak{R}').$$

But $\delta_{0,V(\mathfrak{R}),n}(\mathfrak{R}) = \delta_{0,0,n}(\mathfrak{R}) = \mathfrak{R}$, $\delta_{0,V(\mathfrak{R}'),n}(\mathfrak{R}') = \Omega_{p,p,n} \wedge \Omega_{0,p,n} = \Omega_{0,p,n}$ and $\Omega_{0,p,n} \vee \mathfrak{R} = \mathfrak{R}$. Thus we would have $\mathfrak{R} = (\mathfrak{R} \vee \mathfrak{R}') \wedge \Omega_{0,p+1,n}$ and this would imply that $\mathfrak{R}$ is in the image of $\phi_{0,p,p,p+1,n}$. Thus the embedding does not preserve joins.

LEMMA 3.7. *For all $n \geqq 1$, both $[\Omega_{0,1,n}, \Omega_{0,2,n}]$ and $[\Omega_{1,1,n}, \Omega_{1,2,n}]$ are isomorphic to $\omega + 1$, i.e., to a countable ascending chain with unit adjoined.*

Proof. The proof follows immediately from Lemmas 3.3, 3.5 and 3.6, and the fact that

$$T_{1,2} = \{\underbrace{(1, 1, \ldots 1}_{m}, 0, 0, \ldots) | m \geqq 1\}.$$

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3.5. The relationship between $\mathfrak{L}_n$ and $\mathfrak{L}_m$. For $n \neq m$, $\mathfrak{L}_n$ and $\mathfrak{L}_m$ are disjoint. $\mathfrak{L} = \bigcup_{n\in N}\mathfrak{L}_n \cup \{E\}$. If $n|m$, we define a mapping $\beta_{n,m} : \mathfrak{L}_m \rightarrow \mathfrak{L}_n$ as follows: for $\mathfrak{K} \in \mathfrak{L}_m$ with $V(\mathfrak{K}) = r$ and $U(\mathfrak{K}) = s$, $\beta_{n,m}(\mathfrak{K}) = \mathfrak{K} \wedge \Omega_{r,s,n}$. Then for any t, u with $\mathfrak{K} \subseteq \Omega_{t,u,m}$ we have that

$$\mathfrak{K} \wedge \Omega_{t,u,n} = (\mathfrak{K} \wedge \Omega_{r,s,m}) \wedge \Omega_{t,u,n} = \mathfrak{K} \wedge \Omega_{r,s,n} = \beta_{n,m}(\mathfrak{K}).$$

It follows from this that $\beta_{n,m}$ is a meet-homomorphism. Moreover, by the corollary to Theorem 3.1, if $\mathfrak{K} \in \mathfrak{L}_m$ then $\mathfrak{K} = \beta_{n,m}(\mathfrak{K}) \vee \Omega_{0,1,m}$; thus, $\beta_{n,m}$ is one-to-one. Thus, to show that $\beta_{n,m}$ is a lattice monomorphism of $\mathfrak{L}_m$ into $\mathfrak{L}_n$, it suffices to show that it preserves joins.

Let $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathfrak{L}_m$. We may assume without loss of generality that $r = V(\mathfrak{K}_1) \leqq V(\mathfrak{K}_2) = s$. Let $u = \max\{U(\mathfrak{K}_1), U(\mathfrak{K}_2)\}$. Then

$$\beta_{n,m}(\mathfrak{K}_1 \vee \mathfrak{K}_2) = (\mathfrak{K}_1 \vee \mathfrak{K}_2) \wedge \Omega_{s,u,n}$$

and

$$\beta_{n,m}(\mathfrak{K}_1) \vee \beta_{n,m}(\mathfrak{K}_2) = (\mathfrak{K}_1 \wedge \Omega_{r,u,n}) \vee (\mathfrak{K}_2 \wedge \Omega_{s,u,n}).$$

It follows from Lemma 3.2 that $\beta_{n,m}(\mathfrak{K}_1) \vee \beta_{n,m}(\mathfrak{K}_2) = \beta_{n,m}(\mathfrak{K}_1 \vee \mathfrak{K}_2)$.

Thus $\beta_{n,m}$ is a lattice monomorphism of $\mathfrak{L}_m$ into $\mathfrak{L}_n$ with the property that for each $\mathfrak{K} \in \mathfrak{L}_m$, $\beta_{n,m}(\mathfrak{K}) \vee \Omega_{0,1,m} = \mathfrak{K}$. Moreover, $\beta_{n,m}$ retains the skeleton; $\beta_{n,m}(\Omega_{r,s,m}) = \Omega_{r,s,n}$. Clearly, if $n|m$ and $m|p$ then $\beta_{n,p} = \beta_{n,m} \circ \beta_{m,p}$.

THEOREM 3.3. *The mapping* $\mathfrak{K} \rightsquigarrow (\beta_{1,D(\mathfrak{K})}(\mathfrak{K}), D(\mathfrak{K}))$ *is an embedding of* $\mathfrak{L} - \{E\}$ *as a meet subsemilattice into* $\mathfrak{L}_1 \times N^+$.

Proof. Since $\beta_{1,n}$ is one-to-one for each $n \in N$, the mapping in question is one-to-one. Since for non-trivial equational classes $\mathfrak{K}_1$, $\mathfrak{K}_2$, $D(\mathfrak{K}_1 \wedge \mathfrak{K}_2)$ is the greatest common divisor of $D(\mathfrak{K}_1)$ and $D(\mathfrak{K}_2)$, it is enough to show that

$$\beta_{1,D(\mathfrak{K}_1\wedge\mathfrak{K}_2)}(\mathfrak{K}_1 \wedge \mathfrak{K}_2) = \beta_{1,D(\mathfrak{K}_1)}(\mathfrak{K}_1) \wedge \beta_{1,D(\mathfrak{K}_2)}(\mathfrak{K}_2).$$

If $r, u \in N$ are chosen such that

$$\mathfrak{K}_1 \subseteq \Omega_{r,u,D(\mathfrak{K}_1)}, \; \mathfrak{K}_2 \subseteq \Omega_{r,u,D(\mathfrak{K}_2)},$$

then

$$\begin{aligned} \beta_{1,D(\mathfrak{K}_1\wedge\mathfrak{K}_2)}(\mathfrak{K}_1 \wedge \mathfrak{K}_2) &= \mathfrak{K}_1 \wedge \mathfrak{K}_2 \wedge \Omega_{r,u,1} \\ &= (\mathfrak{K}_1 \wedge \Omega_{r,u,1}) \wedge (\mathfrak{K}_2 \wedge \Omega_{r,u,1}) \\ &= \beta_{1,D(\mathfrak{K}_1)}(\mathfrak{K}_1) \wedge \beta_{1,D(\mathfrak{K}_2)}(\mathfrak{K}_2). \end{aligned}$$

This completes the proof.

It will be seen in the next section that this embedding does not preserve joins.

Combining the results of this section with those of Section 2, we see that $\mathfrak{L}$ is isomorphic to a meet subsemilattice of $\mathfrak{L}_{1,0} \times I^+ \times N^+$ with a unit adjoined.

THEOREM 3.4. *For equational classes* $\mathfrak{K}_1, \mathfrak{K}_2$, $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$ *if and only if* $D(\mathfrak{K}_1)|D(\mathfrak{K}_2)$, $V(\mathfrak{K}_1) \leqq V(\mathfrak{K}_2)$ *and*

$$\beta_{1,D(\mathfrak{K}_1)}(\delta_{0,V(\mathfrak{K}_1),D(\mathfrak{K}_1)}(\mathfrak{K}_1)) \subseteq \beta_{1,D(\mathfrak{K}_2)}(\delta_{0,V(\mathfrak{K}_2),D(\mathfrak{K}_2)}(\mathfrak{K}_2)).$$

Proof. If $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$ then $D(\mathfrak{R}_1)|D(\mathfrak{R}_2)$, $V(\mathfrak{R}_1) \leqq V(\mathfrak{R}_2)$ and $U(\mathfrak{R}_1) \leqq U(\mathfrak{R}_2)$; thus,

$$\begin{aligned}\beta_{1,D(\mathfrak{R}_1)}(\delta_{0,V(\mathfrak{R}_1),D(\mathfrak{R}_1)}(\mathfrak{R}_1)) &= \mathfrak{R}_1 \wedge \Omega_{0,U(\mathfrak{R}_1),1}\\ &= \mathfrak{R}_1 \wedge \Omega_{0,U(\mathfrak{R}_2),1} \subseteq \mathfrak{R}_2 \wedge \Omega_{0,U(\mathfrak{R}_2),1}\\ &= \beta_{1,D(\mathfrak{R}_2)}(\delta_{0,V(\mathfrak{R}_2),D(\mathfrak{R}_2)}(\mathfrak{R}_2)).\end{aligned}$$

Conversely, if $D(\mathfrak{R}_1)|D(\mathfrak{R}_2)$, $V(\mathfrak{R}_1) \leqq V(\mathfrak{R}_2)$ and

$$\beta_{1,D(\mathfrak{R}_1)}(\delta_{0,V(\mathfrak{R}_1),D(\mathfrak{R}_1)}(\mathfrak{R}_1)) \subseteq \beta_{1,D(\mathfrak{R}_2)}(\delta_{0,V(\mathfrak{R}_2),D(\mathfrak{R}_2)}(\mathfrak{R}_2))$$

then

$$\begin{aligned}\mathfrak{R}_1 &= \beta_{1,D(\mathfrak{R}_1)}(\delta_{0,V(\mathfrak{R}_1),D(\mathfrak{R}_1)}(\mathfrak{R}_1)) \vee \Omega_{V(\mathfrak{R}_1),V(\mathfrak{R}_1),D(\mathfrak{R}_1)}\\ &\subseteq \beta_{1,D(\mathfrak{R}_1)}(\delta_{0,V(\mathfrak{R}_1),D(\mathfrak{R}_1)}(\mathfrak{R}_1)) \vee \Omega_{V(\mathfrak{R}_2),V(\mathfrak{R}_2),D(\mathfrak{R}_2)}\\ &\subseteq \beta_{1,D(\mathfrak{R}_2)}(\delta_{0,V(\mathfrak{R}_2),D(\mathfrak{R}_2)}(\mathfrak{R}_2)) \vee \Omega_{V(\mathfrak{R}_2),V(\mathfrak{R}_2),D(\mathfrak{R}_2)}\\ &= \mathfrak{R}_2.\end{aligned}$$

This completes the proof.

3.6. Another mapping between intervals of the lattice. For $r \leqq s < t$ and $n|m$, $n \neq m$, let $\alpha_{r,s,t,n,m}$ be the restriction of $\beta_{n,m}$ to $[\Omega_{r,s,m}, \Omega_{r,t,m}]$. Then $\alpha_{r,s,t,n,m}$ is a lattice monomorphism of $[\Omega_{r,s,m}, \Omega_{r,t,m}]$ into $[\Omega_{r,s,n}, \Omega_{r,t,n}]$. We will investigate for which values of r, s, t, n, m, $\alpha_{r,s,t,n,m}$ actually maps onto the whole interval $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.

LEMMA 3.8. *If $r > 0$ and $r + n < t$, then $\alpha_{r,s,t,n,m}$ does not map onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.*

Proof. Let $e = ((r, r+n, s), (r+n, r, s))$ and let $\mathfrak{R} = e^* \wedge \Omega_{r,t,n}$. Then $\mathfrak{R} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$. If $\mathfrak{R}$ is in the image of $\alpha_{r,s,t,n,m}$, then $\mathfrak{R} = \alpha_{r,s,t,n,m}(\mathfrak{R}')$ for some $\mathfrak{R}' \in [\Omega_{r,s,m}, \Omega_{r,t,m}]$ and then by Theorem 3.1, $\mathfrak{R} \vee \Omega_{r,s,m} = \mathfrak{R}'$; thus, $\mathfrak{R} = \alpha_{r,s,t,n,m}(\mathfrak{R} \vee \Omega_{r,s,m}) = (\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n}$. Thus it is enough to show that $\mathfrak{R} \neq (\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n}$. Clearly $e \in \mathfrak{R}^*$. We will show that $e \notin ((\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n})^*$.

Assume that $e \in ((\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n})^* = (\mathfrak{R}^* \cap \Omega_{r,s,m}{}^*) \vee \Omega_{r,t,n}{}^*$. Then there exist $\tau_1, \tau_2 \in F(\omega)$ such that

$$((r, r+n, s), \tau_1),\ (\tau_2, (r+n, r, s)) \in \mathfrak{R}^* \cap \Omega_{r,s,m}{}^* \text{ and } (\tau_1, \tau_2) \in \Omega_{r,t,n}{}^*.$$

But $((r, r+n, s), \tau_1) \in \mathfrak{R}^*$ implies that there exists $\tau_3, \tau_4 \in F(\omega)$ such that $((r, r+n, s), \tau_3), (\tau_1, \tau_4) \in \Gamma e$, $(\tau_3, \tau_4) \in \Omega_{r,t,n}{}^*$. Since $((r, r+n, s), \tau_3) \in \Gamma e$, it follows that, if $r + n \neq s$ and $r \neq s$ then $\tau_3 = (r, r+n, s)$ or $(r+n, r, s)$, if $r = s$ then $\tau_3 = (r, r+n, r)$, $(r+n, r, r)$ or $(r, r, r+n)$ and if $r + n = s$ then $\tau_3 = (r, r+n, r+n)$, $(r+n, r, r+n)$ or $(r+n, r+n, r)$. In any case, since $r + n < t$ and $s < t$ and $(\tau_3, \tau_4) \in \Omega_{r,t,n}{}^*$, it follows that (τ_3, τ_4) is trivial. Thus $((r, r+n, s), \tau_1) \in \Gamma e$. But then the same argument yields $\tau_1 = \tau_2$. But this implies that

$$((r, r+n, s), (r+n, r, s)) \in \mathfrak{R}^* \cap \Omega_{r,s,m}{}^* \subseteq \Omega_{r,s,m}{}^*$$

and this is a contradiction. This completes the proof.

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LEMMA 3.9. *$\alpha_{r,s,t,n,m}$ restricted to $[\Omega_{r,s,m}, \Omega_{r,t,m}] \cap \gamma$ is an isomorphism of $[\Omega_{r,s,m}, \Omega_{r,t,m}] \cap \gamma$ onto $[\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \gamma$.*

Proof. It is clear from the definition of $\alpha_{r,s,t,n,m}$ that it maps S-classes to S-classes. Thus it is enough to show that for every $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \gamma$ there exists $\mathfrak{K}' \in [\Omega_{r,s,m}, \Omega_{r,t,m}] \cap \gamma$ such that $\mathfrak{K} = \alpha_{r,s,t,n,m}(\mathfrak{K}')$.

Let $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \gamma$. By Lemma 3.6 there exists $T \subseteq T_{s,t}$ such that $\mathfrak{K} = T(n)^* \wedge \Omega_{r,t,n}$. Since $n|m$, $T(m)^* \wedge \Omega_{r,t,n} = T(n)^* \wedge \Omega_{r,t,n}$. Thus $\mathfrak{K} = T(m)^* \wedge \Omega_{r,t,m} \wedge \Omega_{r,t,n} = \alpha_{r,s,t,n,m}(T(m)^* \wedge \Omega_{r,t,m})$ and this completes the proof.

COROLLARY. *If $r + n \geqq t$ then $\alpha_{r,s,t,n,m}$ maps onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.*

Proof. The proof follows from the lemma, and Lemma 3.5.

Since $\alpha_{0,s,t,n,m} \circ \phi_{0,1,s,t,m} = \phi_{0,1,s,t,n} \circ \alpha_{1,s,t,n,m}$ and since $\phi_{0,1,s,t,n}$ and $\phi_{0,1,s,t,m}$ are isomorphisms, it follows that $\alpha_{0,s,t,n,m}$ maps onto $[\Omega_{0,s,n}, \Omega_{0,t,n}]$ if and only if $\alpha_{1,s,t,n,m}$ maps onto $[\Omega_{1,s,n}, \Omega_{1,t,n}]$. From the above results we have that $\alpha_{r,s,t,n,m}$ maps onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$ for $r > 0$ if and only if $r + n \geqq t$. Thus $\alpha_{r,s,t,n,m}$ maps onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$ if and only if $r = 0$ and $n + 1 \geqq t$ or $r > 0$ and $r + n \geqq t$.

It follows from this that the embedding of $\mathfrak{L} - \{E\}$ into $\mathfrak{L}_1 \times N^+$ described in the last section does not preserve joins: let $\mathfrak{K}_1 \in \mathfrak{L}_1$ such that $\mathfrak{K}_1 \notin$ image of $\beta_{1,n}$ and let $\mathfrak{K}_2 = \Omega_{0,1,n}$. Then

$$\beta_{1,1}(\mathfrak{K}_1) \vee \beta_{1,n}(\mathfrak{K}_2) = \mathfrak{K}_1 \vee (\Omega_{0,1,n} \wedge \Omega_{0,1,1}) = \mathfrak{K}_1 \vee \Omega_{0,1,1} = \mathfrak{K}_1,$$

but $\mathfrak{K}_1 \neq \beta_{1,n}(\mathfrak{K}_1 \vee \mathfrak{K}_2)$ since $\mathfrak{K}_1 \notin$ image of $\beta_{1,n}$.

3.7. The sublattice of Schwabauer classes. It has already been mentioned that γ, the set of all S-classes, forms a distributive sublattice of $\mathfrak{L}$. In this section, this and the fact that γ is a maximal modular sublattice will be proved. We first give the following characterization of S-classes:

LEMMA 3.10. *$\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ is an S-class if and only if it satisfies: (1) for all u with $r < u \leqq s$, $\mathfrak{K}$ is in the image of $\phi_{r,s,t,u,n}$ and (2) for all $m > n$ with $n|m$, $\mathfrak{K}$ is in the image of $\alpha_{r,s,t,n,m}$.*

Proof. If $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \gamma$ then (1) and (2) follow from Lemma 3.6, Corollary 1 and Lemma 3.9, respectively.

On the other hand, if $\mathfrak{K}$ satisfies (1) and (2), then choose $m > n$ such that $r + m \geqq t$ and $n|m$. Then by (2), $\mathfrak{K} = \mathfrak{K}' \wedge \Omega_{r,t,n}$ for some $\mathfrak{K}' \in [\Omega_{r,s,m}, \Omega_{r,t,m}]$. By Lemma 3.5, $\mathfrak{K}'$ is an S-class. Thus $\mathfrak{K}$ is an S-class and this completes the proof.

Let $\gamma_n = \{\mathfrak{K} \in \gamma | D(\mathfrak{K}) = n\} = \mathfrak{L}_n \cap \gamma$. Then the γ_n are pairwise disjoint and $\gamma = \bigcup_{n \in N} \gamma_n \cup \{E\}$. Moreover, from Lemma 3.9, $\beta_{n,m}$ restricted to γ_m is an isomorphism of γ_m onto γ_n. This implies that the mapping

$$\mathfrak{K} \rightsquigarrow (\beta_{1,D(\mathfrak{K})}(\mathfrak{K}), D(\mathfrak{K}))$$

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is a meet monomorphism of $\gamma - \{E\}$ onto $\gamma_1 \times N^+$. But a mapping from one lattice to another that is one-to-one, onto, and meet preserving is also join preserving, i.e., it is a lattice isomorphism.

It follows that γ is lattice isomorphic with $\gamma_1 \times N^+$ with a unit adjoined.

LEMMA 3.11. *$[\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \gamma$ is distributive for all $t \geqq 1$.*

Proof. For $T \subseteq T_{1,t}$, define $\bar{T}$ to be the set of those sequences $(u_i)_{i\in N} \in T_{1,t}$ such that $((u_i)_{i\in N}, (u_1 + 1, u_2, u_3, \ldots)) \in \Gamma(T(1) \cup \Omega_{0,t,1}^*)$. Then $(u_i)_{i\in N} \in \bar{T}$ if and only if there exists $(v_i)_{i\in N} \in T$ such that

$$\{(v_i)_{i\in N}, (v_1 + 1, v_2, v_3, \ldots)\} \cup \Omega_{0,t,1}^* \rightarrow ((u_i)_{i\in N}, (u_1 + 1, u_2, u_3, \ldots)).$$

Thus the set of all $T \subseteq T_{1,t}$ such that $T = \bar{T}$ is closed under unions and intersections. Moreover, if $T_1, T_2 \subseteq T_{1,t}$ and $T_1 = \bar{T}_1$, $T_2 = \bar{T}_2$ then

$$(T_1(1)^* \wedge \Omega_{0,t,1}) \wedge (T_2(1)^* \wedge \Omega_{0,t,1}) = (T_1 \cup T_2)(1)^* \wedge \Omega_{0,t.1}$$

and

$$(T_1(1)^* \wedge \Omega_{0,t,1}) \vee (T_2(1)^* \wedge \Omega_{0,t,1}) = (T_1 \cap T_2)(1)^* \wedge \Omega_{0,t,1}.$$

Since for each $\mathfrak{K} \in [\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \gamma$ there exists $T \subseteq T_{1,t}$ such that $T = \bar{T}$ and $\mathfrak{K} = T(1)^* \wedge \Omega_{0,t,1}$, it follows that $[\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \gamma$ is isomorphic to a sublattice of the power set of $T_{1,t}$ and hence is distributive.

COROLLARY. *$\gamma_{1,0} = \mathfrak{L}_{1,0} \cap \gamma$ is distributive.*

Proof. This follows immediately from the lemma and the fact that $\{[\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \gamma | t \geqq 1\}$ forms an ascending chain and

$$\gamma_{1,0} = \bigcup_{t\geqq 1} [\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \gamma.$$

Since for $p < q$, $\delta_{p,q,1}$ maps S-classes to S-classes, it follows that the mapping $\mathfrak{K} \rightsquigarrow (\delta_{0,V(\mathfrak{K}),1}(\mathfrak{K}), V(\mathfrak{K}))$ is a meet monomorphism of γ_1 into $\gamma_{1,0} \times I^+$. Moreover, this mapping preserves joins: let

$$\mathfrak{K}_1, \mathfrak{K}_2 \in \gamma_1, V(\mathfrak{K}_1) = p, V(\mathfrak{K}_2) = q.$$

We may assume without loss of generality that $p \leqq q$. Let

$$u = \max\{U(\mathfrak{K}_1), U(\mathfrak{K}_2)\}.$$

Then

$$\begin{aligned} \delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2) \vee \Omega_{q,q,1} &= \delta_{0,p,1}(\mathfrak{K}_1) \vee \Omega_{p,p,1} \vee \delta_{0,q,1}(\mathfrak{K}_2) \vee \Omega_{q,q,1} \\ &= \mathfrak{K}_1 \vee \mathfrak{K}_2. \end{aligned}$$

But $\mathfrak{K}_2 \supseteq \Omega_{q,q,1}$; thus, $\delta_{0,q,1}(\mathfrak{K}_2) \supseteq \Omega_{0,q,1}$ and thus

$$\delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2) \in [\Omega_{0,q,1}, \Omega_{0,u,1}] \cap \gamma.$$

It follows from Corollary 1 of Lemma 3.6 that there exists

$$\mathfrak{K} \in [\Omega_{q,q,1}, \Omega_{q,u,1}] \cap \gamma \quad \text{with} \quad \delta_{0,q,1}(\mathfrak{K}) = \delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2).$$

But then $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \delta_{0,p,1}(\mathfrak{R}_1) \vee \delta_{0,q,1}(\mathfrak{R}_2) \vee \Omega_{q,q,1} = \delta_{0,q,1}(\mathfrak{R}) \vee \Omega_{q,q,1} = \mathfrak{R}$. Thus $\delta_{0,p,1}(\mathfrak{R}_1) \vee \delta_{0,q,1}(\mathfrak{R}_2) = \delta_{0,q,1}(\mathfrak{R}_1 \vee \mathfrak{R}_2)$.

It follows that γ_1 is lattice isomorphic to a sublattice of $\gamma_{1,0} \times I^+$. But $\gamma_{1,0}$ is distributive; thus, γ_1 is also distributive.

Thus, since γ is isomorphic to $\gamma_1 \times N^+$ with a unit adjoined, we can state the following:

THEOREM 3.5. *γ is distributive.*

THEOREM 3.6. *γ is maximal modular.*

Proof. Let $\mathfrak{R}$ be any equational class not in γ, $\mathfrak{R} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$, say. Choose m such that $n|m$ and $r + m \geqq t$. Then $\mathfrak{R}$ is not in the image of $\alpha_{r,s,t,n,m}$ and thus $\mathfrak{R} \neq (\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n}$. But this implies that the sublattice of $\mathfrak{L}$ generated by $\gamma \cup \{\mathfrak{R}\}$ is not modular. Thus γ is a maximal modular sublattice.

One might well ask whether the set of maximal distributive sublattices of $\mathfrak{L}$ coincides with the set of maximal modular sublattices of $\mathfrak{L}$; this is the case if and only if every modular sublattice of $\mathfrak{L}$ is distributive. However, by a result of [**1**], $\mathfrak{L}$ has a sublattice isomorphic to the partition lattice on a three-element set; this lattice is the five-element modular, non-distributive lattice.

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