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OPTIMAL LOWER BOUND ON THE SUPREMAL STRICT p-NEGATIVE TYPE OF A FINITE METRIC SPACE

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Abstract

Determining meaningful lower bounds on the supremal strict p-negative type of classes of finite metric spaces is a difficult nonlinear problem. In this paper we use an elementary approach to obtain the following result: given a finite metric space (X, d) there is a constant $\zeta > 0$, dependent only on $n = |X|$ and the scaled diameter $\mathfrak{D} = (\text{diam } X)/\min\{d(x, y) \mid x \neq y\}$ of X (which we may assume is > 1), such that (X, d) has p-negative type for all $p \in [0, \zeta]$ and strict p-negative type for all $p \in [0, \zeta)$. In fact, we obtain

$$\zeta = \frac{\ln\big(1/(1-\Gamma)\big)}{\ln \mathfrak{D}} \quad \text{where } \Gamma = \frac{1}{2}\left(\frac{1}{\lfloor n/2 \rfloor} + \frac{1}{\lceil n/2 \rceil}\right).$$

A consideration of basic examples shows that our value of ζ is optimal provided that $\mathfrak{D} \leq 2$. In other words, for each $\mathfrak{D} \in (1, 2]$ and natural number $n \geq 3$, there exists an n-point metric space of scaled diameter $\mathfrak{D}$ whose supremal strict p-negative type is exactly ζ. The results of this paper hold more generally for all finite semi-metric spaces since the triangle inequality is not used in any of the proofs. Moreover, ζ is always optimal in the case of finite semi-metric spaces.

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1. Introduction

The notion of p-negative type given in Definition 2.1(a) was first isolated in the early 1900s in the works of mathematicians such as Menger [14], Moore [15] and Schoenberg [17, 18]. One of the main interests at the time was obtaining characterizations of subsets of Hilbert space up to isometry. For example, Schoenberg showed that a metric space is isometric to a subset of Hilbert space if and only if it has 2-negative type. In the 1960s Bretagnolle *et al.* [1] obtained a spectacular generalization of Schoenberg's result to the category of Banach spaces: a Banach space is linearly isometric to a subspace of some L_p-space (for a fixed p, $0 < p \leq 2$) if and only if it has p-negative type. More recently, difficult questions concerning p-negative type, such as the *Goemans–Linial conjecture*, have figured prominently in theoretical computer science. Some monographs and papers which help illustrate the landscape of results and problems in these directions include those of Deza and Laurent [2], Khot and Vishnoi [11] (who solved the *Goemans–Linial conjecture* negatively), Lee and Naor [12], Prassidis and Weston [16], and Wells and Williams [19].

The related notion of strict p-negative type given in Definition 2.1(b) has been studied rather less well than its classical counterpart and most known results deal with the case $p = 1$. Examples of papers which illustrate this concentration on the case $p = 1$ include those of Hjorth *et al.* [8, 9], Doust and Weston [3, 4], and Prassidis and Weston [16]. The theme of this paper is to focus instead on the limiting case $p = 0$. A natural quantification of the nontrivial p-negative type inequalities which is given in Definition 2.6 allows us to bootstrap the case $p = 0$ to obtain the results described in our abstract. This is done using Lagrange multipliers.

In the next section we present some background material on the notions of (strict) negative type and (strict) generalized roundness. This includes a discussion of the recently developed notion of the (*normalized*) *p-negative type gap* Γ_X^p of a metric space (X, d). This concept is due to Doust and Weston [3] and is formally described in Definition 2.6. The informal idea of Γ_X^p is to provide a natural measure of the degree of strictness of the nontrivial p-negative type inequalities for any metric space (X, d) that happens to have p-negative type.

The purpose of Section 3 is to compute the 0-negative type gap Γ_X^0 of each finite metric space (X, d) exactly. In Theorem 3.2 we obtain the rather succinct formula

$$\Gamma_X^0 = \frac{1}{2}\left(\frac{1}{\lfloor n/2 \rfloor} + \frac{1}{\lceil n/2 \rceil}\right) > 0,$$

where $n \geq 2$ denotes $|X|$. Here, given a real number x, we are using $\lfloor x \rfloor$ to denote the largest integer that does not exceed x, and $\lceil x \rceil$ to denote the smallest integer number which is not less than x.

The paper culminates in Section 4 with our main results. We show that each finite semi-metric space has strict p-negative type on an interval of the form $[0, \zeta)$ where $\zeta > 0$ depends only on the cardinality and the scaled diameter of the underlying space. We also show that ζ provides an optimal lower bound on the supremal strict p-negative type of finite metric spaces whose scaled diameter $\mathfrak{D}$ lies in the interval $(1, 2]$. In the case of finite semi-metric spaces we point out that ζ is in fact *always* optimal. These results appear in Theorem 4.1 and Remarks 4.2 and 4.3.

Throughout this paper the set of natural numbers $\mathbb{N}$ is taken to consist of all positive integers and sums indexed over the empty set are always taken to be zero.

2. Classical (strict) p-negative type

We begin by recalling some theoretical features of (strict) p-negative type. More detailed accounts may be found in the monographs of Deza and Laurent [2] and Wells and Williams [19]. These works emphasize the interplay between p-negative type inequalities and isometric embeddings as well as indicating applications to more contemporary areas of interest such as combinatorial optimization.

DEFINITION 2.1. Let $p \geq 0$ and let (X, d) be a metric space. Then:

(a) (X, d) has *p-negative type* if and only if for all natural numbers $k \geq 2$, all finite subsets $\{x_1, \ldots, x_k\} \subseteq X$ and all choices of real numbers $\eta_1, \ldots, \eta_k$ with $\eta_1 + \cdots + \eta_k = 0$, we have

$$\sum_{1 \leq i, j \leq k} d(x_i, x_j)^p \eta_i \eta_j \leq 0; \tag{2.1}$$

(b) (X, d) has *strict p-negative type* if and only if it has p-negative type and the associated inequalities (2.1) are all strict except in the trivial case $(\eta_1, \ldots, \eta_k) = (0, \ldots, 0)$.

A basic classical property of p-negative type is that it holds on closed intervals. If (X, d) is a metric space, then (X, d) has p-negative type for all p such that $0 \leq p < \wp$, where $\wp = \sup\{p_* \mid (X, d) \text{ has } p_*\text{-negative type}\}$. It is an open problem whether the corresponding such interval property holds for *strict* p-negative type.

It turns out that it is possible to reformulate both ordinary and strict p-negative type in terms of an invariant known as *generalized roundness* from the uniform theory of Banach spaces. Generalized roundness was introduced by Enflo [6] in order to solve (in the negative) *Smirnov's problem*: is every separable metric space uniformly homeomorphic to a subset of Hilbert space? The analog of this problem for coarse embeddings was later raised by Gromov [7] and solved negatively by Dranishnikov *et al*. [5]. Prior to introducing generalized roundness in Definition 2.3(a) we develop some intermediate technical notions in order to streamline the exposition throughout the remainder of this paper.

DEFINITION 2.2. Let q, t be arbitrary natural numbers and let X be any set.

(a) A *(q, t)-simplex* in X is a $(q + t)$-vector $(a_1, \ldots, a_q, b_1, \ldots, b_t) \in X^{q+t}$ whose coordinates consist of $q + t$ distinct vertices $a_1, \ldots, a_q, b_1, \ldots, b_t \in X$. Such a simplex will be denoted by $D = [a_j; b_i]_{q,t}$.

(b) A *load vector* for a (q, t)-simplex $D = [a_j; b_i]_{q,t}$ in X is an arbitrary vector $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t) \in \mathbb{R}_+^{q+t}$ that assigns a positive weight $m_j > 0$ or $n_i > 0$ to each vertex a_j or b_i of D, respectively.

(c) A *loaded (q, t)-simplex* in X consists of a (q, t)-simplex $D = [a_j; b_i]_{q,t}$ in X together with a load vector $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t)$ for D. Such a loaded simplex will be denoted by $D(\vec{\omega})$ or $[a_j(m_j); b_i(n_i)]_{q,t}$ as the need arises.

(d) A *normalized (q, t)-simplex* in X is a loaded (q, t)-simplex $D(\vec{\omega})$ in X whose load vector $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t)$ satisfies the two normalizations:

$$m_1 + \cdots + m_q = 1 = n_1 + \cdots n_t.$$

Such a vector $\vec{\omega}$ will be called a *normalized load vector* for D.

Rather than give the original definition of generalized roundness p from Enflo [6] we present an equivalent reformulation in Definition 2.3(a) that is due to Lennard *et al*. [13] and Weston [20]. (See also Prassidis and Weston [16].)

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DEFINITION 2.3. Let $p \geq 0$ and let (X, d) be a metric space. Then:

(a) (X, d) has *generalized roundness* p if and only if for all $q, t \in \mathbb{N}$ and all normalized (q, t)-simplices $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{q,t}$ in X we have

$$\sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p \leq \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p; \tag{2.2}$$

(b) (X, d) has *strict generalized roundness* p if and only if it has generalized roundness p and the associated inequalities (2.2) are all strict.

Two key aspects of generalized roundness for the purposes of this paper are the following equivalences. Part (a) is due to Lennard *et al.* [13] and part (b) was later observed by Doust and Weston [3].

THEOREM 2.4. *Let $p \geq 0$ and let (X, d) be a metric space. Then:*

(a) *(X, d) has p-negative type if and only if it has generalized roundness p;*

(b) *(X, d) has strict p-negative type if and only if it has strict generalized roundness p.*

Based on Definition 2.3(a) and Theorem 2.4 we introduce two numerical parameters $\gamma_D^p(\vec{\omega})$ and Γ_X^p that are designed to quantify the *degree of strictness* of the nontrivial p-negative type inequalities.

DEFINITION 2.5. Let $p \geq 0$ and (X, d) be a metric space. Let q, t be natural numbers and $D = [a_j; b_i]_{q,t}$ be a (q, t)-simplex in X. Denote by $N_{q,t}$ the set of all normalized load vectors $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t) \subset \mathbb{R}_+^{q+t}$ for D. Then the *(normalized) p-negative type simplex gap* of D is defined to be the function $\gamma_D^p : N_{q,t} \to \mathbb{R}$ where

$$\gamma_D^p(\vec{\omega}) = \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p - \sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p - \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p$$

for each $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t) \in N_{q,t}$.

Note that $\gamma_D^p(\vec{\omega})$ is just taking the difference between the right-hand side and the left-hand side of the inequality (2.2). So, by Theorem 2.4, (X, d) has strict p-negative type if and only if $\gamma_D^p(\vec{\omega}) > 0$ for each normalized (q, t)-simplex $D(\vec{\omega}) \subseteq X$.

DEFINITION 2.6. Let $p \geq 0$. Let (X, d) be a metric space with p-negative type. We define the *(normalized) p-negative type gap* of (X, d) to be the nonnegative quantity

$$\Gamma_X^p = \inf_{D(\vec{\omega})} \gamma_D^p(\vec{\omega}),$$

where the infimum is taken over all normalized (q, t)-simplices $D(\vec{\omega})$ in X.

Note (for example) that if $\Gamma_X^p > 0$, then (X, d) has strict p-negative type. Doust and Weston [3] have given an example of an infinite metric space to show that the converse of this statement is not true in general. In other words, there exist infinite metric spaces (X, d) with strict p-negative type and with $\Gamma_X^p = 0$. It is not at all clear whether the same phenomenon can occur for finite metric spaces that have strict p-negative type.

REMARK 2.7. Suppose that (X, d) is a metric space with p-negative type for some $p \geq 0$. There are two ways in which we may view the parameter $\Gamma = \Gamma_X^p$. By definition, Γ is the largest nonnegative constant so that

$$\Gamma + \sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p \leq \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p \tag{2.3}$$

for all normalized (q, t)-simplices $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{q,t}$ in X. Alternatively, Γ is the largest nonnegative constant so that

$$\frac{\Gamma}{2}\left(\sum_{\ell=1}^{k} |\eta_\ell|\right)^2 + \sum_{1 \leq i,j \leq k} d(x_i, x_j)^p \eta_i \eta_j \leq 0 \tag{2.4}$$

for all natural numbers $k \geq 2$, all finite subsets $\{x_1, \ldots, x_k\} \subseteq X$ and all choices of real numbers $\eta_1, \ldots, \eta_k$ with $\eta_1 + \cdots + \eta_k = 0$. The fact that Γ is scaled on the left-hand side of (2.4) simply reflects that the classical p-negative type inequalities are not (by definition) normalized whereas the generalized roundness inequalities are normalized. The equivalence of (2.3) and (2.4) is noted by Doust and Weston [3, 4].

Recall that a *finite metric tree* is a finite connected graph that has no cycles, endowed with an edge weighted path metric. Hjorth *et al.* [9] have shown that finite metric trees have strict 1-negative type. Therefore, it makes sense to try to compute the 1-negative type gap of any given finite metric tree. This has been done recently by Doust and Weston [3]. However, a modicum of additional notation is necessary before stating their result. The set of all edges in a metric tree (T, d), considered as unordered pairs, will be denoted by $E(T)$, and the metric length $d(x, y)$ of any given edge $e = (x, y) \in E(T)$ will be denoted by $|e|$.

THEOREM 2.8 (Doust and Weston [3]). *Let (T, d) be a finite metric tree. Then the (normalized) 1-negative type gap $\Gamma = \Gamma_T^1$ of (T, d) is given by the following formula:*

$$\Gamma = \left\{ \sum_{e \in E(T)} |e|^{-1} \right\}^{-1}.$$

In particular, $\Gamma > 0$.

Although strict 1-negative type has been relatively well studied, the properties of strict p-negative type for $p \neq 1$ remain rather obscure and, indeed, there are a large number of intriguing open problems which beg further investigation. See, for example, Prassidis and Weston [16, Section 6] which lists some such problems.

3. Determining the 0-negative type gap of a finite metric space

One interesting feature of 0-negative type is that the metric is *forgotten* in the families of inequalities (2.1) and (2.2) since we have $d(x, y)^0 = 1$ for all x, y with $x \neq y$. In this section we concentrate on the limiting case $p = 0$ of negative type in the context of finite metric spaces.

Forgetting the metric allows the exact computation of the 0-negative type gap Γ_X^0 for each finite metric space (X, d). This is done in Theorem 3.2. As one would expect, the resulting formula for $\Gamma_X^0 > 0$ depends only on $|X|$. The most critical computations actually take place in the following technical lemma. Recall that $N_{q,t}$ denotes the set of vectors $\vec{\omega} = (m_1, \dots, m_q, n_1, \dots, n_t) \in \mathbb{R}_+^{q+t}$ that satisfy the two constraints $m_1 + \cdots + m_q = 1 = n_1 + \cdots + n_t$.

LEMMA 3.1. *Let (X, d) be a metric space and let $q, t \in \mathbb{N}$ such that $q + t \leq |X|$. Then, for each (q, t)-simplex $D = [a_j; b_i]_{q,t} \subseteq X$, we have*

$$\min_{\vec{\omega} \in N_{q,t}} \gamma_D^0(\vec{\omega}) = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{t}\right).$$

In particular, this minimum depends only on q and t, and not on the specific vertices of the simplex $D \subseteq X$.

PROOF. The overall idea of the proof is to implement Lagrange's multiplier theorem on a large scale. Consider a given (q, t)-simplex $D = [a_j; b_i]_{q,t} \subseteq X$ such that $q + t \leq |X|$. According to Definition 2.5, the 0-negative type simplex gap γ_D^0 is currently only defined on the constraint surface $N_{q,t} \subset \mathbb{R}_+^{q+t}$. Let γ denote the formal extension of γ_D^0 to all of the open quadrant $\mathbb{R}_+^{q+t}$. In other words,

$$\gamma(\vec{\omega}) = \sum_{j,i=1}^{q,t} m_j n_i - \sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} - \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2}$$

for all $\vec{\omega} = (m_1, \dots, m_q, n_1, \dots, n_t) \in \mathbb{R}_+^{q+t}$. Note that both γ and $\gamma_{|N_{q,t}} = \gamma_D^0$ depend only on $\vec{\omega}$, q and t, and not on the specific vertices of the simplex $D \subseteq X$. This is because we are dealing with the limiting case of p-negative type: $p = 0$.

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To complete the proof we introduce two Lagrange multipliers λ_1, λ_2 and proceed to solve the system

$$\begin{cases} \dfrac{\partial}{\partial m_j}\left(\gamma(\vec{\omega}) - \lambda_1 \cdot \displaystyle\sum_{j_1=1}^{q} m_{j_1} - \lambda_2 \cdot \sum_{i_1=1}^{t} n_{i_1}\right) = 0, & 1 \leq j \leq q \\ \dfrac{\partial}{\partial n_i}\left(\gamma(\vec{\omega}) - \lambda_1 \cdot \displaystyle\sum_{j_1=1}^{q} m_{j_1} - \lambda_2 \cdot \sum_{i_1=1}^{t} n_{i_1}\right) = 0, & 1 \leq i \leq t \end{cases} \tag{3.1}$$

subject to the two constraints imposed by the condition $\vec{\omega} \in N_{q,t}$.

The partial derivatives appearing in (3.1) are easily computed and (together with the two constraints) lead to the following equations:

$$\sum_{i=1}^{t} n_i - \sum_{j_1 \neq j} m_{j_1} - \lambda_1 = 0 \quad \forall j,\ 1 \leq j \leq q \tag{3.2}$$

$$\sum_{j=1}^{q} m_j - \sum_{i_1 \neq i} n_{i_1} - \lambda_2 = 0 \quad \forall i,\ 1 \leq i \leq t \tag{3.3}$$

$$n_1 + \cdots + n_t = 1 \tag{3.4}$$

$$m_1 + \cdots + m_q = 1. \tag{3.5}$$

By adding the q equations of (3.2) and applying the constraint (3.4) we obtain

$$q - (q-1)\sum_{j=1}^{q} m_j - q\lambda_1 = 0.$$

Hence, $\lambda_1 = 1/q$ by the constraint (3.5). So (further) from (3.2) and (3.5) we therefore have

$$m_j = 1 - \sum_{j_1 \neq j} m_{j_1} = \frac{1}{q} \quad \forall\, j,\ 1 \leq j \leq q.$$

Similarly, $n_i = 1/t$ for all i, $1 \leq i \leq t$.

We may now conclude from Lagrange's multiplier theorem that

$$\begin{aligned} \min_{\vec{\omega} \in N_{q,t}} \gamma(\vec{\omega}) &= qt \cdot \frac{1}{qt} - \frac{q(q-1)}{2} \cdot \frac{1}{q^2} - \frac{t(t-1)}{2} \cdot \frac{1}{t^2} \\ &= 1 - \frac{1}{2} \cdot \frac{q-1}{q} - \frac{1}{2} \cdot \frac{t-1}{t} \\ &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{t}\right), \end{aligned}$$

thereby establishing the statement of the lemma. □

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THEOREM 3.2. *Let (X, d) be a finite metric space with cardinality $n = |X| \geq 2$. Then the (normalized) 0-negative type gap Γ_X^0 of (X, d) is given by the following formula:*

$$\Gamma_X^0 = \frac{1}{2}\left(\frac{1}{\lfloor n/2 \rfloor} + \frac{1}{\lceil n/2 \rceil}\right).$$

PROOF. We begin by considering a fixed natural number m such that $2 \leq m \leq n$. Suppose that D is an arbitrary (q, t)-simplex in X such that $q + t = m$. We may assume that $q \leq t$ (by relabeling the simplex if necessary). Let $F(q)$ denote $\min_{\vec{\omega} \in N_{q,t}} \gamma_D^0(\vec{\omega})$. We proceed to minimize F as a function of q. According to Lemma 3.1:

$$F(q) = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{m - q}\right).$$

Consideration of $F'(q)$ shows that F minimizes when $q = \lfloor m/2 \rfloor$ (in which case $t = \lceil m/2 \rceil$). Finally, as a function of m, the expression

$$(\min F)(m) = \frac{1}{2}\left(\frac{1}{\lfloor m/2 \rfloor} + \frac{1}{\lceil m/2 \rceil}\right)$$

decreases (strictly) as m increases. We therefore obtain the stated formula for Γ_X^0 by considering the largest possible value of m allowed in this setting: $m = n$. □

REMARK 3.3. Consider a finite metric space (X, d) with $n = |X|$. By way of definition, we say that a normalized (q, t)-simplex $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{q,t}$ in X is *extreme for the zero* 0*-negative type gap of* (X, d) if and only if $\gamma_D(\vec{\omega}) = \Gamma_X^0$. The proofs of Lemma 3.1 and Theorem 3.2 show that $D(\vec{\omega})$ is extreme if and only if $q = \lfloor n/2 \rfloor$, $t = \lceil n/2 \rceil$, $m_j = 1/q$ for all j $(1 \leq j \leq q)$, and $n_i = 1/t$ for all i $(1 \leq i \leq t)$.

The next result makes the point that (unlike all finite metric spaces) no infinite metric space (X, d) can have a positive 0-negative type gap Γ_X^0.

COROLLARY 3.4. *No infinite metric space has a positive* 0*-negative type gap.*

PROOF. Let $n \to \infty$ in Theorem 3.2 (whence $\Gamma_X^0 \to 0^+$). □

4. The supremal strict p-negative type of a finite metric space

In this section we present the main results of this paper in Theorem 4.1 and Remark 4.2. In the proof of Theorem 4.1 we employ the notation $(\min F)(m)$ that was introduced in the proof of Theorem 3.2. The overarching strategy is to exploit the *positive* 0-negative type gap of each finite metric space (X, d). Recall that the *diameter* of a metric space (X, d) is the quantity $\operatorname{diam} X = \max_{x,y \in X} d(x, y)$.

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THEOREM 4.1. *Let (X, d) be a finite metric space with cardinality $n = |X| \geq 3$ (to avoid the trivial case $n = 2$). Let $\mathfrak{D} = (\operatorname{diam} X)/\min\{d(x, y) \mid x \neq y\}$ denote the scaled diameter of (X, d). Then (X, d) has p-negative type for all $p \in [0, \zeta]$ where*

$$\zeta = \frac{\ln(1/(1 - \Gamma))}{\ln \mathfrak{D}} \quad \textit{with } \Gamma = \Gamma_X^0.$$

Moreover, (X, d) has strict p-negative type for all $p \in [0, \zeta)$.

PROOF. We may assume that the metric d is not a positive multiple of the discrete metric on X. (Otherwise, (X, d) has strict p-negative type for all $p \geq 0$.) Hence $\mathfrak{D} > 1$. We may also assume that $\min\{d(x, y) \mid x \neq y\} = 1$ by scaling the metric d in the obvious way, if necessary. This means that $\mathfrak{D}$ is now the diameter of our rescaled metric space (which we continue to denote by (X, d)). Let Γ denote Γ_X^0.

Consider an arbitrary normalized (q, t)-simplex $D = [a_j(m_j); b_i(n_i)]_{q,t}$ in X. Necessarily, $m = q + t \leq n$. For any given $p \geq 0$, let

$$L(p) = \sum_{j_1 < j_2} m_{j_1} m_{j_2} d(a_{j_i}, a_{j_2})^p + \sum_{i_1 < i_2} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p,$$

and

$$R(p) = \sum_{j,i} m_j n_i d(a_j, b_i)^p.$$

By definition of the 0-negative type gap $\Gamma = \Gamma_X^0$ we have

$$L(0) + \Gamma \leq R(0). \tag{4.1}$$

The strategy of the proof is to argue that

$$L(p) < L(0) + \Gamma \quad \text{and} \quad R(0) \leq R(p) \tag{4.2}$$

provided that $p > 0$ is sufficiently small. The net effect from (4.1) and (4.2) is then $L(p) < R(p)$ or, put differently, that (X, d) has strict p-negative type. As all nonzero distances in (X, d) are at least one we automatically obtain the second inequality of (4.2) for all $p > 0$: $R(0) \leq R(p)$. Therefore, we only need to concentrate on the first inequality of (4.2). First of all note that

$$\begin{aligned} L(p) - L(0) &= \sum_{j_1 < j_2} m_{j_1} m_{j_2} (d(a_{j_1}, a_{j_2})^p - 1) + \sum_{i_1 < i_2} n_{i_1} n_{i_2} (d(b_{i_1}, b_{i_2})^p - 1) \\ &\leq \left(\sum_{j_1 < j_2} m_{j_1} m_{j_2} + \sum_{i_1 < i_2} n_{i_1} n_{i_2} \right) \cdot (\mathfrak{D}^p - 1) \end{aligned}$$

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$$
\begin{aligned}
&\leq \left(\frac{q(q-1)}{2}\cdot\frac{1}{q^2}+\frac{t(t-1)}{2}\cdot\frac{1}{t^2}\right)\cdot(\mathfrak{D}^p-1)\\
&= \left(1-\frac{1}{2}\left(\frac{1}{q}+\frac{1}{t}\right)\right)\cdot(\mathfrak{D}^p-1)\\
&\leq \left(1-\frac{1}{2}\left(\frac{1}{\lfloor m/2\rfloor}+\frac{1}{\lceil m/2\rceil}\right)\right)\cdot(\mathfrak{D}^p-1)\\
&= (1-(\min F)(m))\cdot(\mathfrak{D}^p-1)\\
L(p)-L(0) &\leq (1-\Gamma)\cdot(\mathfrak{D}^p-1) \qquad (4.3)
\end{aligned}
$$

by (slight modifications of) the computations in the proofs of Lemma 3.1 and Theorem 3.2. Now observe that

$$(1-\Gamma)\cdot(\mathfrak{D}^p-1)\leq\Gamma \quad \text{if and only if} \quad p\leq\frac{\ln(1/(1-\Gamma))}{\ln\mathfrak{D}}. \qquad (4.4)$$

By combining (4.3) and (4.4) we obtain the first inequality of (4.2) for all $p>0$ such that

$$p<\zeta=\frac{\ln(1/(1-\Gamma))}{\ln\mathfrak{D}}.$$

Hence, $L(p)<R(p)$ for any such p. It is also clear from (4.2), (4.3) and (4.4) that $L(\zeta)\leq R(\zeta)$. In total these observations complete the proof of the theorem. □

REMARK 4.2. The following example illustrates that the value of $\zeta=\zeta(n,\mathfrak{D})$ obtained in Theorem 4.1 is optimal provided that $\mathfrak{D}\leq 2$. More precisely, for each natural number $n\geq 3$, there exists an n-point metric space (X,d) of scaled diameter $\mathfrak{D}=2$ whose maximal p-negative type and supremal strict p-negative type are both equal to $\zeta(n,2)$. The idea is to consider a class of simple metrics on certain complete bipartite graphs. Consider a natural number $n\geq 3$. Set $q=\lfloor n/2\rfloor$ and $t=\lceil n/2\rceil$ (whence $q+t=n$). Let the set X consist of n distinct vertices $a_1,\ldots,a_q,b_1,\ldots,b_t$. Define a metric d on X as follows: $d(a_j,b_i)=1$ for all i and j, $d(a_{j_1},a_{j_2})=2$ for all $j_1\neq j_2$ and $d(b_{i_1},b_{i_2})=2$ for all $i_1\neq i_2$. This metric space has scaled diameter $\mathfrak{D}=2$. Now consider the normalized (q,t)-simplex $D=[a_j(1/q);b_i(1/t)]_{q,t}$ in X. If inequality (2.2) holds with exponent $p\geq 0$ for the normalized (q,t)-simplex D, then p is subject to the following condition

$$
\begin{aligned}
\frac{q(q-1)}{2}\cdot\frac{1}{q^2}\cdot 2^p+\frac{t(t-1)}{2}\cdot\frac{1}{t^2}\cdot 2^p &= 2^p\cdot\frac{1}{2}\cdot\left(\frac{q-1}{q}+\frac{t-1}{t}\right)\\
&= 2^p\cdot(1-\Gamma)\\
&\leq 1
\end{aligned}
$$

where

$$\Gamma=\frac{1}{2}\left(\frac{1}{q}+\frac{1}{t}\right)=\frac{1}{2}\left(\frac{1}{\lfloor n/2\rfloor}+\frac{1}{\lceil n/2\rceil}\right).$$

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It follows that the p-negative type of (X, d) cannot exceed

$$\zeta(n, 2) = \frac{\ln(1/1 - \Gamma)}{\ln 2}.$$

Hence, the maximal p-negative type and supremal strict p-negative type of (X, d) are both equal to $\zeta(n, 2)$ by Theorem 4.1. We note that this example can be naturally modified to accommodate any scaled diameter $\mathfrak{D} \in (1, 2]$. The obstruction to the case $\mathfrak{D} > 2$ comes from the triangle inequality (which would be violated in the above setting). However, no such obstruction exists for semi-metric spaces. Recall that a *semi-metric* is required to satisfy all of the axioms of a metric except (possibly) the triangle inequality. In this respect we are following Khamsi and Kirk [10].

REMARK 4.3. In closing we note that Theorems 3.2 and 4.1 hold (more generally) for all finite semi-metric spaces (X, d). This is because the triangle inequality has played no role in any of the definitions or computations of this paper. Moreover, in the case of finite semi-metric spaces, the value of $\zeta(n, \mathfrak{D})$ from Theorem 4.1 is always optimal. This follows from the obvious modifications to the examples given in Remark 4.2.

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