A GENERALIZATION OF AN INVERSION FORMULA FOR THE GAUSS TRANSFORMATION

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1. Introduction. In an earlier paper [3] we considered an inversion formula for the Gauss transformation G defined by

$$(1.1) \qquad (G\varphi)(x) = (4\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/4} \varphi(y)dy .$$

We noted there that formally G is inverted by

$$(1.2) \qquad e^{-D^2} \text{ where } D = \frac{d}{dx} ,$$

and we showed that if e^{-D^2} is interpreted via the power series for the exponential function, that is if

$$(1.3) \qquad e^{-D^2} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(2n)}(x) ,$$

then under certain conditions on φ ,

$$(1.4) \qquad e^{-D^2} f(x) = \varphi(x) , \text{ where } f = G\varphi .$$

Now from [1; 10.13(19)] ,

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$$(1.5) \qquad e^{-(tz+t^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!} t^n ,$$

so that formally

$$e^{-D^2} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!} D^n e^{zD} f(x) ,$$

and thus using the usual rule that $e^{zD} f(x) = f(x+z)$ we obtain formally

$$(1.6) \qquad e^{-D^2} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!} f^{(n)}(x+z) .$$

We shall show in section two that with this new interpretation of e^{-D^2}, then under certain conditions on φ, (1.4) holds. The interpretation of e^{-D^2} given by (1.6) reduces to that of (1.3) when $z = 0$ since

$$H_{2n+1}(0) = 0 \text{ and } H_{2n}(0) = (-1)^n (2n)!/n! .$$

Another special case of (1.6), namely $z = -x$ is worthy of note, since then the coefficients of the Hermite polynomials become independent of x.

In [3] we also considered the Abel summability of (1.3). This is equivalent to considering

$$\lim_{t \to 1-} e^{-tD^2} f(x) ,$$

where $e^{-tD^2} f(x)$ is defined through the power series for the exponential function. Using (1.5) as previously we would have

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(1.7) $$e^{-tD^2} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!} f^{(n)}(x+t^{1/2}z)t^{n/2} ,$$

and we shall study, in section three, the behaviour of series (1.7) as $t \to 1-$. In section four we give some applications of (1.6).

2. Convergence theory. In theorem one we show that the series (1.5) is equiconvergent with a well known integral and in theorem two we specialize the results of theorem one to obtain convergence results.

THEOREM 1. Let φ be integrable over every finite interval, and suppose that

(2.1) $$\int_n^{\infty} e^{-v^2/2} v^{-5/3} \{|\varphi(x+z+2v)|+|\varphi(x+z-2v)|\} dv = o(n^{-1})$$

as $n \to \infty$.

Then $(G\varphi)(u)$ exists for all real u, and if $f = G\varphi$ and $s_n(x, z)$ denotes the n^{th} partial sum of the series (1.6) then for any $\delta > 0$

$$\lim_{n\to\infty} (s_n(x, z) - \pi^{-1} \int_{x-\delta}^{x+\delta} \frac{\sin((n/2)^{1/2}(x-t))}{x-t} \varphi(t)dt) = 0 .$$

Proof. Clearly $|x+z-y|^{5/3} e^{-(u-y)^2/4 + (x+z-y)^2/8}$ is a bounded function of y. Let k be its maximum. Then

$$\int_{-\infty}^{\infty} e^{-(u-y)^2/4} |\varphi(y)| dy = \int_{-\infty}^{x+z-2} + \int_{x+z-2}^{x+z+2} + \int_{x+z+2}^{\infty}$$

$$\leq k \int_{-\infty}^{x+z-2} e^{-(x+z-y)^2/8} |x+z-y|^{-5/3} |\varphi(y)| dy$$

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$$+ \int_{x+z-2}^{x+z+2} |\varphi(y)|\,dy + k \int_{x+z+2}^{\infty} e^{-(x+z-y)^2/8} |x+z-y|^{-5/3} |\varphi(y)|\,dy$$

$$= k \int_{1}^{\infty} e^{-v^2/2} v^{-5/3} \{|\varphi(x+z+2v)| + |\varphi(x+z-2v)|\}\,dv$$

$$+ \int_{x+z-2}^{x+z+2} |\varphi(y)|\,dy < \infty .$$

Hence $(G\varphi)(u)$ exists for all u. Further from [2; chap. VIII, § 2.2] $f = G\varphi$ has derivatives of all orders which can be obtained by differentiating under the integral sign. Since from [1; 10.13(7)],

$$\frac{d^n}{dx^n} e^{-x^2/4} = (-1)^n 2^{-n} H_n(x/2) e^{-x^2/4}$$

it follows that

$$f^{(n)}(x) = \frac{(-1)^n}{2^n (4\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4} H_n((x-y)/2)\, \varphi(y)\,dy , \tag{2.2}$$

so that

$$\frac{(-1)^n}{n!} f^{(n)}(x+z) = \frac{1}{2^n n!\,(4\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-(x+z-y)^2/4} H_n((x+z-y)/2) \varphi(y)\,dy$$

$$= \frac{1}{2^n n!\,\pi^{1/2}} \int_{-\infty}^{\infty} e^{-v^2} H_n(v)\, \varphi(x+z-2v)\,dv .$$

Comparing this last expression with [4; 9.1.3], we see that the coefficient of $H_n(z/2)$ in (1.6) is equal to the n^{th} Hermite coefficient of $\varphi(x+z-2v)$. Hence (1.6) is the Hermite expansion, evaluated at $z/2$, of $\varphi(x+z-2v)$. But by [4, Theorem 9.1.6], since (2.1) is satisfied the Hermite expansion of $\varphi(x+z-2v)$,

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evaluated at $z/2$, is equiconvergent with

$$\frac{1}{\pi}\int_{(z-\delta)/2}^{(z+\delta)/2} \varphi(x+z-2v)\,\frac{\sin((2n)^{1/2}((z/2)-v))}{(z/2)-v}\,dv$$

$$= \frac{1}{\pi}\int_{x-\delta}^{x+\delta} \varphi(t)\,\frac{\sin((n/2)^{1/2}(x-t))}{x-t}\,dt \ ;$$

that is

$$\lim_{n\to\infty}\left(s_n(x,z) - \frac{1}{\pi}\int_{x-\delta}^{x+\delta} \varphi(t)\,\frac{\sin((n/2)^{1/2}(x-t))}{x-t}\,dt\right) = 0 \ .$$

Using Theorem 1 we can derive many sets of conditions that the series (1.6) converge to $\varphi(x)$. We state some of these as our next theorem.

THEOREM 2. Suppose φ is integrable over every finite interval, and that

$$\int_1^\infty e^{-v^2/2}\, v^{-2/3}\,\{|\varphi(x+z+2v)| + |\varphi(x+z-2v)|\}\,dv < \infty \ .$$

Then $(G\varphi)(u)$ exists for all real u, and if $f = G\varphi$

$$\text{(i)}\quad \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!}\, f^{(n)}(x+z) = \{\varphi(x+) + \varphi(x-)\}/2$$

if φ is of bounded variation in a neighbourhood of x, and

$$\text{(ii)}\quad \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!}\, f^{(n)}(x+z) = \varphi(x)$$

if for some $\Delta > 0$, $\int_0^\Delta |\varphi(x+t) + \varphi(x-t) - 2\varphi(x)|\,\frac{dt}{t} < \infty$.

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Proof. Clearly

$$\int_n^\infty e^{-v^2/2} v^{-5/3} \{|\varphi(x+z+2v)| + |\varphi(x+z-2v)|\} dv$$

$$\leq n^{-1} \int_n^\infty e^{-v^2/2} v^{-2/3} \{|\varphi(x+z+2v)| + |\varphi(x+z-2v)|\} dv = o(n^{-1}) .$$

Hence by Theorem 1, $(G\varphi)(u)$ exists and (1.6) is equiconvergent with

$$\frac{1}{\pi} \int_{x-\delta}^{x+\delta} \varphi(t) \frac{\sin((n/2)^{1/2}(x-t))}{x-t} dt .$$

But from [5; v. II, chap. XVI, 1.3], [5; v. I, chap. II, 6.1], and [5; v. I, chap. II, 8.1]

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{x-\delta}^{x+\delta} \varphi(t) \frac{\sin((n/2)^{1/2}(x-t))}{x-t} dt$$

$$= (\varphi(x+) + \varphi(x-))/2 \quad \text{or} \quad \varphi(x)$$

under precisely the conditions described in (i) and (ii).

3. Theorem 3 deals with e^{-tD^2}. It should be noticed that z can depend on t.

THEOREM 3. Suppose φ is integrable over every finite interval, that

$$\int_{-\infty}^\infty e^{-(x+t^{1/2}z-y)^2/8} |\varphi(y)| dy < \infty \quad \text{for } t_o < t < 1 ,$$

and that

$$\int_x^y [\varphi(v) - \varphi(x)] dv = o(y-x) \qquad \text{as } y \to x .$$

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Then

$$\lim_{t \to 1-} e^{-tD^2} f(x) = \varphi(x)$$

Proof. The existence of $(G\varphi)(u)$ follows as in Theorem 1. Suppose $t_o < t < 1$. Then from (2.2), and using [1; 10.13(22)]

$$e^{-tD^2} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n H_n(z/2)}{n!} f^{(n)}(x+t^{1/2}z)t^{n/2}$$

$$= \sum_{n=0}^{\infty} (4\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(x+t^{1/2}z-y)^2/4} \frac{H_n(z/2)H_n((x+t^{1/2}z-y)/2)}{2^n n!} t^{n/2} \varphi(y)dy$$

$$= (4\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(x+t^{1/2}z-y)^2/4} \sum_{n=0}^{\infty} \frac{H_n(z/2)H_n((x+t^{1/2}z-y)/2)}{2^n n!} t^{n/2} \varphi(y)dy$$

$$= (4\pi(1-t))^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/4(1-t)} \varphi(y)dy ,$$

providing we justify the interchange of summation and integration. For this it suffices to show that

$u(x, t)$

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-(x+t^{1/2}z-y)^2/4} \frac{|H_n(z/2)||H_n((x+t^{1/2}z-y)/2)|}{2^n n!} t^{n/2} |\varphi(y)| dy < \infty .$$

But from [1; 10.18(19)],

$$u(x, t) \leq k^2 e^{z^2/8} \sum_{n=0}^{\infty} t^{n/2} \int_{-\infty}^{\infty} e^{-(x+t^{1/2}z-y)^2/8} |\varphi(y)| dy$$

$$= \frac{k^2 e^{z^2/8}}{(1-t^{1/2})} \int_{-\infty}^{\infty} e^{-(x+t^{1/2}z-y)^2/8} |\varphi(y)| dy < \infty ,$$

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and the interchange is justified.

But from [2; chap. VIII, Theorem 7.2]

$$\lim_{t \to 1-} (4\pi(1-t))^{-1/2} \int_{-\infty}^{\infty} e^{-(x-y)^2/4(1-t)} \varphi(y)dy = \varphi(x)$$

provided

$$\int_x^y [\varphi(v) - \varphi(x)]dv = o(y-x) .$$

4. Applications. Various expansions can be deduced from Theorem 2. For example, it follows from [1; 10.13(30)] that if

$$\varphi(x) = H_n(ax), -1/2 < a < 1/2 , \quad \text{then}$$

$$(G\varphi)(x) = (1-4a^2)^{n/2} H_n(ax/(1-4a^2)^{1/2}) .$$

Then, since from [1; 10.13(14)],

$$\frac{d^r}{dx^r} H_n(bx) = \begin{cases} (2b)^r n! \, H_{n-r}(bx)/(n-r)! \, , & r \leq n , \\ 0 & , \quad r > n , \end{cases}$$

it follows from Theorem 2 that

$$H_n(ax)$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-1)^r (2a)^r (1-4a^2)^{(n-r)/2} H_r(z/2) H_{n-r}(a(x+z)/(1-4a^2)^{1/2}) ,$$

or setting $a = (\cos \varphi)/2$, $0 < \varphi < \pi$, $\varphi \neq \pi/2$,
$x = 2(s+t \tan \varphi)$, $z = -2s$

$$H_n(s \cos \varphi + t \sin \varphi) = \sum_{r=0}^{n} \binom{n}{r} \cos^r \varphi \sin^{n-r} \varphi \, H_r(s) \, H_{n-r}(t) ,$$

which is a special case of [1; 10.13(40)], clearly valid now for $0 \le \varphi \le \pi$.

Also, from [2; chap. VIII, § 2.6] if $\varphi(x) = e^{-ax^2}$, $a > -1/4$, then $(G(\varphi))(x) = (1+4a)^{-1/2} e^{-ax^2/(1+4a)}$. Hence, since if $b > 0$

$$\frac{d^r}{dx^r} e^{-bx^2} = (-1)^r b^{r/2} e^{-bx^2} H_r(b^{1/2}x),$$

theorem two yields, if $a > 0$,

$$e^{-ax^2}$$
$$= (1+4a)^{-1/2} e^{-a(x+z)^2/(1+4a)} \sum_{n=0}^{\infty} \frac{H_n(z/2) H_n((x+z)(a/(1+4a))^{1/2})}{n!} (a/(1+4a))^{n/2}$$

a result equivalent to [1; 10.13(22)].

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