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HEYDE'S CHARACTERIZATION THEOREM FOR DISCRETE ABELIAN GROUPS

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Abstract

Let X be a countable discrete abelian group with automorphism group $\mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent X-valued random variables with distributions μ_1 and μ_2, respectively. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$ and $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Assuming that the conditional distribution of the linear form L_2 given L_1 is symmetric, where $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ and $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$, we describe all possibilities for the μ_j. This is a group-theoretic analogue of Heyde's characterization of Gaussian distributions on the real line.

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1. Introduction

Many studies have been devoted to characterizing Gaussian distributions on the real line. Specifically, in 1970 Heyde proved the following theorem, which characterizes a Gaussian distribution by the symmetry of the conditional distribution of one linear form given another.

THEOREM 1.1 (Heyde [10]; see also [11, Section 13.4.1]). *Suppose that $n \geq 2$, $\xi_1, \xi_2, \ldots, \xi_n$, are independent random variables, and let α_j and β_j be nonzero constants, where $j = 1, 2, \ldots, n$, such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ whenever $i \neq j$. If the conditional distribution of L_2 given L_1 is symmetric, where $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ and $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$, then all the random variables ξ_j are Gaussian.*

The articles [3–5, 12] (see also [6, Ch. VI]) were devoted to finding group-theoretic analogues of Heyde's theorem. The present article continues this research.

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ASSUMPTIONS 1.2. Suppose that $\mathrm{Aut}(X)$ is the set of topological automorphisms of a second countable locally compact abelian group X. Suppose also that $n \geq 2$ and ξ_j are independent X-valued random variables with distributions μ_j, where $j = 1, 2, \ldots, n$, and that $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ are such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ whenever $i \neq j$. Here, whenever we make a statement involving the $\pm$ symbol, we mean that both cases hold. Define the linear forms L_1 and L_2 by $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$.

We formulate the following problem.

PROBLEM 1.3. Describe groups X for which the symmetry of the conditional distribution of L_2 given L_1 implies that all of the μ_j either are Gaussian distributions or belong to a class of distributions which can be considered as a natural analogue of the class of Gaussian distributions.

Problem 1.3 has been studied in different important subclasses of the class of locally compact abelian groups, but has not yet been solved in general. In [3], Problem 1.3 was completely solved for the class of finite abelian groups, and in [5] it was solved for the class of countable discrete abelian groups. For these classes of groups, the class of idempotent distributions can be regarded as a natural analogue of the class of Gaussian distributions. In both cases, a corresponding class of groups can easily be described; it consists of groups that contain no elements of order two.

We now formulate the following general problem.

PROBLEM 1.4. Let X be a second countable locally compact abelian group. Assume that the conditional distribution of L_2 given L_1 is symmetric. Describe the possible distributions μ_j.

Problem 1.4 was solved within the class of finite abelian groups in [12]. In this article, we solve Problem 1.4 for the class of countable discrete abelian groups. We note that the solution of Problem 1.4 in [12] was based on the finiteness of the automorphism group of a finite group; however, for a general discrete abelian group, the automorphism group may be infinite. Therefore, our solution of Problem 1.4 in the class of countable discrete abelian groups requires new and different reasoning.

We shall use various well-known facts from abstract harmonic analysis, the structure theory of locally compact abelian groups [9], and the theory of infinite abelian groups [7, 8].

First, let us fix some notation. Let X be a second countable locally compact abelian group. We denote by Y the character group X^* of X. Let $\langle x, y\rangle$ be the value of a character $y \in Y$ at an element $x \in X$. For $\alpha \in \mathrm{Aut}(X)$, we define the adjoint automorphism $\tilde{\alpha} \in \mathrm{Aut}(Y)$ via the formula

$$\langle x, \tilde{\alpha}y\rangle = \langle \alpha x, y\rangle \quad \forall x \in X,\ \forall y \in Y.$$

We denote by I the identity automorphism of a group. A subgroup G of the group X is said to be characteristic if G is invariant under all topological automorphisms of X.

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Given a subgroup H of Y, we denote its annihilator $\{x \in X \mid \forall y \in H, \langle x, y\rangle = 1\}$ by $A(X, H)$. We denote by b_X the subgroup of all compact elements of X. Given subsets A and B of X, we denote the sum set $\{x \in X \mid x = a + b \text{ with } a \in A, b \in B\}$ by $A + B$.

For any integer n, we put $X^{(n)} = \{x \in X \mid \exists x' \in X \text{ such that } x = nx'\}$ and $X_{(n)} = \{x \in X \mid nx = 0\}$. A group X is said to be bounded if the orders of the elements of X are bounded, that is, if there exists n such that $X = X_{(n)}$. For a discrete abelian group X and a prime number p, let X_p be the p-component of X, that is, the subgroup of X consisting of elements whose orders are powers of p.

Given a distribution μ, we define its characteristic function $\widehat{\mu}$ by

$$\widehat{\mu}(y) = \int_X \langle x, y\rangle \, d\mu(x)$$

and denote its support by $\sigma(\mu)$. We recall that if H is a closed subgroup of Y and $\widehat{\mu}(y) = 1$ for all $y \in H$, then $\widehat{\mu}(y + h) = \widehat{\mu}(y)$ for all $y \in Y$ and $h \in H$, and $\sigma(\mu) \subseteq A(X, H)$. We define a distribution $\overline{\mu}$ by the formula $\overline{\mu}(B) = \mu(-B)$ for all Borel sets $B \subseteq X$. Let $I(X)$ be the set of idempotent distributions on X, that is, the set of translates of the Haar distributions m_K of compact subgroups K of X. We note that the characteristic function of the Haar distribution m_K is of the form

$$\widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in A(Y, K), \\ 0 & \text{if } y \notin A(Y, K). \end{cases}$$

We denote by E_x the degenerate distribution concentrated at a point $x \in X$.

2. Main results

The main result of this article is the following theorem.

THEOREM 2.1. *Suppose that Assumptions 1.2 hold; moreover, assume that X is countable and discrete and that $n = 2$. If the conditional distribution of L_2 given L_1 is symmetric, then $\mu_j = \rho_j * \pi_j$, where $\sigma(\rho_j) \subseteq X_2$ and $\pi_j \in I(X)$, for $j = 1, 2$.*

To prove Theorem 2.1, we need some auxiliary results.

The next theorem, which solves Problem 1.3 for the class of countable discrete abelian groups, was proved in [5]. For convenience, we formulate this theorem in the following form.

LEMMA 2.2 [5]. *Let X be a countable discrete abelian group with no elements of order two. Let ξ_1 and ξ_2 be independent X-valued random variables with distributions μ_1 and μ_2, respectively. Suppose that $\delta, I \pm \delta \in \mathrm{Aut}(X)$. If the conditional distribution of L_2 given L_1 is symmetric, where $L_2 = \xi_1 + \delta\xi_2$ and $L_1 = \xi_1 + \xi_2$, then $\mu_j = E_{k_j} * m_F$, where $k_j \in X$ and F is a finite subgroup of X such that $\delta(F) = F$.*

Let Y be an arbitrary abelian group, let f be a function on Y, and let $h \in Y$. We denote by Δ_h the finite difference operator

$$\Delta_h f(y) = f(y+h) - f(y) \quad \forall y \in Y.$$

A function f on Y is called a polynomial if

$$\Delta_h^{n+1} f(y) = 0$$

for some natural number n and for all $y, h \in Y$.

LEMMA 2.3 [1]. *Let Y be a locally compact abelian group and f a continuous polynomial on Y. Then f is constant on b_Y.*

LEMMA 2.4 [4]. *Suppose that Assumptions 1.2 hold. The conditional distribution of L_2 given L_1 is symmetric if and only if the characteristic functions of the distributions μ_j satisfy the functional equation*

$$\prod_{j=1}^{n} \widehat{\mu}_j(\tilde{\alpha}_j u + \tilde{\beta}_j v) = \prod_{j=1}^{n} \widehat{\mu}_j(\tilde{\alpha}_j u - \tilde{\beta}_j v) \quad \forall u, v \in Y. \tag{2.1}$$

Lemma 2.4 reduces the solution of Problems 1.3 and 1.4 to the study of solutions to equation (2.1) in the class of characteristic functions.

It is well known that any locally compact abelian group X is topologically isomorphic to a group of the form $\mathbb{R}^m \times G$, where $m \geq 0$ and G contains a compact open subgroup (see [9, Section 24.30]).

PROPOSITION 2.5 [5]. *Suppose that Assumptions 1.2 hold and, moreover, that $X = \mathbb{R}^m \times G$, where $m \geq 0$ and G contains a compact open subgroup. Assume that the conditional distribution of L_2 given L_1 is symmetric. Then each of the random variables ξ_j can be replaced by a translate ξ'_j in such a way that $\sigma(\mu'_j) \subseteq \mathbb{R}^m \times b_G$ for all j and the conditional distribution of L'_2 given L'_1 is symmetric, where $L'_2 = \beta_1 \xi'_1 + \cdots + \beta_n \xi'_n$ and $L'_1 = \alpha_1 \xi'_1 + \cdots + \alpha_n \xi'_n$.*

For the class of countable discrete abelian groups, Proposition 2.5 may be strengthened. The following statement is the crucial point in the proof of Theorem 2.1 and is of interest in its own right.

PROPOSITION 2.6. *Suppose that Assumptions 1.2 hold and, moreover, that X is a countable discrete abelian group. Then each of the random variables ξ_j can be replaced by a translate ξ'_j in such a way that, for some $k \geq 2$, $\sigma(\mu'_j) \subseteq X_{(k)}$ for all j and the conditional distribution of L'_2 given L'_1 is symmetric, where $L'_2 = \beta_1 \xi'_1 + \cdots + \beta_n \xi'_n$ and $L'_1 = \alpha_1 \xi'_1 + \cdots + \alpha_n \xi'_n$.*

PROOF. Taking into account Proposition 2.5, we may assume from the beginning that X is a torsion group. We will prove that in this case there exists $k \geq 2$ such that $\sigma(\mu_j) \subseteq X_{(k)}$ for all j. Since $X_{(k)}$ is a characteristic subgroup, we can pass to the new random variables $\alpha_j\xi_j$ and reduce the proof of Proposition 2.6 to consideration of the case where $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, for some $\delta_j \in \mathrm{Aut}(X)$. The condition $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$ is transformed into the condition $\delta_i \pm \delta_j \in \mathrm{Aut}(X)$ for all $i \neq j$. By Lemma 2.4, the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions $\widehat{\mu}_j$ satisfy (2.1), which takes the form

$$\prod_{j=1}^{n} \widehat{\mu}_j\big(u + \widetilde{\delta}_j v\big) = \prod_{j=1}^{n} \widehat{\mu}_j\big(u - \widetilde{\delta}_j v\big) \quad \forall u, v \in Y. \tag{2.2}$$

Set $\nu_j = \mu_j * \overline{\mu}_j$. Then $\widehat{\nu}_j(y) = |\widehat{\mu}_j(y)|^2 \geq 0$ for all $y \in Y$. Obviously, the characteristic functions $\widehat{\nu}_j$ also satisfy (2.2). Let U be a neighborhood of zero in the group Y such that $\widehat{\nu}_j(y) > 0$ for all $y \in U$. Set $\varphi_j(y) = -\log \widehat{\nu}_j(y)$ for all $y \in U$. We restrict ourselves to the case where $n = 2$; the case of arbitrary n is dealt with similarly. Rewriting (2.2) for $n = 2$, we obtain

$$\widehat{\mu}_1\big(u + \widetilde{\delta}_1 v\big)\widehat{\mu}_2\big(u + \widetilde{\delta}_2 v\big) = \widehat{\mu}_1\big(u - \widetilde{\delta}_1 v\big)\widehat{\mu}_2\big(u - \widetilde{\delta}_2 v\big) \quad \forall u, v \in Y. \tag{2.3}$$

Let V be a symmetric neighborhood of zero in the group Y such that for any choice of automorphisms $\lambda_j \in \{I, \widetilde{\delta}_1, \widetilde{\delta}_2\}$, with $j = 1, \ldots, 8$, the following inclusion holds:

$$\sum_{j=1}^{8} \lambda_j(V) \subseteq U.$$

Since X is a discrete torsion group, its character group Y is compact and totally disconnected. Hence there exists an open subgroup W of Y such that $W \subseteq V$.

We conclude from (2.3) that the functions φ_j satisfy the equation

$$\varphi_1\big(u + \widetilde{\delta}_1 v\big) + \varphi_2\big(u + \widetilde{\delta}_2 v\big) - \varphi_1\big(u - \widetilde{\delta}_1 v\big) - \varphi_2\big(u - \widetilde{\delta}_2 v\big) = 0 \quad \forall u, v \in W. \tag{2.4}$$

We use the finite difference method to solve (2.4). Take an arbitrary element k_1 of W. Substitute $u + \widetilde{\delta}_2 k_1$ for u and $v + k_1$ for v in (2.4); then subtract (2.4) from the resulting equation. This gives

$$\Delta_{l_{11}}\varphi_1\big(u + \widetilde{\delta}_1 v\big) + \Delta_{l_{12}}\varphi_2\big(u + \widetilde{\delta}_2 v\big) - \Delta_{l_{13}}\varphi_1\big(u - \widetilde{\delta}_1 v\big) = 0 \quad \forall u, v \in W, \tag{2.5}$$

where $l_{11} = (\widetilde{\delta}_1 + \widetilde{\delta}_2)k_1$, $l_{12} = 2\widetilde{\delta}_2 k_1$ and $l_{13} = (\widetilde{\delta}_2 - \widetilde{\delta}_1)k_1$. Take an arbitrary element k_2 of W, and substitute $u + k_2$ for u and $v + k_2$ for v in (2.5). Subtracting (2.5) from the resulting equation, we get

$$\Delta_{l_{21}}\Delta_{l_{11}}\varphi_1\big(u + \widetilde{\delta}_1 v\big) + \Delta_{l_{22}}\Delta_{l_{12}}\varphi_2\big(u + \widetilde{\delta}_2 v\big) = 0 \quad \forall u, v \in W, \tag{2.6}$$

where $l_{21} = 2\widetilde{\delta}_1 k_2$ and $l_{22} = (\widetilde{\delta}_1 + \widetilde{\delta}_2)k_2$. Take an arbitrary element k_3 of W. Substitute $u - \widetilde{\delta}_2 k_3$ for u and $v + k_3$ for v in (2.6). Subtracting (2.6) from the resulting equation yields

$$\Delta_{l_{31}}\Delta_{l_{21}}\Delta_{l_{11}}\varphi_1\big(u + \widetilde{\delta}_1 v\big) = 0 \quad \forall u, v \in W, \tag{2.7}$$

where $l_{31} = (\widetilde{\delta}_1 - \widetilde{\delta}_2)k_3$. Putting $v = 0$ in (2.7), we find that

$$\Delta_{l_{31}}\Delta_{l_{21}}\Delta_{l_{11}}\varphi_1(u) = 0 \quad \forall u \in W. \tag{2.8}$$

Note that the k_j are arbitrary elements of W and that $\delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. Taking into account (2.8) and the representations for l_{11}, l_{21} and l_{31}, it is not difficult to prove that there exists an open subgroup P of W such that on the subgroup $H = P \cap Y^{(2)}$, the function φ_1 satisfies the equation

$$\Delta_h^3\varphi_1(y) = 0 \quad \forall h,\, y \in H, \tag{2.9}$$

that is, the function φ_1 is a continuous polynomial on the subgroup H. Using similar reasoning, we can show that the function φ_2 satisfies (2.9) as well.

Since Y is a compact group, the subgroup $Y^{(2)}$ is compact too. Since the subgroup P is open, it is closed and hence compact. Therefore the subgroup H is compact. Taking into account Lemma 2.3 and the condition $\widehat{\mu}_j(0) = 1$, this implies that $\varphi_j = 0$ in H. Hence $\widehat{\nu}_j = 1$ in H, and so $\sigma(\nu_j) \subseteq A(X, H)$. Set $G = A(X, H)$. Note that $A(X, Y^{(2)}) = X_{(2)}$. Since $H = P \cap Y^{(2)}$, the subgroup G is the subgroup generated by the subgroups $X_{(2)}$ and $A(X, P)$. Note that $(Y/P)^* \approx A(X, P)$. Since the quotient Y/P is finite, the annihilator $A(X, P)$ is also finite. Thus the subgroup G is generated by the subgroup $X_{(2)}$ and some finite group. Hence the subgroup G is bounded. On the other hand, it is well known that if a distribution is concentrated on a Borel-measurable subgroup, then each of its divisors must be concentrated on a coset of this subgroup (see, for instance, [2, Proposition 2.5]). Thus, since $\sigma(\nu_j) \subseteq G$, the distribution μ_j is concentrated on a coset $x_j + G$, where $x_j \in X$. Since X is a torsion group, x_j is an element of finite order. Hence the subgroup generated by G and x_j is bounded, that is, there exists k such that all supports $\sigma(\mu_j)$ are subsets of $X_{(k)}$. Proposition 2.6 is therefore proved. □

REMARK 2.7. In light of Proposition 2.5, the study of Problems 1.3 and 1.4 for countable discrete abelian groups reduces to that for countable discrete abelian torsion groups. Note that a countable discrete torsion abelian group can have quite complicated structure (see, for instance, [7]). At the same time, Proposition 2.6 reduces the study of Problems 1.3 and 1.4 from the class of countable discrete torsion abelian groups to the class of bounded countable discrete abelian groups. The structure of a bounded countable discrete abelian group is very simple. In particular, by the Baer–Prüfer theorem, each such group is a weak direct product of cyclic groups (see [7, Section 17.2]).

Suppose that the conditions of Proposition 2.6 are valid. As is evident from the proof of Proposition 2.6, the distributions μ_j are concentrated on a subgroup generated by $X_{(2)}$ and a finite subgroup. We will check that this statement cannot be strengthened.

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PROPOSITION 2.8. *Let X be a countable discrete abelian group generated by $X_{(2)}$ and a finite subgroup. Suppose that δ, $I \pm \delta \in \mathrm{Aut}(X)$. Then there exist independent identically distributed X-valued random variables ξ_1 and ξ_2 with distribution μ such that the conditional distribution of L_2 given L_1 is symmetric, where $L_2 = \xi_1 + \delta\xi_2$ and $L_1 = \xi_1 + \xi_2$; furthermore, the support $\sigma(\mu)$ of μ is equal to X.*

PROOF. It is obvious that X is a bounded group. Therefore, by the Baer–Prüfer theorem (see [7, Section 17.2]), the group X can be decomposed into a weak direct product of cyclic groups. It follows that X can be represented in the form $B \times C$, where $B = B_{(2)}$ and C is a finite group.

Let ξ_1 and ξ_2 be independent identically distributed X-valued random variables whose distribution μ is equal to $\rho * m_C$, where ρ is a distribution on the subgroup $X_{(2)}$ such that $\sigma(\rho) = X_{(2)}$. It is clear that $\sigma(\mu) = X$. We verify that the conditional distribution of L_2 given L_1 is symmetric. By Lemma 2.4, it suffices to verify that the characteristic function of the distribution μ satisfies (2.1), which in this case takes the form

$$\widehat{\mu}(u+v)\widehat{\mu}(u+\varepsilon v) = \widehat{\mu}(u-v)\widehat{\mu}(u-\varepsilon v) \quad \forall u, v \in Y, \tag{2.10}$$

where $\varepsilon = \widetilde{\delta}$. Since $\widehat{\mu} = \widehat{\rho}\,\widehat{m}_C$, it suffices to show that both $\widehat{\rho}$ and $\widehat{m}_C$ satisfy (2.10).

We verify that the characteristic function $\widehat{\rho}$ satisfies (2.10). Note that $A(Y, X_{(2)}) = Y^{(2)}$. Since $\sigma(\rho) = X_{(2)}$, we have $\widehat{\rho}(y+h) = \widehat{\rho}(y)$ for all $y \in Y$ and $h \in Y^{(2)}$. Hence, $\widehat{\rho}(u+v) = \widehat{\rho}(u-v)$ and $\widehat{\rho}(u+\varepsilon v) = \widehat{\rho}(u-\varepsilon v)$ for all $u, v \in Y$. Thus, the function $\widehat{\rho}$ satisfies (2.10).

We verify now that the characteristic function $\widehat{m}_C$ also satisfies (2.10). From [5], it suffices to verify that $\gamma(C) = C$, where $\gamma = (I+\delta)^{-1}(I-\delta)$. This is equivalent to showing that $\widetilde{\gamma}(A(Y, C)) = A(Y, C)$. It is obvious that $A(Y, C) \approx B^*$. Since $B = B_{(2)}$, the equality $B^* = B^*_{(2)}$ holds. Hence, the automorphism $\widetilde{\gamma}$ acts on the subgroup $A(Y, C)$ as the identity. Therefore $\widetilde{\gamma}(A(Y, C)) = A(Y, C)$, and so $\gamma(C) = C$. □

PROOF OF THEOREM 2.1. It is obvious that we may assume without loss of generality that $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_1, \delta_2, \delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. It is also obvious that we may suppose that $\delta_1 = I$. Set $\delta_2 = \delta$. By Lemma 2.4, the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions $\widehat{\mu}_j$ satisfy (2.1), which takes the form

$$\widehat{\mu}_1(u+v)\widehat{\mu}_2(u+\varepsilon v) = \widehat{\mu}_1(u-v)\widehat{\mu}_2(u-\varepsilon v) \quad \forall u, v \in Y, \tag{2.11}$$

where $\varepsilon = \widetilde{\delta}$.

Taking into account Proposition 2.6, we can assume that X is a bounded group, that is, there exists $k \geq 2$ such that $X = X_{(k)}$.

Since X is a bounded group, all but finitely many p-components of X are trivial. Decompose the group X into a finite direct product of its p-components:

$$X = \mathop{\mathbf{P}}_{p \in \mathcal{P}} X_p,$$

where $\mathcal{P}$ is a finite set of prime numbers.

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Set $G = X_2$ and $K = \mathbf{P}_{p>2}\, X_p$, so that $X = G \times K$. If $G = \{0\}$, then the assertion of the theorem follows from Lemma 2.2. Assume that $G \neq \{0\}$. Then $Y = H \times L$, where $H \approx G^*$ and $L \approx K^*$. Write the element y of the group Y as (h, l), where $h \in H$ and $l \in L$. Since the subgroups G and K are characteristic, so are the subgroups H and L. Hence, any automorphism $\varepsilon \in \mathrm{Aut}(Y)$ can be written in the form $\varepsilon(h, l) = (\varepsilon_H h, \varepsilon_L l)$, where $(h, l) \in Y$.

Put $u = (h, l)$, $v = (h', l')$, $\widehat{\mu}_1 = f$ and $\widehat{\mu}_2 = g$, and rewrite (2.11) in the form

$$f(h + h', l + l')g(h + \varepsilon_H h', l + \varepsilon_L l') = f(h - h', l - l')g(h - \varepsilon_H h', l - \varepsilon_L l') \tag{2.12}$$

for all (h, l), $(h', l') \in Y$. Substituting $h = h' = 0$ into (2.12), we get

$$f(0, l + l')g(0, l + \varepsilon_L l') = f(0, l - l')g(0, l - \varepsilon_L l') \quad \forall l, l' \in L. \tag{2.13}$$

By Lemma 2.2, any solution of (2.13) has the form

$$f(0, l) = \langle k_1, l\rangle \widehat{m}_F(l), \qquad g(0, l) = \langle k_2, l\rangle \widehat{m}_F(l) \qquad \forall l \in L, \tag{2.14}$$

where F is a finite subgroup of the group K and $k_1, k_2 \in K$.

Substituting (2.14) into (2.13) gives $2(k_1 + \delta k_2) \in F$. Since $K_{(2)} = \{0\}$, we have $k_1 + \delta k_2 \in F$. Set $k = k_1 + \delta k_2$. It is clear that the representation (2.14) does not change if we substitute $k_1' = k_1 - k$ for k_1. But then $k_1' + \delta k_2 = 0$. It is easy to see that in this case the characteristic functions $\tilde{f}(0, l) = \langle -k_1', l\rangle$ and $\tilde{g}(0, l) = \langle -k_2, l\rangle$ satisfy (2.13). Replace the distributions μ_j by their translates $\mu_1' = \mu_1 * E_{-k_1'}$ and $\mu_2' = \mu_2 * E_{-k_2}$, and denote by f' and g' the characteristic functions of the distributions μ_j'. It is clear that

$$f'(0, l) = g'(0, l) = \begin{cases} 1 & \text{if } l \in B, \\ 0 & \text{if } l \notin B, \end{cases} \tag{2.15}$$

where $B = A(L, F)$. Hence, $\sigma(\mu_j') \subseteq G \times F$. Since the subgroup G is characteristic and $\delta(F) = F$ by Lemma 2.2, we have $\delta(G \times F) = G \times F$. Thus, we can assume that the group X is of the form $X = G \times K$, where G is a 2-prime group and K is a finite group with no elements of order two. Moreover, we have

$$f'(0, l) = g'(0, l) = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases} \tag{2.16}$$

Putting $h' = h$ and $l' = l$ into (2.12), we get

$$f(2(h, l))g((I + \varepsilon)(h, l)) = g((I - \varepsilon)(h, l)) \quad \forall (h, l) \in Y. \tag{2.17}$$

We will prove by induction on m, where 2^m is the order of the element h, that $f(h, l) = g(h, l) = 0$ for $l \neq 0$. Set $Y_m = H_{(2^m)} \times L$ for $m \geq 1$, and let $Y_0 = L$. Note that Y_m, with $m \geq 0$, is a characteristic subgroup.

It follows from (2.16) that $f'(h,l)=g'(h,l)=0$ for $(h,l)\in Y_0$ when $l\neq 0$. Assume that when $l\neq 0$, $f'(h,l)=g'(h,l)=0$ for $(h,l)\in Y_m$. Consider the restriction of (2.17) to Y_{m+1}. Then $2(h,l)\in Y_m$. Hence, $f'(2(h,l))=0$ if $l\neq 0$. It then follows from (2.17) that $g'((I-\varepsilon)(h,l))=0$ if $(h,l)\in Y_{m+1}$ and $l\neq 0$. Since $I-\varepsilon\in \mathrm{Aut}(Y)$, we deduce that $g'(h,l)=0$ if $(h,l)\in Y_{m+1}$ and $l\neq 0$. Arguing similarly, we see that $f'(h,l)=0$ if $(h,l)\in Y_{m+1}$ and $l\neq 0$. Since G is a bounded subgroup, there exists k such that $G^{(2^k)}=\{0\}$. Hence, $H_{(2^k)}=H$ and $Y_k=Y$. Thus

$$f'(h,l)=\begin{cases} f_0(h) & \text{if } l=0,\\ 0 & \text{if } l\neq 0,\end{cases}\qquad g'(h,l)=\begin{cases} g_0(h) & \text{if } l=0,\\ 0 & \text{if } l\neq 0,\end{cases}$$

where $f_0(h)=f'(h,0)$ and $g_0(h)=g'(h,0)$. Hence, $f'(h,l)=f_0(h)f_1(l)$ and $g'(h,l)=g_0(h)g_1(l)$, where $f_1(l)=f'(0,l)$ and $g_1(l)=g'(0,l)$. The functions f_0 and g_0 on Y are the characteristic functions of distributions ρ_1 and ρ_2 such that $\sigma(\rho_j)\subseteq G$. The functions f_1 and g_1 are the characteristic functions of the distribution m_K. Thus $\mu_j=\rho_j * m_K$ when $j=1,2$. Returning to the original distributions, we obtain the required result. □

REMARK 2.9. The proof of Theorem 2.1 implies the following statement. Assume that the conditions of Theorem 2.1 hold and that L_1 and L_2 are of the form $L_1=\xi_1+\xi_2$ and $L_2=\xi_1+\delta\xi_2$. Then $\mu_j=\rho_j * m_F * E_{x_j}$, where $\sigma(\rho_j)\subseteq X_2$, F is a finite subgroup containing no elements of order two, and $x_j\in X$.

REMARK 2.10. Applying Propositions 2.5 and 2.6 and reasoning as in [12, Proof of Theorem 3], we deduce the following assertion.

Suppose that Assumptions 1.2 hold; moreover, assume that $X=\mathbb{R}\times D$, where D is a countable discrete abelian group, and that $n=2$. If the conditional distribution of L_2 given L_1 is symmetric, then $\mu_j=\gamma_j*\rho_j*\pi_j$, where the γ_j are Gaussian distributions on $\mathbb{R}$, $\sigma(\rho_j)\subseteq D_2$, and $\pi_1,\pi_2\in I(X)$.

References

[1] G. M. Feldman, 'Marcinkiewicz and Lukacs theorems on abelian groups', *Theory Probab. Appl.* **34** (1989), 290–297.

[2] G. M. Feldman, *Arithmetic of Probability Distributions and Characterization Problems on Abelian Groups*, Translations of Mathematical Monographs, 116 (American Mathematical Society, Providence, RI, 1993).

[3] G. M. Feldman, 'On the Heyde theorem for finite abelian groups', *J. Theoret. Probab.* **17** (2004), 929–941.

[4] G. M. Feldman, 'On a characterization theorem for locally compact abelian groups', *Probab. Theory Related Fields* **133** (2005), 345–357.

[5] G. M. Feldman, 'On the Heyde theorem for discrete abelian groups', *Studia Math.* **177**(1) (2006), 67–79.

[6] G. M. Feldman, *Functional Equations and Characterization Problems on Locally Compact Abelian Groups*, EMS Tracts in Mathematics, 5 (European Mathematical Society, Zürich, 2008).

[7] L. Fuchs, *Infinite Abelian Groups*, Vol. 1 (Academic Press, New York, 1970).

[8] L. Fuchs, *Infinite Abelian Groups*, Vol. 2 (Academic Press, New York, 1973).

[9] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. 1 (Springer, Berlin, 1963).

[10] C. C. Heyde, 'Characterization of the normal law by the symmetry of a certain conditional distribution', *Sankhya Ser. A* **32** (1970), 115–118.

[11] A. M. Kagan, Yu. V. Linnik and C. R. Rao, *Characterization Problems in Mathematical Statistics*, Wiley Series in Probability and Mathematical Statistics (John Wiley & Sons, New York, 1973).

[12] M. V. Mironyuk and G. M. Fel'dman, 'On a characterization theorem on finite abelian groups', *Siberian Math. J.* **46**(2) (2005), 315–324.

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