

RESEARCH ARTICLE

# Hyperbolic tessellations and generators of $K_3$ for imaginary quadratic fields

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## Abstract

We develop methods for constructing explicit generators, modulo torsion, of the  $K_3$ -groups of imaginary quadratic number fields. These methods are based on either tessellations of hyperbolic 3-space or on direct calculations in suitable pre-Bloch groups and lead to the very first proven examples of explicit generators, modulo torsion, of any infinite  $K_3$ -group of a number field. As part of this approach, we make several improvements to the theory of Bloch groups for  $K_3$  of any field, predict the precise power of 2 that should occur in the Lichtenbaum conjecture at  $-1$  and prove that this prediction is valid for all abelian number fields.

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**1. Introduction**

**1.1. The general context**

Let  $F$  be a number field with ring of algebraic integers  $\mathcal{O}_F$ . Then, for each integer  $m \geq 2$ , the algebraic  $K$ -group  $K_m(\mathcal{O}_F)$  of Quillen is a fundamental invariant of  $F$ , constituting a natural generalisation of the ideal class group of  $\mathcal{O}_F$  if  $m$  is even and the group of units of  $\mathcal{O}_F$  if  $m$  is odd. By fundamental work of Quillen [49] and Borel [7], this abelian group is known to be finite if  $m$  is even and finitely generated if  $m$  is odd.

In addition, as a natural generalisation of the analytic class number formula, Lichtenbaum [42] has conjectured that the leading coefficient  $\zeta_F^*(1 - m)$  in the Taylor expansion at  $s = 1 - m$  of the Dedekind zeta function  $\zeta_F(s)$  of  $F$  should satisfy

$$\zeta_F^*(1 - m) = \pm 2^{n_{m,F}} \frac{|K_{2m-2}(\mathcal{O}_F)|}{|K_{2m-1}(\mathcal{O}_F)_{\text{tor}}|} R_m(F). \tag{1.1}$$

Here we write  $|X|$  for the cardinality of a finite set  $X$ ,  $K_{2m-1}(\mathcal{O}_F)_{\text{tor}}$  for the torsion subgroup of  $K_{2m-1}(\mathcal{O}_F)$ ,  $R_m(F)$  for the covolume of the image of  $K_{2m-1}(\mathcal{O}_F)$  under the Beilinson regulator map and  $n_{m,F}$  for an undetermined integer.

Borel [8] proved that the quotient of  $\zeta_F^*(1 - m)$  by  $R_m(F)$  is rational (see Theorem 2.1), but the identity (1.1) has been proved unconditionally only for  $F$  an abelian extension of  $\mathbb{Q}$  (cf. Remark 2.8). Moreover, it still is a difficult problem to explicitly compute, except in special cases, either  $|K_{2m-2}(\mathcal{O}_F)|$  or  $R_m(F)$ , or to give explicit generators of  $K_{2m-1}(\mathcal{O}_F)$  modulo torsion.

As our contribution to a solution, in this article we clarify the integer  $n_{m,F}$ , determine the precise relation between various Bloch groups without ignoring any torsion, develop techniques for checking divisibility in such groups and, in the Bloch group of each imaginary quadratic field  $F$ , algorithmically construct an element that gives rise to a subgroup of index  $|K_2(\mathcal{O}_F)|$  in the quotient group  $K_3(\mathcal{O}_F)/K_3(\mathcal{O}_F)_{\text{tor}}$ . (The number fields  $F$  of lowest degree with infinite  $K_3(\mathcal{O}_F)$  are precisely the imaginary quadratic ones.) These results are of independent interest, but combining them allows us to give  $|K_2(\mathcal{O}_F)|$ , the value of  $R_2(F)$  and explicit generators of  $K_3(\mathcal{O}_F)/K_3(\mathcal{O}_F)_{\text{tor}}$ , for all imaginary quadratic fields  $F$  of absolute discriminant at most 1000. These data are available online [62] and give  $|K_2(\mathcal{O}_F)|$  for several interesting new cases. We note that it contains the very first proven examples of explicit generators of a nontrivial  $K_3(\mathcal{O}_F)/K_3(\mathcal{O}_F)_{\text{tor}}$  for any number field  $F$ , solving a problem open ever since  $K_3$ -groups were introduced.

Our construction of the element in the Bloch group for an imaginary quadratic field  $F$  is based on an ideal tessellation of hyperbolic 3-space on which  $\mathrm{GL}_2(\mathcal{O}_F)$  acts. After original work in Dupont-Sah [21] and Neumann-Zagier [47], where the gluing condition for hyperbolic tetrahedra was formulated in an algebraic way, the first calculations of Bloch elements from hyperbolic tessellations were, to the best of our knowledge, implicit in preliminary versions of [64], as well as in [65] and [28]. (Those tessellations give rise to elements in a Bloch group; see, e.g., [43].)

Other explicit constructions, generally based on a triangulation of a hyperbolic manifold of finite volume and its homology, or on group homology, often lead to elements in a Bloch group for  $\mathbb{C}$  or  $\mathbb{Q}$  instead of a number field (see, e.g., [18]), sometimes even tensored with  $\mathbb{Q}$  (see, e.g., [30, Theorem 1.2], which also discusses higher dimensional analogues). Among the earlier results on elements of Bloch groups that are closest to our own are [46, Theorem 6.1], which gives an element associated to an oriented hyperbolic 3-manifold of finite volume, and [4, Corollary 6.2.1], which obtains such elements from group homology in an essentially geometric way. By contrast, we argue directly on the combinatorics of the tessellation under the action of  $\mathrm{GL}_2(\mathcal{O}_F)$ , resulting in a much simpler construction. In particular, our tetrahedra are not ‘decorated’, we do not need any hyperbolic 3-manifold, or group homology, and we construct our element directly for a given imaginary quadratic field  $F$  in such a way that it can be explicitly computed algorithmically.

We want to highlight a very interesting by-product of our methods and calculations. For a field  $F$ , Bloch, in the seminal work [5], constructed, modulo some torsion, a subgroup of  $K_3(F)^{\mathrm{ind}}$ , the ‘quotient of indecomposables’ of  $K_3(F)$ . This inspired the paper [55], in which Suslin defines, for infinite  $F$ , a ‘Bloch group’  $B(F)$  that describes  $K_3(F)^{\mathrm{ind}}$  modulo some specific torsion. Although it superficially looks very similar to the group from [5], the precise relation between them is mysterious because Bloch’s construction is based on relative  $K$ -theory, whereas Suslin’s is based on group homology. Still, they are expected to be very closely related.

Instead of the construction of [5] we use a variation based on an idea in a 1990 letter of Bloch to Deninger, as worked out in [14, 19], that gives a subgroup of  $K_3(F)^{\mathrm{ind}}$  modulo torsion. (We note in passing that the precise relation between this variation and the original construction in [5] is not known even though both use relative  $K$ -theory.) We map  $B(F)$  to the group from [14] in Theorem 3.25 but it is not a priori clear whether this is compatible with the relations of both groups with  $K_3(F)^{\mathrm{ind}}$ .

If  $F$  is an imaginary quadratic field, then from the tessellation we obtain an element in  $B(F)$  modulo some torsion, which under our map gives an element in  $K_3(F)^{\mathrm{ind}}$  modulo torsion. From the precise statement of (1.1) for  $F$  as obtained in Section 2 we can then verify for many such  $F$  that our map induces an isomorphism between  $B(F)$  modulo torsion and the subgroup from [14] and that the latter is the whole of  $K_3(F)^{\mathrm{ind}}$  modulo torsion. This provides the first concrete evidence that such statements might hold for all fields. For more details we refer to Section 3.5.

## 1.2. The main results

We now discuss the main contents of this article in some more detail.

In Section 2 we address the issue of the undetermined exponent  $n_{m,F}$  in (1.1) by proving that the Tamagawa number conjecture that was formulated by Bloch and Kato in [6] and later extended by Fontaine and Perrin-Riou in [27] predicts a precise, and more or less explicit, formula for it. For  $m = 2$  we can make this conjectural formula completely explicit by using a result of Levine [41]. Using results of Huber and Kings [32], of Greither and the first author [17] and of Flach [25], relating to the Tamagawa number conjecture, we can then prove the (unconditional) validity of (1.1) for all  $m$  if  $F$  is abelian over  $\mathbb{Q}$ , with a precise expression for  $n_{m,F}$ .

This result is essential for our subsequent computations but is also of independent interest. However, the arguments in Section 2 are technical in nature and because these methods are not used elsewhere in the article we invite any reader whose main interest is the determination of explicit generators of  $K_3$ -groups to read this section up to the end of Subsection 2.1 and then pass on to Section 3.

In Section 3 we shall introduce, for any field  $F$ , a pre-Bloch group  $\overline{\mathfrak{p}}(F)$  based on (possibly degenerate) configurations of points. Our approach differs slightly but crucially from that in [29, §3], but this ostensibly minor improvement is essential to finding explicit generators of  $K_3$ -groups because we do not ignore any torsion. We also analyse the corresponding variant of the second exterior power of  $F^\times$  in detail, bearing applications in computer calculations in mind.

If  $F$  has at least four elements, we shall relate  $\overline{\mathfrak{p}}(F)$  to the pre-Bloch group  $\mathfrak{p}(F)$  of Suslin [55] (cf. [61, VI.5]) in a precise way and use this to determine the torsion subgroup of the resulting modified Bloch group  $\overline{B}(F)$  for a number field  $F$ . For an imaginary quadratic field  $F$  it turns out that  $\overline{B}(F)$  is torsion free, making it much more suitable for computer calculations than  $B(F)$ .

The section actually starts with a review of some earlier results, including one from [14] that enables us for a field  $F$  to construct a homomorphism  $\psi_F$ , natural up to a universal choice of sign, from  $\overline{B}(F)$  to  $K_3(F)^{\text{ind}}$  modulo torsion. This is the map mentioned at the end of Subsection 1.1, for which it is unclear how it fits in with the relation between  $B(F)$  and  $K_3(F)^{\text{ind}}$  of [55], but this way our computations are compatible with those in [14] and shed light on the relation between the construction of Bloch [5] and of Suslin [55].

In Sections 4 to 6 we specialise to consider the case of an imaginary quadratic field  $k$ .

In this case we shall, in Section 4, use the theory of perfect forms to obtain a tessellation of hyperbolic 3-space  $\mathbb{H}^3$  on which  $\text{PGL}_2(\mathcal{O}_k)$  acts and from this construct an explicit well-defined element  $\beta_{\text{geo}}$  of the group  $\overline{B}(k)$ . Humbert's classical formula for  $\zeta_k(2)$  in terms of the volume of a fundamental domain for the action of  $\text{PGL}_2(\mathcal{O}_k)$  on  $\mathbb{H}^3$  allows us to relate the image under the Beilinson regulator map of  $\psi_k(\beta_{\text{geo}})$  to  $\zeta_k^*(-1)$ . From the validity of a precise form of (1.1) for  $F = k$  and  $m = 2$  it then follows that  $\psi_k(\beta_{\text{geo}})$  generates a subgroup of index  $|K_2(\mathcal{O}_k)|$  of the maximal torsion free quotient  $K_3(k)_{\text{tf}}^{\text{ind}}$  of  $K_3(k)^{\text{ind}}$ .

The proof that  $\beta_{\text{geo}}$  is in  $\overline{B}(k)$  is lengthy and detailed, because it relies on a precise study of the combinatorics of the tessellation constructed in Section 4 and, for this reason, it is deferred to Section 5.

Then in Section 6 we use results from previous sections to describe two concrete approaches to finding an explicit generator of  $K_3(k)_{\text{tf}}^{\text{ind}}$  and the order of  $K_2(\mathcal{O}_k)$  for an imaginary quadratic field  $k$ .

The first approach is discussed in Subsection 6.1. It depends on dividing  $\beta_{\text{geo}}$  by  $|K_2(\mathcal{O}_k)|$  directly in  $\overline{B}(k)$  by generating elements in it using a method involving exceptional  $S$ -units (described in Subsection 6.3) and the defining relations in  $\overline{B}(k)$ . These computations do not use the validity of (1.1), but they rely on, and complement, earlier work of Belabas and the third author in [2] on the orders of such  $K_2(\mathcal{O}_k)$ . In particular, they show that the (divisional) bounds on  $|K_2(\mathcal{O}_k)|$  obtained in loc. cit. (in those cases where the order could not be precisely established) are sharp.

The second approach, discussed in Subsection 6.2, does rely on the known validity of (1.1) for  $F = k$  and  $m = 2$  in an essential way. Combining it with some (in practice sharp) bounds on  $|K_2(\mathcal{O}_k)|$  provided by [2], we can draw algebraic conclusions from numerical calculations on elements of  $\overline{B}(k)$  obtained using exceptional  $S$ -units, which leads to the computation of a generator of  $K_3(k)_{\text{tf}}^{\text{ind}}$  (or of  $\overline{B}(k)$ ) as well as of  $|K_2(\mathcal{O}_k)|$  in many interesting cases. As a concrete example, we show that  $|K_2(\mathcal{O}_k)| = 233$  for  $k = \mathbb{Q}(\sqrt{-4547})$ , thereby verifying a conjecture from [12].

Some of the techniques of Section 6 can be applied to an arbitrary number field  $F$ , for which essentially nothing of a general nature beyond the result of Borel is known. Doing this can be used to test the validity of Lichtenbaum's conjectural formula (1.1) for  $m = 2$ , but for the sake of brevity we shall not pursue these aspects in the present article.

The article then concludes with two appendices. In Appendix A we shall prove several useful results about finite subgroups of  $\text{PGL}_2(\mathcal{O}_k)$  of an imaginary quadratic field  $k$  that are needed in earlier arguments but for which we could not find a suitable reference. Finally, in Appendix B we shall give details of the results of applying the geometrical construction of Section 4 and the approach described in Subsection 6.1 to an imaginary quadratic field  $k$  for which  $|K_2(\mathcal{O}_k)|$  is equal to 22.

1.3. Notations and conventions

As a general convention we let  $F$  denote an arbitrary field (assumed in places to be infinite) or a number field and  $k$  an imaginary quadratic field.

For a number field  $F$  we write  $\mathcal{O}_F$  for its ring of integers,  $D_F$  for its discriminant and  $r_1(F)$  and  $r_2(F)$  for the number of its real and complex places, respectively.

For an imaginary quadratic field  $k$  we set

$$\omega = \omega_k := \begin{cases} \sqrt{D_k/4} & \text{if } D_k \equiv 0 \pmod{4}, \\ (1 + \sqrt{D_k})/2 & \text{if } D_k \equiv 1 \pmod{4} \end{cases}$$

so that  $k = \mathbb{Q}(\omega)$  and  $\mathcal{O}_k = \mathbb{Z}[\omega]$ .

For an abelian group  $M$  we write  $M_{\text{tor}}$  for its torsion subgroup and  $M_{\text{tf}}$  for the quotient group  $M/M_{\text{tor}}$ . The cardinality of a finite set  $X$  will be denoted by  $|X|$ .

2. The conjectures of Lichtenbaum and of Bloch and Kato

It has long been known that the validity of (1.1) follows from that of the conjecture originally formulated by Bloch and Kato in [6] and then reformulated and extended by Fontaine in [26] and by Fontaine and Perrin-Riou in [27] (see Remark 2.5(iii)). However, for the main purpose of this article, it is essential to know not just the validity of (1.1) but also an explicit value of the exponent  $n_{m,F}$ . In this section we shall therefore derive an essentially precise formula for  $n_{m,F}$  from the assumed validity of the above conjecture of Bloch and Kato.

For each subring  $\Lambda$  of  $\mathbb{R}$  and each integer  $a$ , we write  $\Lambda(a)$  for the subset  $(2\pi i)^a \cdot \Lambda$  of  $\mathbb{C}$ .

For a  $\mathbb{Z}_p$ -module  $M$  we identify  $M_{\text{tf}}$  with its image in  $\mathbb{Q}_p \cdot M := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$ , and for a homomorphism of  $\mathbb{Z}_p$ -modules  $\theta : M \rightarrow N$  we write  $\theta_{\text{tf}}$  for the induced homomorphism  $M_{\text{tf}} \rightarrow N_{\text{tf}}$ . We write  $D(\mathbb{Z}_2)$  for the derived category of  $\mathbb{Z}_2$ -modules and  $D^{\text{perf}}(\mathbb{Z}_2)$  for the full triangulated subcategory of  $D(\mathbb{Z}_2)$  comprising complexes that are isomorphic (in  $D(\mathbb{Z}_2)$ ) to a bounded complex of finitely generated  $\mathbb{Z}_2$ -modules. (Note that, because the ring  $\mathbb{Z}_2$  is regular, such complexes are precisely those that are quasi-isomorphic to a perfect complex.)

2.1. Statement of the main result

Throughout this section,  $F$  denotes a number field.

2.1.1.

We first review Borel’s theorem. For this, we fix an integer  $m \geq 2$  and recall that Beilinson’s regulator map

$$\text{reg}_m : K_{2m-1}(\mathbb{C}) \rightarrow \mathbb{R}(m-1)$$

is compatible with the natural actions of complex conjugation on  $K_{2m-1}(\mathbb{C})$  and  $\mathbb{R}(m-1)$ . For each embedding  $\sigma : F \rightarrow \mathbb{C}$ , we consider the composite homomorphism

$$\text{reg}_{m,\sigma} : K_{2m-1}(\mathcal{O}_F) \xrightarrow{\sigma_*} K_{2m-1}(\mathbb{C}) \xrightarrow{\text{reg}_m} \mathbb{R}(m-1),$$

where  $\sigma_*$  denotes the induced map on  $K$ -groups. We let  $\zeta_F^*(1-m)$  be the first nonzero coefficient in the Taylor expansion at  $s = 1 - m$  of the Dedekind zeta function  $\zeta_F(s)$  of  $F$  and set  $d_m(F)$  to be  $r_2(F)$  for even  $m$  and  $r_1(F) + r_2(F)$  for odd  $m$ .

**Theorem 2.1** (Borel’s theorem). *For each integer  $m \geq 2$  the following hold:*

- (i) *The rank of  $K_{2m-2}(\mathcal{O}_F)$  is zero.*
- (ii) *The rank of  $K_{2m-1}(\mathcal{O}_F)$  is  $d_m(F)$ .*

- (iii) Write  $\text{reg}_{m,F}$  for the map  $K_{2m-1}(\mathcal{O}_F) \rightarrow \prod_{\sigma: F \rightarrow \mathbb{C}} \mathbb{R}(m-1)$  given by  $(\text{reg}_{m,\sigma})_{\sigma}$ . Then the image of  $\text{reg}_{m,F}$  is a lattice in the real vector space  $V_{m-1} = \{(c_{\sigma})_{\sigma} | c_{\bar{\sigma}} = \overline{c_{\sigma}}\}$ , and its kernel is  $K_{2m-1}(\mathcal{O}_F)_{\text{tor}}$ .
- (iv) Let  $R_m(F)$  be the covolume of the image of  $\text{reg}_{m,F}$ , with covolumes normalised so that the lattice  $W_{m-1}^+ = \{(c_{\sigma})_{\sigma} | c_{\bar{\sigma}} = \overline{c_{\sigma}} \text{ and } c_{\sigma} \in \mathbb{Z}(m-1)\}$  has covolume 1. Then  $\zeta_F(s)$  vanishes to order  $d_m(F)$  at  $s = 1 - m$ , and  $\zeta_F^*(1 - m) = q_{m,F} \cdot R_m(F)$  for some  $q_{m,F}$  in  $\mathbb{Q}^{\times}$ .

**2.1.2.**

For each pair of integers  $i$  and  $j$  with  $j \in \{1, 2\}$  and  $i \geq j$  and each prime  $p$  there exist natural ‘Chern class’ homomorphisms of  $\mathbb{Z}_p$ -modules

$$K_{2i-j}(\mathcal{O}_F) \otimes \mathbb{Z}_p \rightarrow H^j(\mathcal{O}_F[1/p], \mathbb{Z}_p(i)). \tag{2.2}$$

The first such homomorphism  $c_{F,i,j,p}^S$  was constructed using higher Chern class maps by Soulé [52] (with more details for  $p = 2$  being provided by Weibel [58]), and a second  $c_{F,i,j,p}^{\text{DF}}$  was constructed using étale  $K$ -theory by Dwyer-Friedlander [23]. By [53, Prop. 3], they coincide if  $p \neq 2$ . For  $p = 2$  there is a third, introduced independently by Kahn [34] and by Rognes and Weibel [50], which will play a key role for us. All of these maps are natural in  $\mathcal{O}_F$  and have finite kernels and cokernels (see Lemma 2.19 and Theorem 2.21 for  $c_{F,i,1,2}^S$ ) and hence induce isomorphisms of the associated  $\mathbb{Q}_p$ -vector spaces.

Because we are mostly interested in  $p = 2$ , we usually write  $c_{F,i,j}^S$  and  $c_{F,i,j}^{\text{DF}}$  for  $c_{F,i,j,2}^S$  and  $c_{F,i,j,2}^{\text{DF}}$ . The main result of this section, on the number  $q_{m,F}$  in Theorem 2.1(iv), is then the following. Here  $\det_{\mathbb{Q}_2}(\alpha)$  is the determinant of an automorphism  $\alpha$  of a finite-dimensional  $\mathbb{Q}_2$ -vector space.

**Theorem 2.3.** Fix an integer  $m \geq 2$ . Then the Bloch-Kato conjecture is valid for the motive  $h^0(\text{Spec}(F))(1 - m)$  if and only if one has

$$q_{m,F} = (-1)^{s_m(F)} 2^{r_2(F) + t_m(F)} \cdot \frac{|K_{2m-2}(\mathcal{O}_F)|}{|K_{2m-1}(\mathcal{O}_F)_{\text{tor}}|}. \tag{2.4}$$

Here we set

$$s_m(F) := \begin{cases} [F : \mathbb{Q}] \frac{m}{2} - r_2(F), & \text{if } m \text{ is even,} \\ [F : \mathbb{Q}] \frac{m-1}{2}, & \text{if } m \text{ is odd,} \end{cases}$$

and  $t_m(F) := r_1(F) \cdot t_m^1(F) + t_m^2(F)$  with

$$t_m^1(F) := \begin{cases} -1, & \text{if } m \equiv 1 \pmod{4} \\ -2, & \text{if } m \equiv 3 \pmod{4} \\ 1, & \text{otherwise} \end{cases}$$

and  $t_m^2(F)$  the integer that satisfies

$$2^{t_m^2(F)} := |\text{cok}(c_{F,m,1,\text{tf}}^S)| \cdot 2^{-a_m(F)} \equiv \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot c_{F,m,1}^S) \circ (\mathbb{Q}_2 \cdot c_{F,m,1}^{\text{DF}})^{-1}) \pmod{\mathbb{Z}_2^{\times}},$$

where  $a_m(F)$  is 0 except possibly when both  $m \equiv 3 \pmod{4}$  and  $r_1(F) > 0$ , in which case it is an integer satisfying  $0 \leq a_m(F) < r_1(F)$ .

**Remark 2.5.** (i) The main result of Burgos Gil’s book [13] implies that the  $m$ th Borel regulator of  $F$  is equal to  $2^{d_m(F)} \cdot R_m(F)$ . So (2.4) leads directly to a more precise form of the conjectural formula for  $\zeta_F^*(1 - m)$  in terms of Borel’s regulator that is given by Lichtenbaum in [42].

(ii) The proof of Lemma 2.19(ii) gives a closed formula for the integer  $a_m(F)$  in Theorem 2.3 (also see Remark 2.20 in this regard). In addition, in [41, Th. 4.5], Levine shows that  $c_{F,2,1}^S$ , and hence also

$c_{F,2,1,\text{tr}}^S$  is surjective. It follows that  $t_2^2(F) = 0$ , so that the exponent of 2 in (2.4) is completely explicit in the case  $m = 2$ . However, in order to make the kind of numerical computations we perform in Section 6.2 for  $m > 2$ , it is important to have an explicit upper bound on  $t_m^2(F)$ , and such a bound follows directly from Theorem 2.21.

(iii) Up to sign and an unknown power of 2, Theorem 2.3 is proved by Huber and Kings in [32, Th. 1.4.1] under the assumption that  $c_{F,m,j,p}^S$  is bijective for all odd primes  $p$ , as conjectured by Quillen and Lichtenbaum. Suslin has shown that the latter conjecture is implied by the Bloch-Kato conjecture relating Milnor  $K$ -theory to étale cohomology. Following fundamental work of Voevodsky and Rost, Weibel completed the proof of this last conjecture [60]. So our contribution to Theorem 2.3 consists of specifying the sign (which is easy) and the exponent of 2.

If  $F$  is an abelian field, then the Bloch-Kato conjecture for  $h^0(\text{Spec}(F))(1 - m)$  is known to be valid. Up to the 2-primary part, this was verified independently by Huber and Kings [32] and by Greither and the first author [17], and the 2-primary component was subsequently resolved by Flach [25]. Theorem 2.3 thus leads directly to the following result.

**Corollary 2.6.** *The formula (2.4) is unconditionally valid if  $F$  is an abelian field.*

**Example 2.7.** For an imaginary quadratic field  $k$ , Corollary 2.6 combines with Theorem 2.1(iv), Remark 2.5(ii) and Example 3.1 to unconditionally prove the equality  $\zeta_k^*(-1) = -12^{-1} |K_2(\mathcal{O}_k)| R_2(k)$ .

**Remark 2.8.** Independently of connections to the Bloch-Kato conjecture, the validity of (1.1), but not (2.4), for abelian fields  $F$  was first established by Kolster, Nguyen Quang Do and Fleckinger in [38] (the main result of loc. cit. contains certain erroneous Euler factors, but the necessary correction is provided by Benois and Nguyen Quang Do in [3, §A.3]). The general approach of [38] also provided motivation for the subsequent work of Huber and Kings in [32].

### 2.2. The proof of Theorem 2.3: a first reduction

In the sequel we abbreviate  $r_1(F)$ ,  $r_2(F)$  and  $d_m(F)$  to  $r_1$ ,  $r_2$  and  $d_m$ , respectively. The functional equation of  $\zeta_F(s)$  then has the form

$$\zeta_F(1 - s) = \frac{2^{r_2} \cdot \pi^{[F:\mathbb{Q}]/2}}{|D_F|^{1/2}} \left( \frac{|D_F|}{\pi^{[F:\mathbb{Q}]}} \right)^s \left( \frac{\Gamma(s)}{\Gamma(1 - s)} \right)^{r_2} \left( \frac{\Gamma(s/2)}{\Gamma((1 - s)/2)} \right)^{r_1} \cdot \zeta_F(s) \tag{2.9}$$

with  $\Gamma(s)$  the Gamma function. Because  $\zeta_F(s)$  converges at  $s = m$  we have  $\zeta_F(m) > 0$ . In addition, the function  $\Gamma(s)$  is analytic and strictly positive for  $s > 0$ , is analytic, nonzero and of sign  $(-1)^{\frac{1}{2}-n}$  at each strictly negative half-integer  $n$  and has a simple pole at each strictly negative integer  $n$  with residue of sign  $(-1)^n$ . So from (2.9) we find that  $\zeta_F(s)$  vanishes to order  $d_m$  at  $s = 1 - m$  (as stated in Theorem 2.1(iv)), with leading term of sign equal to  $(-1)^{[F:\mathbb{Q}]\frac{m}{2}-r_2}$  if  $m$  is even and to  $(-1)^{[F:\mathbb{Q}]\frac{m-1}{2}}$  if  $m$  is odd, as per the explicit formula (2.4).

Therefore, in view of Remark 2.5(iii), in order to prove Theorem 2.3 it now suffices to show that the 2-adic component of the Bloch-Kato conjecture for  $h^0(\text{Spec}(F))(1 - m)$  is valid if and only if the 2-adic valuation of the rational number  $\zeta_F^*(1 - m)/R_m(F)$  is as implied by (2.4). After several preliminary steps, this will be proved in Subsection 2.4.

### 2.3. The role of Chern class maps

Regarding  $F$  as fixed we set

$$K_{m,j} := K_{2m-j}(\mathcal{O}_F) \otimes \mathbb{Z}_2 \quad \text{and} \quad H_m^j := H^j(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$$

for each strictly positive integer  $m$  and each  $j \in \{1, 2\}$ . We also set

$$Y_m := H^0(G_{\mathbb{C}/\mathbb{R}}, \prod_{F \rightarrow \mathbb{C}} \mathbb{Z}_2(m-1)),$$

where  $G_{\mathbb{C}/\mathbb{R}}$  acts diagonally on the product via its natural action on  $\mathbb{Z}_2(m-1)$  and via post-composition on the set of embeddings  $F \rightarrow \mathbb{C}$ .

We recall that in [34] Kahn uses the Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky spectral sequence to construct, for each pair of integers  $i$  and  $j$  with  $j \in \{1, 2\}$  and  $i \geq j$ , a homomorphism of  $\mathbb{Z}_2$ -modules  $c_{i,j}^K = c_{F,i,j}^K$ , of the form (2.2) for  $p = 2$ .

We write  $R\Gamma_c(\mathcal{O}_F[1/2], \mathbb{Z}_2(1-m))$  for the compactly supported étale cohomology of  $\mathbb{Z}_2(1-m)$  on  $\text{Spec}(\mathcal{O}_F[1/2])$  (as defined, for example, in [15, (3)]). We recall that this complex belongs to the category  $D^{\text{perf}}(\mathbb{Z}_2)$ , and hence the same holds for its (shifted) linear dual

$$C_m^\bullet := R\text{Hom}_{\mathbb{Z}_2}(R\Gamma_c(\mathcal{O}_F[1/2], \mathbb{Z}_2(1-m)), \mathbb{Z}_2[-2]).$$

In the sequel we shall write  $D(-)$  for the Grothendieck-Knudsen-Mumford determinant functor on  $D^{\text{perf}}(\mathbb{Z}_2)$ , as constructed in [36]. (Note, however, that, because  $\mathbb{Z}_2$  is local, one does not lose any significant information by (suppressing gradings and) regarding the values of  $D(-)$  as free rank one  $\mathbb{Z}_2$ -modules and so this is what we do.) We shall also write  $\bigwedge_{\mathbb{Z}_2}^a M$  for the (standard)  $a$ th exterior power of a  $\mathbb{Z}_2$ -module  $M$  if  $a$  is a nonnegative integer.

**Proposition 2.10.** *The map  $c_{m,1}^K$  combines with the Artin-Verdier duality theorem [45, Chap. II, Th. 3.1] to induce a natural identification of  $\mathbb{Z}_2$ -lattices*

$$D(C_m^\bullet) = (2^{r_1})^{t_m^1(F)} \frac{|K_{m,2}|}{|(K_{m,1})_{\text{tor}}|} \cdot \left( \bigwedge_{\mathbb{Z}_2}^{d_m} K_{m,1,\text{tf}} \right) \otimes_{\mathbb{Z}_2} \text{Hom}_{\mathbb{Z}_2} \left( \bigwedge_{\mathbb{Z}_2}^{d_m} Y_m, \mathbb{Z}_2 \right), \tag{2.11}$$

where the integer  $t_m^1(F)$  is as defined in Theorem 2.3.

*Proof.* We abbreviate  $c_{m,j}^K$  to  $c_j$  and set  $b_m := r_1$  if  $m$  is odd and  $b_m := 0$  if  $m$  is even. Then a straightforward computation of determinants shows that

$$\begin{aligned} D(C_m^\bullet) &= D(H^0(C_m^\bullet)[0]) \otimes D(H^1(C_m^\bullet)[-1]) \\ &= D(H_m^1[0]) \otimes D(H_m^2[-1]) \otimes 2^{-b_m} D(Y_m[-1]) \\ &= (D(\ker(c_1)[0])^{-1} \otimes D(K_{m,1}[0]) \otimes D(\text{cok}(c_1)[0])) \otimes (D(\ker(c_2)[-1])^{-1} \\ &\quad \otimes D(K_{m,2}[-1]) \otimes D(\text{cok}(c_2)[-1])) \otimes 2^{-b_m} D(Y_m[-1]) \\ &= 2^{-b_m} \frac{|\ker(c_1)| \cdot |\text{cok}(c_2)|}{|\ker(c_2)| \cdot |\text{cok}(c_1)|} \frac{|K_{m,2}|}{|(K_{m,1})_{\text{tor}}|} \left( \bigwedge_{\mathbb{Z}_2}^{d_m} K_{m,1,\text{tf}} \right) \otimes_{\mathbb{Z}_2} \text{Hom}_{\mathbb{Z}_2} \left( \bigwedge_{\mathbb{Z}_2}^{d_m} Y_m, \mathbb{Z}_2 \right). \end{aligned}$$

Here the first equality holds because  $C_m^\bullet$  is acyclic outside degrees zero and one, the second follows from the descriptions of Lemma 2.14 below, the third is induced by the tautological exact sequences  $0 \rightarrow \ker(c_j) \rightarrow K_{m,j} \xrightarrow{c_j} H_m^j \rightarrow \text{cok}(c_j) \rightarrow 0$  for  $j = 1, 2$  and the last follows from the fact that for any finitely generated  $\mathbb{Z}_2$ -module  $M$  and integer  $a$  the  $\mathbb{Z}_2$ -lattice  $D(M[a])$  equals  $(|M_{\text{tor}}|)^{-1} \cdot \bigwedge_{\mathbb{Z}_2}^b M_{\text{tf}}$  with  $b = \dim_{\mathbb{Q}_2}(\mathbb{Q}_2 \cdot M)$  for  $a$  even and  $|M_{\text{tor}}| \cdot \text{Hom}_{\mathbb{Z}_2}(\bigwedge_{\mathbb{Z}_2}^b M_{\text{tf}}, \mathbb{Z}_2)$  for  $a$  odd. So it suffices to show that the product of  $2^{-b_m}$  and  $|\ker(c_1)| \cdot |\text{cok}(c_2)| / (|\ker(c_2)| \cdot |\text{cok}(c_1)|)$  is  $(2^{r_1})^{t_m^1(F)}$ .

This is an easy computation using that by [34, Th. 1] there are integers  $r_{1,4}$  and  $r_{1,5}$  such that

$$\begin{cases} |\ker(c_j)| = |\text{cok}(c_j)| = 1, & \text{if } 2m - j \equiv 0, 1, 2, 7 \pmod{8} \\ |\ker(c_j)| = 2^{r_1}, |\text{cok}(c_j)| = 1, & \text{if } 2m - j \equiv 3 \pmod{8} \\ |\ker(c_j)| = 2^{r_{1,4}}, |\text{cok}(c_j)| = 1, & \text{if } 2m - j \equiv 4 \pmod{8} \\ |\ker(c_j)| = 1, |\text{cok}(c_j)| = 2^{r_{1,5}}, & \text{if } 2m - j \equiv 5 \pmod{8} \\ |\ker(c_j)| = 1, |\text{cok}(c_j)| = 2^{r_1}, & \text{if } 2m - j \equiv 6 \pmod{8} \end{cases} \tag{2.12}$$

and  $r_{1,4} = r_{1,5} = 0$  if  $r_1 = 0$ , whereas for  $r_1 > 0$  one has  $r_{1,4} \geq 0, r_{1,5} > 0$  and  $r_{1,4} + r_{1,5} = r_1$ . □

**Remark 2.13.** In [50] Rognes and Weibel use a slightly different approach to Kahn to construct maps of the form  $c_{i,j}^K$ . Their results can be used to give an alternative proof of Proposition 2.10.

In the sequel we write  $\Sigma_\infty, \Sigma_{\mathbb{R}}$  and  $\Sigma_{\mathbb{C}}$  for the sets of Archimedean, real Archimedean and complex Archimedean places of  $F$ , respectively.

**Lemma 2.14.** *The Artin-Verdier duality theorem induces the following identifications.*

- (i)  $H^0(C_m^\bullet) = H^1(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$ .
- (ii)  $H^1(C_m^\bullet)_{\text{tor}} = H^2(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$ .
- (iii)  $H^1(C_m^\bullet)_{\text{lf}}$  is the submodule

$$\bigoplus_{w \in \Sigma_{\mathbb{R}}} 2 \cdot H^0(G_{\mathbb{C}/\mathbb{R}}, \mathbb{Z}_2(m-1)) \oplus \bigoplus_{w \in \Sigma_{\mathbb{C}}} H^0(G_{\mathbb{C}/\mathbb{R}}, \mathbb{Z}_2(m-1) \cdot \sigma_w \oplus \mathbb{Z}_2(m-1) \cdot \overline{\sigma_w})$$

of

$$Y_m = \bigoplus_{w \in \Sigma_{\mathbb{R}}} H^0(G_{\mathbb{C}/\mathbb{R}}, \mathbb{Z}_2(m-1)) \oplus \bigoplus_{w \in \Sigma_{\mathbb{C}}} H^0(G_{\mathbb{C}/\mathbb{R}}, \mathbb{Z}_2(m-1) \cdot \sigma_w \oplus \mathbb{Z}_2(m-1) \cdot \overline{\sigma_w}),$$

where for each place  $w \in \Sigma_{\mathbb{C}}$  we choose a corresponding embedding  $\sigma_w : F \rightarrow \mathbb{C}$  and write  $\overline{\sigma_w}$  for its complex conjugate.

*Proof.* For each  $w$  in  $\Sigma_\infty$  we write  $R\Gamma_{\text{Tate}}(F_w, \mathbb{Z}_2(1-m))$  for the standard complex that computes Tate cohomology for  $F_w$  with coefficients  $\mathbb{Z}_2(1-m)$  and  $R\Gamma_\Delta(F_w, \mathbb{Z}_2(1-m))$  for the mapping fibre of the natural morphism

$$R\Gamma(F_w, \mathbb{Z}_2(1-m)) \rightarrow R\Gamma_{\text{Tate}}(F_w, \mathbb{Z}_2(1-m)).$$

We recall (see, e.g., [15, Prop. 4.1]) that Artin-Verdier duality gives an exact triangle in  $D^{\text{perf}}(\mathbb{Z}_2)$  of the form

$$\begin{aligned} C_m^\bullet &\rightarrow \bigoplus_{w \in \Sigma_\infty} R\text{Hom}_{\mathbb{Z}_2}(R\Gamma_\Delta(F_w, \mathbb{Z}_2(1-m)), \mathbb{Z}_2[-1]) \\ &\rightarrow R\Gamma(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))[2] \rightarrow C_m^\bullet[1]. \end{aligned} \tag{2.15}$$

Explicit computation shows that  $R\text{Hom}_{\mathbb{Z}_2}(R\Gamma_\Delta(F_w, \mathbb{Z}_2(1-m)), \mathbb{Z}_2[-1])$  is represented by the complex

$$\begin{cases} \mathbb{Z}_2(m-1)[-1], & \text{if } w \in \Sigma_{\mathbb{C}} \\ (\mathbb{Z}_2(m-1) \xrightarrow{\delta_m^0} \mathbb{Z}_2(m-1) \xrightarrow{\delta_m^1} \mathbb{Z}_2(m-1) \xrightarrow{\delta_m^0} \dots)[-1], & \text{if } w \in \Sigma_{\mathbb{R}}, \end{cases}$$

with  $\delta_m^i$  equal to multiplication by  $1 + (-1)^{i+m}$  for  $i = 0, 1$ . From this and the fact that  $H^2(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$  is finite, the long exact cohomology sequence of (2.15) directly implies claims (i)



Making explicit the formulation of [16, Conj. 1] (which originates with Fontaine and Perrin-Riou [27, Prop. III.3.2.5]) and the construction of [16, Lem. 18], one finds that the Bloch-Kato conjecture for  $h^0(\text{Spec}(F))(1 - m)$  uses the map  $\beta_m$  rather than  $\text{reg}_{m,F}$ . In addition, if one fixes a topological generator  $\eta$  of  $\mathbb{Z}_2(m - 1)$ , then mapping  $\eta$  to the element of  $\text{Hom}_{\mathbb{Q}}((2\pi i)^{1-m}\mathbb{Q}, \mathbb{Q})$  that sends  $(2\pi i)^{1-m}$  to 1 identifies  $Y_m$  in Lemma 2.14 with  $\mathbb{Z}_2 \otimes \text{Hom}_{\mathbb{Z}}(W_{1-m}^+, \mathbb{Z})$ .

Given these observations, the discussion of [32, §1.2] shows that the Bloch-Kato conjecture asserts the following: if one fixes an identification of  $\mathbb{C}$  with  $\mathbb{C}_2$ , then  $\zeta_F^*(1 - m)$  is a generator over  $\mathbb{Z}_2$  of the image of  $D(C_m^\bullet)$  under the composite isomorphism

$$\mathbb{C}_2 \cdot D(C_m^\bullet) \cong \mathbb{C}_2 \cdot ((\bigwedge_{\mathbb{Z}_2}^{d_m} K_{m,1}) \otimes_{\mathbb{Z}_2} \text{Hom}_{\mathbb{Z}_2}(\bigwedge_{\mathbb{Z}_2}^{d_m} Y_m, \mathbb{Z}_2)) \cong \mathbb{C}_2.$$

Here the first map is constructed as (2.11) was but using  $c_{F,m,1}^S$  instead of  $c_{F,m,1}^K$  and coefficients  $\mathbb{C}_2$  as opposed to  $\mathbb{Z}_2$ , and the second isomorphism is  $\mathbb{C}_2 \otimes_{\mathbb{R}} \beta_{m,*}$ . (Note that  $c_1^S$  has finite kernel and cokernel by Lemma 2.19.)

Combining the above and Proposition 2.10, one then finds that the Bloch-Kato conjecture predicts the image under  $\mathbb{C}_2 \otimes_{\mathbb{R}} \beta_{m,*}$  of the lattice on the right-hand side of (2.11) to be equal to

$$\mathbb{Z}_2 \cdot \zeta_F^*(1 - m) \cdot \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot c_{F,m,1}^S) \circ (\mathbb{Q}_2 \cdot c_{F,m,1}^K)^{-1})^{-1}.$$

Moreover, because  $R_m(F)$  is defined with respect to the lattice  $W_{m-1}^+$  rather than  $\text{Hom}_{\mathbb{Z}}(W_{1-m}^+, \mathbb{Z})$ , the formula (2.18) implies that

$$(\mathbb{C}_2 \otimes_{\mathbb{R}} \beta_{m,*})((\bigwedge_{\mathbb{Z}_2}^{d_m} K_{m,1,\text{tf}}) \otimes_{\mathbb{Z}_2} \text{Hom}_{\mathbb{Z}_2}(\bigwedge_{\mathbb{Z}_2}^{d_m} Y_m, \mathbb{Z}_2)) = 2^{r_2} \cdot R_m(F) \cdot \mathbb{Z}_2 \subset \mathbb{C}_2.$$

Now to deduce (2.4), by direct substitution, we only need to note that Lemma 2.19 implies

$$2^{a_m(F)} |\text{cok}(c_{F,m,1,\text{tf}}^S)|^{-1} \cdot \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot c_{F,m,1}^S) \circ (\mathbb{Q}_2 \cdot c_{F,m,1}^K)^{-1}) \in \mathbb{Z}_2^\times$$

for an integer  $a_m(F)$  as in the statement of Theorem 2.3.

The remainder of Theorem 2.3 is now an immediate consequence of part (ii) of Lemma 2.19.

**Lemma 2.19.**

- (i) One has  $\det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot c_{F,m,1}^{\text{DF}}) \circ (\mathbb{Q}_2 \cdot c_{F,m,1}^K)^{-1}) \in \mathbb{Z}_2^\times$ .
- (ii) The kernel and cokernel of  $c_{F,m,1}^S$  are finite. If  $2^{a_m(F)} = |\text{cok}(c_{F,m,1,\text{tf}}^K)|$  for  $a_m(F) \geq 0$ , then

$$2^{a_m(F)} |\text{cok}(c_{F,m,1,\text{tf}}^S)|^{-1} \cdot \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot c_{F,m,1}^S) \circ (\mathbb{Q}_2 \cdot c_{F,m,1}^{\text{DF}})^{-1}) \in \mathbb{Z}_2^\times.$$

Moreover,  $a_m(F) = 0$  except possibly when both  $m \equiv 3 \pmod{4}$  and  $r_1 > 0$ , in which case one has  $a_m(F) = r_{1,5} - 1$ .

*Proof.* Set  $F' := F(\sqrt{-1})$  and  $\Delta := G_{F'/F}$ . With  $\theta_{F'}$  denoting either  $c_{F',i,1}^S$ ,  $c_{F',i,1}^{\text{DF}}$  or  $c_{F',i,1}^K$  we write  $\theta_F$  for the corresponding map for  $F$ . Then there is a commutative diagram

$$\begin{array}{ccc} K_{2m-1}(\mathcal{O}_{F'}) \otimes \mathbb{Z}_2 & \xrightarrow{\theta_{F'}} & H^1(\mathcal{O}_{F'}[1/2], \mathbb{Z}_2(m)) \\ \uparrow & & \uparrow \\ K_{2m-1}(\mathcal{O}_F) \otimes \mathbb{Z}_2 & \xrightarrow{\theta_F} & H^1(\mathcal{O}_F[1/2], \mathbb{Z}_2(m)) \end{array}$$

with the vertical maps the pullbacks. Because  $\theta_{F'}$  is natural it is  $\Delta$ -equivariant; hence, by using the projection formula we may identify  $\mathbb{Q}_2 \cdot \theta_F$  with

$$H^0(\Delta, \mathbb{Q}_2 \cdot \theta_{F'}) : H^0(\Delta, K_{2m-1}(\mathcal{O}_{F'}) \otimes \mathbb{Q}_2) \rightarrow H^0(\Delta, H^1(\mathcal{O}_{F'}[1/2], \mathbb{Q}_2(m))).$$

By [23, Th. 8.7 and Rem. 8.8] we have that  $c_{F',m,1}^{DF}$  is surjective, and because it is a map between finitely generated  $\mathbb{Z}_2$ -modules of the same rank, the induced map  $c_{F',m,1,tf}^{DF}$  is bijective. This also holds for  $c_{F',m,1}^K$  because  $r_1(F') = 0$  in (2.12), so that  $\varphi := c_{F',m,1,tf}^{DF} \circ (c_{F',m,1,tf}^K)^{-1}$  is a  $\Delta$ -equivariant automorphism of the  $\mathbb{Z}_2[\Delta]$ -lattice  $H^1(\mathcal{O}_{F'}[1/2], \mathbb{Z}_2(m))_{tf}$ . Therefore,

$$\begin{aligned} & \det_{\mathbb{Q}_2}((\mathbb{Q}_2 \cdot c_{F',m,1}^{DF}) \circ (\mathbb{Q}_2 \cdot c_{F',m,1}^K)^{-1}) \\ &= \det_{\mathbb{Q}_2}(H^0(\Delta, \mathbb{Q}_2 \cdot c_{F',m,1}^{DF}) \circ H^0(\Delta, \mathbb{Q}_2 \cdot c_{F',m,1}^K)^{-1}) \\ &= \det_{\mathbb{Q}_2}(H^0(\Delta, \mathbb{Q}_2 \cdot \varphi)) \end{aligned}$$

is in  $\mathbb{Z}_2^\times$ , proving claim (i).

We shall prove in Theorem 2.21 that  $c_{F',m,1,tf}^S$  has finite cokernel. This implies that  $c_{F',m,1}^S$  has finite kernel and cokernel because its source and target are finitely generated  $\mathbb{Z}_2$ -modules of the same rank. For the remainder of the claims in (ii), by (i) it suffices to prove those with  $c_{F',m,1}^{DF}$  replaced by  $c_{F',m,1}^K$ . By (2.12) one also knows that  $c_{F',m,1}^K$ , and hence also  $c_{F',m,1,tf}^K$ , is surjective except possibly if  $m \equiv 3 \pmod{4}$  and  $r_1 > 0$ . In the latter case, the  $\mathbb{Z}_2$ -module  $K_{m,1}$  is torsion free (by [50, Th. 0.6]) and, because  $r_1 > 0$  and  $m$  is odd, it is straightforward to check that  $|H_{m,tor}^1| = 2$  (see, for example, [50, Props. 1.8 and 1.9(b)]). In this case, therefore, the computation (2.12) implies the equality  $|\text{cok}(c_{F',m,1,tf}^K)| = 2^{-1} \cdot |\text{cok}(c_{F',m,1}^K)| = 2^{r_1 \cdot s - 1}$ . The remaining part of claim (ii) now follows immediately by a computation with determinants using any fixed  $\mathbb{Z}_2$ -bases of  $(K_{2m-1}(\mathcal{O}_F) \otimes \mathbb{Z}_2)_{tf}$  and  $H^j(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))_{tf}$ .  $\square$

**Remark 2.20.** In [34, just after Th. 1] Kahn asks whether for  $m \equiv 3 \pmod{4}$  and  $r_1 > 0$  one has  $r_{1,5} = 1$  in (2.12) (so that  $a_m(F) = 0$  for all  $m$ ). He points out that it amounts to asking whether, in this case, the image of  $H^1(\mathcal{O}_F[1/2], \mathbb{Z}_2(m))$  in  $H^1(\mathcal{O}_F[1/2], \mathbb{Z}/2) \subset F^\times / (F^\times)^2$  is contained in the subgroup generated by the classes of  $-1$  and the totally positive elements of  $F^\times$ .

### 2.5. An upper bound for $t_m^2(F)$

We now fix an integer  $m \geq 2$ . For a number field  $E$ , we let  $E' := E(\sqrt{-1})$  and write  $c_E$  for Soulé’s 2-adic Chern class map

$$c_{E,m,1,2}^S : K_{2m-1}(\mathcal{O}_E) \otimes \mathbb{Z}_2 \rightarrow H^1(\mathcal{O}_E[1/2], \mathbb{Z}_2(m)).$$

In the proof of Lemma 2.19 we used that  $\text{cok}(c_{F,tf})$  is finite. Although this is commonly believed to be true, we were not able to locate a proof in the literature (cf. Remark 2.24). In addition, and as already discussed in Remark 2.5(ii), for numerical computations one must have a computable upper bound for its order  $2^{t_m^2(F)}$ . For this, we prove the following result.

**Theorem 2.21.**  $|c_{F,tf}|$  is finite and divides  $[F' : F]^{d_m(F)} ((m - 1)!)^{d_m(F')} |K_{2m-2}(\mathcal{O}_{F'})|$ .

As preparation for its proof, we first consider universal norm subgroups in étale cohomology. To do this we write  $F'_\infty$  for the cyclotomic  $\mathbb{Z}_2$ -extension of  $F'$ , and we let  $F'_n$  be the unique subfield of  $F'_\infty$  with  $[F'_n : F'] = 2^n$  for  $n \geq 0$ . We also set  $\Gamma := G_{F'_\infty/F'}$  and write  $\Lambda$  for the Iwasawa algebra  $\mathbb{Z}_2[[\Gamma]]$ . For each  $\Lambda$ -module  $N$  and integer  $a$  we write  $N(a)$  for the  $\Lambda$ -module  $N \otimes_{\mathbb{Z}_2} \mathbb{Z}_2(a)$  upon which  $\Gamma$  acts diagonally. For a finite extension  $E$  of  $F'$  we set  $\mathcal{O}'_E := \mathcal{O}_E[1/2]$ .

We define the ‘universal norm’ subgroup  $H^\infty(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  of  $H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  to be the image of  $\lim_{\leftarrow n} H^1(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m))$  in  $H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  under the natural projection map, where the limit is taken with respect to the natural corestriction maps.

**Proposition 2.22.** The index of  $H^\infty(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  in  $H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  is a divisor of  $|K_{2m-2}(\mathcal{O}_{F'})|$ .

*Proof.* We write  $C^\infty$  for the object of the derived category of perfect complexes of  $\Lambda$ -modules that is obtained as the inverse limit of the complexes  $R\Gamma(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m))$  with respect to the natural projection

morphisms

$$R\Gamma(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m)) \rightarrow \mathbb{Z}_2[\Gamma_n] \otimes_{\mathbb{Z}_2[\Gamma_{n+1}]}^{\mathbb{L}} R\Gamma(\mathcal{O}'_{F'_{n+1}}, \mathbb{Z}_2(m)) \cong R\Gamma(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m)).$$

We recall that there is a natural isomorphism  $\mathbb{Z}_2 \otimes_{\Lambda}^{\mathbb{L}} C_{\infty}^{\bullet} \cong R\Gamma(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  in  $D^{\text{perf}}(\mathbb{Z}_2)$  and that this induces a natural short exact sequence of  $\mathbb{Z}_2$ -modules

$$0 \rightarrow \mathbb{Z}_2 \otimes_{\mathbb{Z}_2[[\Gamma]]} H^1(C_{\infty}^{\bullet}) \xrightarrow{\pi} H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m)) \rightarrow H^0(\Gamma, H^2(C_{\infty}^{\bullet})) \rightarrow 0. \tag{2.23}$$

In addition, in each degree  $i$  one has  $H^i(C_{\infty}^{\bullet}) = \varprojlim_n H^i(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m))$ , where the limits are taken with respect to the natural corestriction maps. We therefore have  $\text{im}(\pi) = H_{\infty}^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  and the  $\Lambda$ -module  $H^2(C_{\infty}^{\bullet})$  is isomorphic to  $(\varprojlim_n H^2(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(1)))(m - 1)$ .

Now for each  $n \geq 0$ , class field theory identifies  $H^2(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(1))_{\text{tor}}$  with the ideal class group  $\text{Pic}(\mathcal{O}'_{F'_n})$  of  $\mathcal{O}'_{F'_n}$  and  $H^2(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(1))_{\text{tf}}$  with a submodule of the free  $\mathbb{Z}_2$ -module on the set of places of  $F'_n$  that are either Archimedean or 2-adic. Hence, upon passing to the limit over  $n$  and then taking  $\Gamma$ -invariants, we obtain an exact sequence of  $\mathbb{Z}_2$ -modules

$$0 \rightarrow H^0(\Gamma, X'_{\infty}(m - 1)) \rightarrow H^0(\Gamma, H^2(C_{\infty}^{\bullet})) \rightarrow \bigoplus_{v \in \Sigma_2(F') \cup \Sigma_{\infty}(F')} H^0(\Gamma, \mathbb{Z}_2[[\Gamma/\Gamma_v]](m - 1)),$$

where  $X'_{\infty}$  is the Galois group of the maximal unramified pro-2 extension of  $F'_{\infty}$  in which all 2-adic places split completely,  $\Sigma_2$  is the set of 2-adic places of  $F'$  and  $\Gamma_v$  is the decomposition subgroup of  $v$  in  $\Gamma$ . Because  $m \geq 2$  it is also clear that each  $H^0(\Gamma, \mathbb{Z}_2[[\Gamma/\Gamma_v]](m - 1))$  vanishes. We therefore have that  $H^0(\Gamma, X'_{\infty}(m - 1)) = H^0(\Gamma, H^2(C_{\infty}^{\bullet}))$ , and by (2.23) it now suffices to show that  $H^0(\Gamma, X'_{\infty}(m - 1))$  is finite and of order dividing  $|K_{2m-2}(\mathcal{O}_{F'})|$ .

Next we recall that, by a standard ‘Herbrand quotient’ argument in Iwasawa theory (see, for example, [57, Exer. 13.12]), if a finitely generated  $\Lambda$ -module  $N$  is such that  $H_0(\Gamma, N)$  is finite, then  $H^0(\Gamma, N)$  is both finite and of order at most  $|H_0(\Gamma, N)|$ . In addition, because  $F'$  is totally imaginary, the argument in [51, §6, Lem. 1] (see also the discussion in [40, just before Lem. 1.2]) shows that  $H_0(\Gamma, X'_{\infty}(m - 1))$  is naturally isomorphic to the ‘étale wild kernel’

$$\text{WK}_{2m-2}^{\text{ét}}(F') := \ker(H^2(\mathcal{O}'_{F'}, \mathbb{Z}_2(m)) \rightarrow \bigoplus_{w \in \Sigma_2(F') \cup \Sigma_{\infty}(F')} H^2(F'_w, \mathbb{Z}_2(m)))$$

of  $F'$ , where the arrow denotes the natural diagonal localisation map. Hence, to deduce the claimed result, we need only recall that, because  $F'$  is totally imaginary, the group  $H^2(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  is naturally isomorphic to  $K_{2m-2}(\mathcal{O}_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ , which is finite by (2.12).  $\square$

Turning now to the proof of Theorem 2.21, we consider for each  $n$  the following diagram:

$$\begin{CD} K_{2m-1}(\mathcal{O}'_{F'_n}, \mathbb{Z}/2^n) @>{m \cdot c_{F'_n, 2^n}}>> H^1(\mathcal{O}'_{F'_n}, (\mathbb{Z}/2^n)(m)) @>{\iota_n}>> H^1(F'_n, (\mathbb{Z}/2^n)(m)) \\ @VVV @VVV @VVV \\ K_{2m-1}(\mathcal{O}'_{F'}, \mathbb{Z}/2^n) @>{m \cdot c_{F', 2^n}}>> H^1(\mathcal{O}'_{F'}, (\mathbb{Z}/2^n)(m)) @>{\iota}>> H^1(F', (\mathbb{Z}/2^n)(m)). \end{CD}$$

Here we let  $c_{E, 2^n} : K_{2m-1}(\mathcal{O}'_E, \mathbb{Z}/2^n) \rightarrow H^1(\mathcal{O}'_E, (\mathbb{Z}/2^n)(m))$  be the Chern class maps of Soulé, as discussed by Weibel in [58], the arrows  $\iota_n$  and  $\iota$  are the natural inflation maps, the left-hand vertical arrow is the natural transfer map and the remaining vertical arrows are the natural corestrictions. The results of [58, Prop. 2.1.1 and 4.4] imply that the outer rectangle of this diagram commutes. Therefore, the first square also commutes because the maps  $\iota_n$  and  $\iota$  are injective.

Because the first square is compatible with change of  $n$  in the natural way, we may then pass to the inverse limit over  $n$  to obtain a commutative diagram

$$\begin{CD} \lim_{\leftarrow n} K_{2m-1}(\mathcal{O}'_{F'_n}, \mathbb{Z}/2^n) @>(m \cdot c_{F'_n, 2^n})_n>> \lim_{\leftarrow n} H^1(\mathcal{O}'_{F'_n}, \mathbb{Z}_2(m)) \\ @VVV @VVV \\ K_{2m-1}(\mathcal{O}'_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 @>m \cdot c_{F'}>> H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m)). \end{CD}$$

Here we use that, because  $K_{2m-1}(\mathcal{O}'_{F'})$  is finitely generated,  $\lim_{\leftarrow n} K_{2m-1}(\mathcal{O}'_{F'_n}, \mathbb{Z}/2^n)$  identifies with  $K_{2m-1}(\mathcal{O}'_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$  in such a way that the limit  $(m \cdot c_{F'_n, 2^n})_n$  is  $m \cdot c_{F'}$ .

Now the image of the right-hand vertical arrow here is  $H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$  and, because each  $F'_n$  contains all roots of unity of order  $2^n$ , from [58, Cor. 5.6] one knows that the exponent of  $\text{cok}((m \cdot c_{F'_n, 2^n})_n)$  divides  $m!$ . From the commutativity of the above diagram we then deduce that  $\text{im}(m \cdot c_{F'})$  contains  $m! \cdot H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))$ , so that  $\text{im}(c_{F', \text{tf}})$  contains  $(m - 1)! \cdot H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))_{\text{tf}}$ . This inclusion implies that  $\text{cok}(c_{F', \text{tf}})$  is finite of order dividing

$$\left| \frac{H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))_{\text{tf}}}{H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))_{\text{tf}}} \right| \cdot \left| \frac{H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))_{\text{tf}}}{(m - 1)! \cdot H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))_{\text{tf}}} \right|$$

and hence also  $|H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))/H^1_{\infty}(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))| \cdot ((m - 1)!)^{d_m(F')}$ . Proposition 2.22 now implies that Theorem 2.21 is true if  $F = F'$ .

For the remaining case of Theorem 2.21 we assume  $F \neq F'$ , write  $\tau$  for the nontrivial element of  $\Delta$  and note that [58, Prop. 4.4] implies that there is a commutative diagram

$$\begin{CD} K_{2m-1}(\mathcal{O}'_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 @>c_{F', \text{tf}}>> H^1(\mathcal{O}'_{F'}, \mathbb{Z}_2(m))_{\text{tf}} \\ @VV T_{\Delta}^1 V @VV T_{\Delta}^2 V \\ K_{2m-1}(\mathcal{O}'_F) \otimes_{\mathbb{Z}} \mathbb{Z}_2 @>c_{F, \text{tf}}>> H^1(\mathcal{O}'_F, \mathbb{Z}_2(m))_{\text{tf}} \end{CD}$$

where the maps  $T_{\Delta}^i$  are induced by the respective actions of  $1 + \tau \in \mathbb{Z}_2[\Delta]$ . It follows that the index of  $c_{F', \text{tf}}(\text{im}(T_{\Delta}^1))$  in  $\text{im}(T_{\Delta}^2)$  divides  $|\text{cok}(c_{F', \text{tf}})|$ . From the projection formula we see that  $\text{im}(T_{\Delta}^2)$  contains  $2H^1(\mathcal{O}'_F, \mathbb{Z}_2(m))_{\text{tf}}$ , so that its index in  $H^1(\mathcal{O}'_F, \mathbb{Z}_2(m))_{\text{tf}}$  divides  $2^{d_m(F)}$  because the latter is a free  $\mathbb{Z}_2$ -module of rank  $d_m(F)$ . The statement of Theorem 2.21 for  $F$  now follows.

**Remark 2.24.** The argument of Huber and Wildeshaus in [33, Th. B.4.8 and Lem. B.4.7] aims to show, amongst other things, that  $\text{cok}(c_{F, m, 1, 2}^S)$  is finite. However, this argument uses in a key way results of Dwyer and Friedlander from [23, Th. 8.7 and Rem. 8.8] that relate to  $c_{F, m, 1, 2}^{\text{DF}}$  rather than  $c_{F, m, 1, 2}^S$ . To complete this argument one would thus need to investigate the relation between  $\text{cok}(c_{F, m, 1, 2}^{\text{DF}})$  and  $\text{cok}(c_{F, m, 1, 2}^S)$ .

### 3. K-theory, wedge complexes, and configurations of points

Let  $F$  be an infinite field. Then it is well known by work of Bloch and (subsequently) Suslin that  $K_3(F)$  is closely related to the Bloch group  $B(F)$  (as defined in Subsection 3.3.4). However, the group  $B(F)$  often contains nontrivial elements of finite order and so can be difficult for the purposes of explicit computation. With this in mind, in this section we shall introduce, for any field  $F$ , a slight variant  $\bar{B}(F)$  of  $B(F)$  over which we have better control.

We shall also construct a natural (but unique up to a universal choice of sign) homomorphism  $\psi_F$  from  $\bar{B}(F)$  to  $K_3(F)_{\text{tf}}^{\text{ind}}$  (see Theorem 3.25) and are motivated to conjecture, on the basis of extensive

computational evidence, that  $\psi_F$  is bijective if  $F$  is a number field (see Conjecture 3.33). We note, in particular, that these observations provide the first concrete evidence to suggest both that the groups defined by Suslin in terms of group homology and by Bloch in terms of relative  $K$ -theory should be related in a very natural way and also that Bloch's group should account for all of  $K_3^{\text{ind}}(F)$ , at least modulo torsion (cf. Remark 3.34).

If  $F$  is imaginary quadratic, then the groups  $\overline{B}(F)$  and  $K_3(F)_{\text{tf}}^{\text{ind}}$  are both isomorphic to  $\mathbb{Z}$  (see Corollary 3.29 for  $\overline{B}(F)$ ) and we shall later use  $\psi_F$  to reduce the problem of finding a generator of  $K_3(F)_{\text{tf}}^{\text{ind}}$  to computational issues in  $\overline{B}(F)$ .

### 3.1. Towards explicit versions of $K_3(F)$ and $\text{reg}_2$

In this section we review some earlier results that we shall need.

#### 3.1.1.

We first recall some basic facts concerning the  $K_3$ -group of a general field  $F$ . For this, we write  $K_3(F)^{\text{ind}}$  for the quotient of  $K_3(F)$  by the image of the Milnor  $K$ -group  $K_3^M(F)$  of  $F$ .

We recall that if  $F$  is a number field then the abelian group  $K_3^M(F)$  has exponent 1 or 2 and order  $2^{r_1(F)}$  (cf. [59, p.146]), so that  $K_3(F)_{\text{tf}}^{\text{ind}}$  identifies with  $K_3(F)_{\text{tf}}$  and hence is a free abelian group of rank  $r_2(F)$  as a consequence of Theorem 2.1(ii).

We further recall that for any field  $F$  the torsion subgroup of  $K_3(F)^{\text{ind}}$  is explicitly described by Levine in [41, Cor. 4.6].

**Example 3.1.** For an imaginary quadratic field  $k$  in  $\overline{\mathbb{Q}}$ , the latter result gives isomorphisms

$$K_3(k)_{\text{tor}}^{\text{ind}} \simeq H^0(\text{Gal}(\overline{\mathbb{Q}}/k), \mathbb{Q}(2)/\mathbb{Z}(2)) = H^0(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}(2)/\mathbb{Z}(2)) = \mathbb{Z}/24\mathbb{Z},$$

where the first equality is valid because complex conjugation acts trivially on  $\mathbb{Q}(2)/\mathbb{Z}(2)$  and the second follows by explicit computation. For any such field  $k$  the abelian group  $K_3(k)^{\text{ind}} = K_3(k)$  is therefore isomorphic to a direct product of the form  $\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$ .

#### 3.1.2.

For an arbitrary field  $F$  we set

$$\tilde{\lambda}^2 F^\times := \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle (-x) \otimes x \text{ with } x \text{ in } F^\times \rangle},$$

a quotient of the usual exterior power  $F^\times \otimes_{\mathbb{Z}} F^\times / \langle x \otimes y + y \otimes x \text{ with } x, y \text{ in } F^\times \rangle$ . We write  $a \tilde{\lambda} b$  for the class of  $a \otimes b$  in  $\tilde{\lambda}^2 F^\times$ , and note  $a \tilde{\lambda} b + b \tilde{\lambda} a$  is trivial.

We next let  $\mathbb{Z}[F^b]$  be the free abelian group on  $F^b := F \setminus \{0, 1\}$  and define the homomorphism

$$\delta_{2,F}: \mathbb{Z}[F^b] \rightarrow \tilde{\lambda}^2 F^\times \tag{3.2}$$

by sending  $[x]$  to  $(1-x) \tilde{\lambda} x$  for each  $x$  in  $F^b$ .

We write  $D: \mathbb{C}^b \rightarrow \mathbb{R}$  for the Bloch-Wigner dilogarithm. Its value at  $z$  is defined by Bloch in [5] by integrating  $\log |w| \cdot d \arg(1-w) - \log |1-w| \cdot d \arg(w)$  along any path from a point  $z_0$  in  $\mathbb{R}^b$  to  $z$ . We recall that, by differentiating, one easily shows the identities

$$\begin{aligned} D(z) + D(z^{-1}) &= 0, & D(z) + D(1-z) &= 0, & D(z) + D(\bar{z}) &= 0, \\ D(x) - D(y) + D\left(\frac{y}{x}\right) - D\left(\frac{1-y}{1-x}\right) + D\left(\frac{1-y^{-1}}{1-x^{-1}}\right) &= 0, \end{aligned} \tag{3.3}$$

for  $x, y$  and  $z$  in  $\mathbb{C}^b$  with  $x \neq y$ .

We note, in particular, that the third identity here implies that the map  $iD$  from  $\mathbb{C}^b$  to  $\mathbb{R}(1)$  is equivariant with respect to the natural action of complex conjugation.

We quote a result connecting these notions to  $K_3(F)_{\text{tf}}^{\text{ind}}$ . It underlies our construction of elements in  $K_3(F)_{\text{tf}}^{\text{ind}}$  for a number field  $F$  (in particular, in Section 4, for  $F$  imaginary quadratic).

**Theorem 3.4** ([14, Th. 4.1]). *With the above notation, the following claims hold.*

(i) *There exists a homomorphism*

$$\varphi_F : \ker(\delta_{2,F}) \rightarrow K_3(F)_{\text{tf}}^{\text{ind}}$$

*that is natural up to sign and, after fixing a choice of sign, functorial in  $F$ .*

(ii) *If  $F$  is a number field, then the cokernel of  $\varphi_F$  is finite.*

(iii) *There exists a universal choice of sign such that if  $F$  is any number field and  $\sigma : F \rightarrow \mathbb{C}$  is any embedding, then the composition*

$$\text{reg}_\sigma : \ker(\delta_{2,F}) \xrightarrow{\varphi_F} K_3(F)_{\text{tf}}^{\text{ind}} = K_3(F)_{\text{tf}} \xrightarrow{\sigma_*} K_3(\mathbb{C})_{\text{tf}} \xrightarrow{\text{reg}_2} \mathbb{R}(1)$$

*is induced by sending each element  $[x]$  for  $x$  in  $F^b$  to  $iD(\sigma(x))$ .*

### 3.2. Analysing our wedge product

In this section we obtain explicit information on the structure of  $\tilde{\lambda}^2 F^\times$  for a general field  $F$ . With an eye towards implementation for the purposes of numerical calculations, we pay special attention to the case that  $F$  is a number field.

#### 3.2.1.

We first consider the abstract structure of  $\tilde{\lambda}^2 F^\times$ .

**Proposition 3.5.** *For a field  $F$ , we have a filtration*

$$\{0\} = \text{Fil}_0 \subseteq \text{Fil}_1 \subseteq \text{Fil}_2 \subseteq \text{Fil}_3 = \tilde{\lambda}^2 F^\times$$

*with  $\text{Fil}_1$  the image of  $F_{\text{tor}}^\times \otimes F_{\text{tor}}^\times$  and  $\text{Fil}_2$  the image of  $F_{\text{tor}}^\times \otimes F_{\text{tf}}^\times$ . Then*

$$\text{Fil}_2 = \frac{F_{\text{tor}}^\times \otimes F_{\text{tf}}^\times}{\langle (-x) \otimes x \text{ with } x \text{ in } F_{\text{tor}}^\times \rangle}$$

*and there are natural isomorphisms*

$$\begin{aligned} \text{Fil}_1/\text{Fil}_0 &= \tilde{\lambda}^2 F_{\text{tor}}^\times = \frac{F_{\text{tor}}^\times \otimes_{\mathbb{Z}} F_{\text{tor}}^\times}{\langle (-x) \otimes x \text{ with } x \text{ in } F_{\text{tor}}^\times \rangle} \\ \text{Fil}_2/\text{Fil}_1 &\simeq F_{\text{tor}}^\times \otimes F_{\text{tf}}^\times \\ \text{Fil}_3/\text{Fil}_2 &\simeq \frac{F_{\text{tf}}^\times \otimes_{\mathbb{Z}} F_{\text{tf}}^\times}{\langle x \otimes x \text{ with } x \text{ in } F_{\text{tf}}^\times \rangle}, \end{aligned}$$

*with the last two induced by the quotient maps  $F_{\text{tor}}^\times \otimes F_{\text{tor}}^\times \rightarrow F_{\text{tor}}^\times \otimes_{\mathbb{Z}} F_{\text{tor}}^\times$  and  $F_{\text{tor}}^\times \otimes F_{\text{tf}}^\times \rightarrow F_{\text{tor}}^\times \otimes_{\mathbb{Z}} F_{\text{tf}}^\times$ .*

*Proof.* By taking filtered direct limits, it suffices to prove those statements with  $F^\times$  replaced with a finitely generated subgroup  $A$  of  $F^\times$  that contains  $-1$ . We can then obtain a splitting  $A \simeq A_{\text{tor}} \oplus A_{\text{tf}}$  and find that the quotient for  $A$  is isomorphic to

$$\frac{A_{\text{tor}} \otimes A_{\text{tor}} \oplus A_{\text{tor}} \otimes A_{\text{tf}} \oplus A_{\text{tf}} \otimes A_{\text{tor}} \oplus A_{\text{tf}} \otimes A_{\text{tf}}}{\langle ((-u) \otimes u, (-u) \otimes c, c \otimes u, c \otimes c) \text{ with } u \text{ in } A_{\text{tor}} \text{ and } c \text{ in } A_{\text{tf}} \rangle}.$$

Our claims follow for  $A$  if we prove that the intersection of

$$A_{\text{tor}} \otimes A_{\text{tor}} \oplus A_{\text{tor}} \otimes A_{\text{tf}} \oplus A_{\text{tf}} \otimes A_{\text{tor}} \oplus 0$$

with the group in the denominator equals

$$\langle ((-u) \otimes u, u \otimes c, c \otimes u, 0) \text{ with } u \text{ in } A_{\text{tor}} \text{ and } c \text{ in } A_{\text{tf}} \rangle$$

because the latter is the product

$$\langle (-u) \otimes u \text{ with } u \text{ in } A_{\text{tor}} \rangle \times \langle (v \otimes c, c \otimes v) \text{ with } v \text{ in } A_{\text{tor}} \text{ and } c \text{ in } A_{\text{tf}} \rangle \times \{0\}.$$

From the identity  $(-uc) \otimes uc - (-c) \otimes c = (-u) \otimes u + u \otimes c + c \otimes u$  in  $A \otimes A$  it is clear that this intersection contains the given subgroup. In order to show that equality holds, choose a basis  $b_1, \dots, b_s$  of  $A_{\text{tf}}$  and assume that, for some integers  $m_i$ , the last position in

$$\sum_i m_i ((-u_i) \otimes u_i, (-u_i) \otimes c_i, c_i \otimes u_i, c_i \otimes c_i) \tag{3.6}$$

is trivial. If  $b_j$  has coefficient  $a_{i,j}$  in  $c_i$ , then  $\sum_i m_i a_{i,j}^2 = 0$  for each  $j$ ; hence each  $\sum_i m_i a_{i,j}$  is even,

$$\sum_i m_i (-1) \otimes c_i = \sum_j \sum_i m_i a_{i,j} (-1) \otimes b_j$$

is trivial and in the second position of the element in (3.6) we can replace each  $-u_i$  with  $u_i$ . □

**Remark 3.7.** Clearly,  $\text{Fil}_1$  is trivial if  $F$  has characteristic 2. It is also trivial if the characteristic is not equal to 2 but  $F^\times$  contains an element of order 4: If  $u$  in  $F^\times$  has order  $2m$  with  $m$  even, then  $u \tilde{\wedge} u = (-1) \tilde{\wedge} u = m(u \tilde{\wedge} u)$  in  $\tilde{\wedge}^2 F^\times$ , and  $\text{gcd}(m - 1, 2m) = 1$ . Finally, if  $F$  has characteristic not equal to 2, and  $F^\times$  does not contain an element of order 4, then by decomposing  $F_{\text{tor}}^\times$  into its primary components, one sees that  $\text{Fil}_1$  is cyclic of order 2, generated by  $(-1) \tilde{\wedge} (-1)$ .

**Corollary 3.8.** *Let  $F$  be a number field,  $n$  the order of  $F_{\text{tor}}^\times$  and  $c_1, c_2, \dots$  in  $F^\times$  such that they give a basis of  $F_{\text{tf}}^\times$ . Let  $m = 1$  and  $u = 1$  if  $n$  is divisible by 4 and  $m = 2$  and  $u = -1$  otherwise. Then the map*

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} \times \oplus_i \mathbb{Z}/n\mathbb{Z} \times \oplus_{i < j} \mathbb{Z} &\rightarrow \tilde{\wedge}^2 F^\times \\ (a, (b_i)_i, (b_{i,j})_{i,j}) &\mapsto u^a \tilde{\wedge} u + \sum_i u^{b_i} \tilde{\wedge} c_i + \sum_{i < j} c_i^{b_{i,j}} \tilde{\wedge} c_j \end{aligned}$$

is an isomorphism.

*Proof.* The domain has a filtration  $\text{Fil}'_l$  for  $l = 0, 1, 2$  and 3 by taking the last  $3 - l$  positions to be trivial, and under the homomorphism we map  $\text{Fil}'_l$  to  $\text{Fil}_l$  as in Proposition 3.5 so it induces a homomorphism  $\text{Fil}'_l/\text{Fil}'_{l-1} \rightarrow \text{Fil}_l/\text{Fil}_{l-1}$  for  $l = 1, 2$  and 3. For  $l = 1$  this is an isomorphism by Remark 3.7 and for  $l = 2$  and  $l = 3$  by Proposition 3.5. We now apply the five lemma. □

**Remark 3.9.** (i) One can get finitely many of the  $c_i$  in the corollary by taking a basis of the free part of the  $S$ -units for a finite set  $S$  of primes of the ring of integers  $\mathcal{O}$  of  $F$ . If one extends  $S$  to  $S'$ , then one can add more  $c_j$  in order to obtain a similar basis for the  $S'$ -units.

(ii) For a generator  $u$  of  $F_{\text{tor}}^\times$  of order  $2l$  and finitely many of the  $c_i$  in the corollary, which together generate a subgroup  $A$  of  $F^\times$ , the isomorphism of the corollary becomes explicit on the image of  $\tilde{\wedge}^2 A$  by writing its elements in terms of the generators and using  $c_i \tilde{\wedge} c_j + c_j \tilde{\wedge} c_i = 0$  if  $i \neq j$ ,  $c_i \tilde{\wedge} c_i = (-1) \tilde{\wedge} c_i$ ,  $u \tilde{\wedge} c_i + c_i \tilde{\wedge} u = 0$ , as well as that  $u \tilde{\wedge} u$  equals  $(-1) \tilde{\wedge} (-1)$  for  $l$  odd and is trivial for  $l$  even.

**3.2.2.**

Any field extension  $F \rightarrow F'$  induces a homomorphism from  $\tilde{\lambda}^2 F^\times$  to  $\tilde{\lambda}^2 (F')^\times$ . We determine its kernel for  $F = \mathbb{Q}$  and  $F'$  imaginary quadratic. This will be important for Theorem 4.7.

**Lemma 3.10.** *Let  $d$  be a positive square-free integer, and let  $k = \mathbb{Q}(\sqrt{-d})$ . If  $d \neq 1$ , then the kernel of the map  $\tilde{\lambda}^2 \mathbb{Q}^\times \rightarrow \tilde{\lambda}^2 k^\times$  has order 2, with  $(-1) \tilde{\lambda} (-d)$  as nontrivial element. If  $d = 1$  then the kernel is noncyclic of order 4 and is generated by  $(-1) \tilde{\lambda} (-1)$  and  $(-1) \tilde{\lambda} 2$ .*

*Proof.* The given elements are easily seen to be in the kernel (for  $\mathbb{Q}(\sqrt{-1})$  use that  $2 = \sqrt{-1}(1 - \sqrt{-1})^2$  and that  $\sqrt{-1} \tilde{\lambda} \sqrt{-1}$  is trivial by Remark 3.7). In addition, by Proposition 3.5 (or Corollary 3.8 with the prime numbers as  $c_i$ ) the elements generate a subgroup of the stated order. It is therefore enough to check the size of the kernel.

Clearly, from the description in Proposition 3.5 the kernel is contained in  $\text{Fil}_2$  on  $\tilde{\lambda}^2 \mathbb{Q}$ . Because the map on the  $\text{Fil}_1$ -pieces is surjective by Remark 3.7, with kernel of order 2 if  $d = 1$  and trivial otherwise, we only have to show that the kernel for  $\text{Fil}_2/\text{Fil}_1$  has order 2.

For this we use the description of  $\text{Fil}_2/\text{Fil}_1$  in Proposition 3.5. If  $|k_{\text{tor}}^\times| = 2m$ , then the kernel corresponds to the kernel of the map  $\mathbb{Q}_{\text{tf}}^\times/2 \rightarrow k_{\text{tf}}^\times/2m$  given by raising to the  $m$ th power, because  $-1$  is the  $m$ th power of a generator of  $k_{\text{tor}}^\times$ . We solve  $a^m = u\alpha^{2m}$  with  $a$  in  $\mathbb{Q}^\times$ ,  $u$  in  $k_{\text{tor}}^\times$  and  $\alpha$  in  $k^\times$  or, equivalently,  $a = v\alpha^2$  for some  $v$  in  $k_{\text{tor}}^\times$  because  $u$  is an  $m$ th power in  $k^\times$ .

After some calculation, we find for  $d \neq 1$  that  $a$  is of the form  $\pm b^2$  or  $\pm d \cdot b^2$  for some  $b$  in  $\mathbb{Q}^\times$ , and for  $d = 1$  that it is of the form  $\pm b^2$  or  $\pm 2b^2$ . In either case, this leads to two elements in  $\mathbb{Q}_{\text{tf}}^\times/2$  that are in the kernel, as required. □

**3.3. Configurations of points, and a modified Bloch group**

In order to be able to apply Theorem 3.4 in our geometric construction of elements in the indecomposable  $K_3$ -groups of imaginary quadratic fields in Section 4, it is convenient to make technical modifications of well-known constructions of Suslin [55] and Goncharov [29, p. 73]. As a result, we shall be able to be more precise about torsion in the resulting Bloch groups and some of the homomorphisms involved.

However, in order to be able to take finite nontrivial torsion in stabilisers of points in  $\mathbb{P}_F^1$  into account and to be able to work with groups like  $\text{PGL}_2(F)$  instead of  $\text{GL}_2(F)$  whenever necessary, we are forced to work in somewhat greater generality.

**3.3.1.**

Let  $F$  be a field and fix two subgroups  $\nu \subseteq \nu'$  of  $F^\times$ . (Typically, we have in mind  $\nu = \{1\}$  or  $\{\pm 1\}$ , and  $\nu'$  the torsion subgroup of the units of the ring of algebraic integers in a number field.) Let  $\Delta = \text{GL}_2(F)/\nu$ . Let  $\mathcal{L}$  be the set of orbits for the action of  $\nu'$  on  $F^2 \setminus \{(0, 0)\}$  given by scalar multiplication, which has a natural map to  $\mathbb{P}_F^1$ . The extreme cases  $\nu' = F^\times$  and  $\nu' = \{1\}$  give  $\mathcal{L} = \mathbb{P}_F^1$  and  $\mathcal{L} = F^2 \setminus \{(0, 0)\}$ , respectively. For  $n \geq 0$  we let  $C_n(\mathcal{L})$  be the free abelian group with as generators  $(n + 1)$ -tuples  $(l_0, \dots, l_n)$  of elements in  $\mathcal{L}$  such that if  $l_{i_1}$  and  $l_{i_2}$  have the same image in  $\mathbb{P}_F^1$ , then  $l_{i_1} = l_{i_2}$  (see Remark 3.15 for an explanation of this condition). We shall call such a tuple  $(l_0, \dots, l_n)$  with all  $l_i$  distinct in  $\mathcal{L}$  (or, equivalently, in  $\mathbb{P}_F^1$ ) nondegenerate, and we shall call it degenerate otherwise. Then  $\Delta$  acts on  $C_n(\mathcal{L})$  as  $\nu \subseteq \nu'$ , and with the usual boundary map  $d: C_n(\mathcal{L}) \rightarrow C_{n-1}(\mathcal{L})$  for  $n \geq 1$  given by

$$d(l_0, \dots, l_n) = \sum_{i=0}^n (-1)^i (l_0, \dots, \widehat{l}_i, \dots, l_n),$$

where  $\widehat{l}_i$  indicates that the term  $l_i$  is omitted, we get a complex

$$\dots \xrightarrow{d} C_4(\mathcal{L}) \xrightarrow{d} C_3(\mathcal{L}) \xrightarrow{d} C_2(\mathcal{L}) \xrightarrow{d} C_1(\mathcal{L}) \xrightarrow{d} C_0(\mathcal{L}) \tag{3.11}$$

of  $\mathbb{Z}[\Delta]$ -modules.

For three nonzero points  $p_0, p_1$  and  $p_2$  in  $F^2$  with distinct images in  $\mathbb{P}_F^1$ , we define  $\text{cr}_2(p_0, p_1, p_2)$  in  $\tilde{\lambda}^2 F^\times$  by the rules:

- $\text{cr}_2(gp_0, gp_1, gp_2) = \text{cr}_2(p_0, p_1, p_2)$  for every  $g$  in  $\text{GL}_2(F)$ ;
- $\text{cr}_2((1, 0), (0, 1), (a, b)) = a \tilde{\lambda} b$ .<sup>1</sup>

Using direct calculations it is immediately verified that one has  $\text{cr}_2((0, 1), (1, 0), (a, b)) = b \tilde{\lambda} a$  and  $\text{cr}_2((1, 0), (a, b), (0, 1)) = (-ab^{-1}) \tilde{\lambda} b^{-1} = b \tilde{\lambda} a$ , so that  $\text{cr}_2$  is alternating. It is also clear that if we scale one of the  $p_i$  by  $\lambda$  in  $v'$ , then  $\text{cr}_2(p_0, p_1, p_2)$  changes by a term  $\lambda \tilde{\lambda} c$  with  $c$  in  $F^\times$ . Let

$$\tilde{\lambda}^2 F^\times / v' \tilde{\lambda} F^\times = \frac{\tilde{\lambda}^2 F^\times}{\langle \lambda \tilde{\lambda} c \text{ with } \lambda \text{ in } v' \text{ and } c \text{ in } F^\times \rangle}. \tag{3.12}$$

We then define a homomorphism

$$f_{2,F} : C_2(\mathcal{L}) \rightarrow \tilde{\lambda}^2 F^\times / v' \tilde{\lambda} F^\times$$

by letting it be trivial on a degenerate generator  $(l_0, l_1, l_2)$  and by mapping a nondegenerate generator  $(l_0, l_1, l_2)$  to  $\text{cr}_2(p_0, p_1, p_2)$  with  $p_i$  a point in  $l_i$ . (We suppress  $v'$  from the notation.)

We next define a homomorphism

$$f_{3,F} : C_3(\mathcal{L}) \rightarrow \mathbb{Z}[F^b]$$

as follows. On a degenerate generator  $(l_0, l_1, l_2, l_3)$  we let  $f_{3,F}$  be trivial, and we let it map a nondegenerate generator  $(l_0, l_1, l_2, l_3)$  to  $[\text{cr}_3(\bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_3)]$ , the generator for the cross-ratio  $\text{cr}_3$  of the images of the points in  $\mathbb{P}_F^1$ . Recall that  $\text{cr}_3$  is defined by rules similar to those for  $\text{cr}_2$ :

- $\text{cr}_3(g\bar{l}_0, g\bar{l}_1, g\bar{l}_2, g\bar{l}_3) = \text{cr}_3(\bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_3)$  for every  $g$  in  $\text{GL}_2(F)$ ;
- $\text{cr}_3([1, 0], [0, 1], [1, 1], [x, 1]) = x$  for  $x$  in  $F^b$ .

**Remark 3.13.** From the  $\text{GL}_2(F)$ -equivariance of  $\text{cr}_3$  one sees by a direct calculation that, for  $l_0, l_1, l_2, l_3$  different nonzero points in  $F^2$ ,

$$\text{cr}_3(\bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_3) = \frac{\det\begin{pmatrix} l_1 & l_3 \end{pmatrix} \det\begin{pmatrix} l_2 & l_4 \end{pmatrix}}{\det\begin{pmatrix} l_1 & l_4 \end{pmatrix} \det\begin{pmatrix} l_2 & l_3 \end{pmatrix}}.$$

As is well known, from this, or by a direct calculation, we see that permuting the four points can give the following related possibilities for a cross-ratio:  $x, 1 - x^{-1}, (1 - x)^{-1}$  for even permutations and  $1 - x, x^{-1}, (1 - x^{-1})^{-1}$  for odd ones, with the subgroup  $V_4$  of  $S_4$  acting trivially.

### 3.3.2.

In the next result we consider the homomorphism

$$\delta_{2,F}^{v'} : \mathbb{Z}[F^b] \rightarrow \tilde{\lambda}^2 F^\times / v' \tilde{\lambda} F^\times$$

that sends each element  $[x]$  for  $x$  in  $F^b$  to the class of  $(1 - x) \tilde{\lambda} x$ . If  $v'$  is trivial then this is still the map  $\delta_{2,F}$  of (3.2).

**Lemma 3.14.** *The following diagram commutes:*

$$\begin{array}{ccc} C_3(\mathcal{L}) & \xrightarrow{d} & C_2(\mathcal{L}) \\ \downarrow f_{3,F} & & \downarrow f_{2,F} \\ \mathbb{Z}[F^b] & \xrightarrow{\delta_{2,F}^{v'}} & \tilde{\lambda}^2 F^\times / v' \tilde{\lambda} F^\times \end{array} .$$

<sup>1</sup>Goncharov, in [29, §3], maps this to  $(-1) \tilde{\lambda} (-1) + b \tilde{\lambda} a$ .



*Proof.* We note first that for each nondegenerate generator  $(l_0, \dots, l_4)$  of  $C_4(\mathcal{L})$  one has

$$(f_{3,F} \circ d)((l_0, \dots, l_4)) = \sum_{i=0}^4 (-1)^i \text{cr}_3(\overline{l_0}, \dots, \widehat{\overline{l_i}}, \dots, \overline{l_4}),$$

where  $\overline{l_0}, \dots, \overline{l_4}$  are distinct points in  $\mathbb{P}_F^1$  and  $\widehat{\overline{l_i}}$  indicates that the term  $\overline{l_i}$  is omitted.

In view of the invariance of  $\text{cr}_3$  under the action of  $\text{GL}_2(F)$  and the fact that for  $\text{cr}_3$  we can use points in  $\mathbb{P}_F^1$ , we may assume that the points are  $(1, 0), (0, 1), (1, 1), (x, 1)$  and  $(y, 1)$  for  $x \neq y$  in  $F^b$ . Then under  $f_{3,F} \circ d$  this yields the element (3.18).

Let now  $(l_0, \dots, l_4)$  be a degenerate generator. Then its image under  $f_{3,F} \circ d$  is trivial if  $\{l_0, \dots, l_4\}$  has at most three elements because then all of the terms in  $d(l_0, \dots, l_4)$  are degenerate. On the other hand, if  $\{l_0, \dots, l_4\}$  has four elements, then after cancelling possible identical terms in  $d(l_0, \dots, l_4)$  and applying  $\text{cr}_3$  to the result we see that it is of the form

$$[\text{cr}_3(\overline{m_1}, \dots, \overline{m_4})] - \text{sgn}(\sigma)[\text{cr}_3(\overline{m_{\sigma(1)}}, \dots, \overline{m_{\sigma(4)}})]$$

for a permutation  $\sigma$  in  $S_4$  with sign  $\text{sgn}(\sigma)$  and four distinct points  $\overline{m_i}$  in  $\mathbb{P}_F^1$ . The subgroup generated by these images coincides with the subgroup generated by the terms (3.19). (This shows, in particular, that the map  $f_{3,F}$  is alternating.) □

**Remark 3.20.**

- (i) If  $v'$  is finite of order  $a$ , then multiplying an element in  $\overline{B}(F)_{v'}$  in the bottom row of (3.16) by  $a$  gives an element in  $\overline{B}(F)$ .
- (ii) For  $\sigma : F \rightarrow \mathbb{C}$  an embedding of a number field, the map  $\mathbb{Z}[F^b] \rightarrow \mathbb{R}(1)$  in Theorem 3.4(iii), which maps a generator  $[x]$  to  $iD(\sigma(x))$ , by (3.3) induces a map  $\mathbb{D}_\sigma : \overline{\mathfrak{p}}(F) \rightarrow \mathbb{R}(1)$ .

**3.3.4.**

We next show that if  $|F| \geq 4$  then  $\overline{B}(F)$  as defined above is naturally isomorphic to a quotient of the ‘Bloch group’  $B(F)$  that is defined and studied by Suslin in [55] if  $F$  is infinite and treated in [61, Chap. VI, §5] for  $|F| \geq 4$ . This result motivates us to regard  $\overline{B}(F)$  as a modified Bloch group (which explains our choice of notation). In fact, we shall establish the precise relation between our groups  $\overline{B}(F)$  and  $\overline{\mathfrak{p}}(F)$  and the corresponding groups  $B(F)$  and  $\mathfrak{p}(F)$ .

Following those two sources, for a field  $F$  with  $|F| \geq 4$  we set its pre-Bloch group to be

$$\mathfrak{p}(F) = \frac{\mathbb{Z}[F^b]}{\langle [x] - [y] + \left[\frac{y}{x}\right] + \left[\frac{1-x}{1-y}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] \text{ with } x, y \text{ in } F^b, x \neq y \rangle}.$$

We then define its Bloch group  $B(F)$  to be the kernel of the homomorphism

$$\begin{aligned} \mathfrak{p}(F) &\rightarrow (F^\times \otimes F^\times)_\sigma \\ [x] &\mapsto x \overset{\sigma}{\otimes} (1-x), \end{aligned} \tag{3.21}$$

where we set

$$(F^\times \otimes F^\times)_\sigma := \frac{F^\times \otimes F^\times}{\langle x \otimes y + y \otimes x \text{ with } x, y \text{ in } F^\times \rangle}$$

and write  $a \overset{\sigma}{\otimes} b$  for the class in the quotient of an element  $a \otimes b$ .

We further recall the existence of an exact sequence

$$0 \rightarrow \text{Tor}(F^\times, F^\times)^\sim \rightarrow K_3(F)^{\text{ind}} \rightarrow B(F) \rightarrow 0, \tag{3.22}$$

natural in  $F$ , where  $\text{Tor}(F^\times, F^\times)^\sim$  denotes the unique nontrivial extension of  $\text{Tor}(F^\times, F^\times)$  by  $\mathbb{Z}/2\mathbb{Z}$  for  $F$  of characteristic different from 2 and  $\text{Tor}(F^\times, F^\times)$  otherwise and  $K_3(F)^{\text{ind}}$  is the cokernel of the natural homomorphism from the Milnor  $K$ -group  $K_3^M(F)$  to  $K_3(F)$ . We also recall that the element  $c_F = [x] + [1 - x]$  of  $B(F)$  is independent of  $x$  in  $F^b$  and has order dividing 6 by [55, Lem. 1.3, 1.5] or [61, VI.5.4] and that  $c_{\mathbb{Q}}$  in  $B(\mathbb{Q})$  has order 6 [55, Prop. 1.1].

3.3.5.

Lemma 3.17 implies that  $\bar{\mathfrak{p}}(F)$  is obtained by quotienting out  $\mathfrak{p}(F)$  by the subgroup generated by all elements of the form  $[x] + [x^{-1}]$  with  $x$  in  $F^b$  and  $[y] + [1 - y]$  with  $y$  in  $F^b$ .

Because the latter elements generate the same group as the element  $c_F$  defined above, we are motivated to consider the following short exact sequence of complexes (with vertical differentials):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \langle [x] + [x^{-1}] \text{ with } x \text{ in } F^b \rangle & \longrightarrow & \mathfrak{p}(F)/\langle c_F \rangle & \longrightarrow & \bar{\mathfrak{p}}(F) \longrightarrow 0 \\
 & & \downarrow f & & \downarrow & & \downarrow \partial_{2,F} \\
 0 & \longrightarrow & \langle x \overset{\sigma}{\otimes} (-x) \text{ with } x \text{ in } F^b \rangle & \longrightarrow & (F^\times \otimes F^\times)_\sigma & \longrightarrow & \tilde{\Lambda}^2 F^\times \longrightarrow 0.
 \end{array}$$

**Theorem 3.23.** *If  $|F| \geq 4$ , then the homomorphism  $f$  above is bijective. In particular, the diagram induces an isomorphism  $B(F)/\langle c_F \rangle \rightarrow \bar{B}(F)$ .*

*Proof.* A calculation shows that  $f$  maps the class of  $[x] + [x^{-1}]$  to  $x \overset{\sigma}{\otimes} (-x)$ , so  $f$  is surjective.

To prove that  $f$  is injective, recall that by [55, Lem. 1.2] or [61, VI.5.4], the map  $F^\times \rightarrow \mathfrak{p}(F)$  sending  $x$  to  $[x] + [x^{-1}]$  if  $x \neq 1$  and 1 to 0 is a homomorphism with  $(F^\times)^2$  in its kernel. We shall consider its composition with the quotient map to  $\mathfrak{p}(F)/\langle c_F \rangle$ , giving a surjective homomorphism

$$g: F^\times \rightarrow \langle [x] + [x^{-1}] \text{ with } x \text{ in } F^b \rangle,$$

with the target in  $\mathfrak{p}(F)/\langle c_F \rangle$ . If  $-1$  is a square, then we already know that  $[-1] + [-1] = 0$  in  $\mathfrak{p}(F)$ . If  $-1$  is not a square, then  $2 \neq 0$ , so  $2[-1] = 2c_F - 2[2] = 2c_F + 2[\frac{1}{2}] = 3c_F$  in  $\mathfrak{p}(F)$  (cf. [55, Lem. 1.4] or [61, VI.5.4]). In either case, we have that  $\{\pm 1\} \cdot (F^\times)^2 \subseteq \ker(g)$  and that  $\text{im}(g)$  is the subgroup generated by the classes of  $[x] + [x^{-1}]$  with  $x$  in  $F^b$ . We also want to consider  $\ker(f \circ g)$ . For this, we fix a basis  $\mathcal{B}$  of  $F^\times/(F^\times)^2$  as  $\mathbb{F}_2$ -vector space, making sure to include  $-1$  in  $\mathcal{B}$  if  $-1$  is not a square in  $F^\times$ . For  $b$  in  $\mathcal{B}$ , the homomorphism  $F^\times/(F^\times)^2 \rightarrow \mathbb{F}_2 \cdot b \simeq \mathbb{F}_2$  obtained from the projection onto  $\mathbb{F}_2 \cdot b$  can be applied twice in the tensor product in order to give a composite homomorphism  $F^\times \otimes F^\times \rightarrow F^\times/(F^\times)^2 \otimes F^\times/(F^\times)^2 \rightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \simeq \mathbb{F}_2$ . This induces a homomorphism  $h_b: (F^\times \otimes F^\times)_\sigma \rightarrow \mathbb{F}_2$ , mapping  $x \otimes y$  to the product of the coefficients of  $b$  in the classes of  $x$  and  $y$  in  $F^\times/(F^\times)^2$ . If  $x$  in  $F^\times$  is in  $\ker(f \circ g)$ , then  $h_b(x \overset{\sigma}{\otimes} (-x)) = 0$  for all  $b$ . If  $-1$  is a square, this means that  $x$  is a square. If  $-1$  is not a square, then  $x$  or  $-x$  must be a square. In either case, it follows that  $\ker(f \circ g) \subseteq \{\pm 1\} \cdot (F^\times)^2$ . Because  $\{\pm 1\} \cdot (F^\times)^2 \subseteq \ker(g)$  and  $g$  is surjective, it follows that  $f$  is injective.

Now that  $f$  is an isomorphism, the snake lemma implies the isomorphism in the theorem. □

**Remark 3.24.** Note that in the above proof it also follows that  $\ker(g) = \{\pm 1\} \cdot (F^\times)^2$ . Therefore,  $g$  induces an isomorphism from  $F^\times/\{\pm 1\} \cdot (F^\times)^2$  to the subgroup of  $\mathfrak{p}(F)/\langle c_F \rangle$  generated by the classes of  $[x] + [x^{-1}]$  with  $x$  in  $F^b$ , given by mapping the class of  $x$  to the class of  $[x] + [x^{-1}]$ , and  $f \circ g$  induces an isomorphism from  $F^\times/\{\pm 1\} \cdot (F^\times)^2$  to the subgroup of  $(F^\times \otimes F^\times)_\sigma$  generated by the  $x \overset{\sigma}{\otimes} (-x)$ , mapping the class of  $x$  to the class of  $x \overset{\sigma}{\otimes} (-x)$ .

3.3.6.

We can now state the main result of this section. It concerns the map  $\varphi_F$  in Theorem 3.4.

**Theorem 3.25.** *The map  $\varphi_F$  induces a homomorphism  $\psi_F: \overline{B}(F) \rightarrow K_3(F)_{\text{tf}}^{\text{ind}}$  for any field  $F$ .*

*Proof.* In view of Lemma 3.17, it suffices to show that  $\varphi_F$  is trivial on all elements of the form (3.18) and (3.19).

If  $F$  is a number field  $F$  then this follows from Theorem 3.4(iii), (3.3) and Borel’s theorem, Theorem 2.1, by letting  $\sigma$  run through all embeddings of  $F$  into  $\mathbb{C}$ .

In order to see that it holds for all fields  $F$  as in Theorem 3.4, we can tensor with  $\mathbb{Q}$ , in which case the construction underlying the construction of the map  $\varphi_F$  in Theorem 3.4 is the simplest case of the constructions that are made by the second author in [19].

One then verifies that the elements in (3.18) and (3.19) are trivial by working over  $\mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$  or  $\mathbb{Z}[x, x^{-1}, (1-x)^{-1}, y, y^{-1}, (1-y)^{-1}, (x-y)^{-1}]$  as the base schemes, along the lines of the proofs of [19, Prop. 6.1] and [20, Lem. 5.2]. We leave the precise details of this argument to an interested reader. □

**Remark 3.26.** In this remark we let  $\nu'$  be trivial and explain the advantages of the definitions that we have adopted in comparison to those used by Goncharov in [29].

For this, we recall that in (3.8) of loc. cit. a key role is played by the map in (3.21) that sends a generator  $[x]$  to  $x \overset{\sigma}{\otimes} (1-x)$ . Our group  $\tilde{\lambda}^2 F^\times$  is a quotient of  $(F^\times \otimes F^\times)_\sigma$  (cf. the diagram just before Theorem 3.23) and  $\delta_{2,F}$  maps  $[x]$  to the inverse of the image of  $x \overset{\sigma}{\otimes} (1-x)$  in  $\tilde{\lambda}^2 F^\times$ .

Now the map that Goncharov constructs from nondegenerate triples of nonzero points in  $F^2$  to the right-hand side of (3.21) is not itself  $\text{GL}_2(F)$ -equivariant because letting a matrix with determinant  $c$  act changes the result by  $c \otimes^\sigma (-c)$ . In addition, the calculation with the points  $(a, 0)$ ,  $(0, b)$ ,  $(1, 1)$  and  $(xc, c)$  in the proof of Lemma 3.14 would similarly result in the element  $x \overset{\sigma}{\otimes} (1-x) + c \overset{\sigma}{\otimes} (-c)$ , which is not what one wants.

Whilst these problems could be simply resolved by multiplying any of the relevant maps by a factor of two, this would in the end lead to either a smaller subgroup of  $K_3(F)_{\text{tf}}^{\text{ind}}$  if we multiply  $f_{3,F}$  by 2 or a (new) Bloch group that is too large (if we multiply Goncharov’s boundary map by 2; cf. [46] and many other papers). It is therefore better to avoid the problem by replacing the right-hand side of (3.21) as the target of the boundary map by its quotient  $\tilde{\lambda}^2 F^\times$ .

But the elements of the form  $[x] + [x^{-1}]$  that are in the kernel of  $\delta_{2,F}$  could then result in a potentially large and undesired subgroup in the kernel of the boundary map, even modulo the 5-term relations (3.18) (see Theorem 3.23 and its proof). To avoid this, we have also imposed the relations (3.19) when defining  $\overline{p}(F)$  by working with degenerate configurations.

### 3.4. Torsion elements in Bloch groups

In this section we study the torsion subgroup of the modified Bloch group  $\overline{B}(F)$  of a number field  $F$  by means of a comparison with the Bloch group  $B(F)$  defined by Suslin (and recalled in Subsection 3.3.4).

In this way, we find that  $\overline{B}(F)$  is torsion free if  $F$  is equal to either  $\mathbb{Q}$  or an imaginary quadratic number field, or is generated over  $\mathbb{Q}$  by a root of unity (of any given order). In the case of imaginary quadratic fields, this fact will then play an important role in Section 4.

#### 3.4.1.

For the sake of simplicity, we formulate and prove the next result only for number fields. In its statement, if  $p$  is a prime number, then we denote the  $p$ -primary torsion subgroup of a finitely generated abelian group by means of the subscript  $p$ . Because all torsion groups here are finite and cyclic, this determines their structures.

**Proposition 3.27.** *Let  $F \subset \overline{\mathbb{Q}}$  be a number field. For a prime  $p$ , let  $p^s$  be the number of  $p$ -power roots of unity in  $F$ , and let  $r$  be the largest integer such that the maximal totally real subfield  $\mathbb{Q}(\mu_{p^r})^+$  of  $\mathbb{Q}(\mu_{p^r})$  is contained in  $F$ . Then the orders of the  $p$ -power torsion subgroups in the various groups are as follows.*

Prime	$ \text{Tor}(F^\times, F^\times)_p^\sim $	$ K_3(F)_p^{\text{ind}} $	$ B(F)_p $	$ \overline{B}(F)_p $	Condition
$p \geq 5$	$p^s$	$p^r$	$p^r$	$p^r$	$\zeta_p \notin F$
			1	1	$\zeta_p \in F$
$p = 3$	$3^s$	$3^r$	$3^r$	$3^{r-1}$	$\zeta_3 \notin F$
			1	1	$\zeta_3 \in F$
$p = 2$	$2^{s+1}$	$2^{r+1}$	$2^{r-1}$	$2^{r-2}$	$\zeta_4 \notin F$
			1	1	$\zeta_4 \in F$

(Note  $r \geq 2$  and  $s \geq 1$  if  $p = 2$ , and  $r \geq 1$  if  $p = 3$ .)

*Proof.* We compute  $|K_3(F)_p^{\text{ind}}|$  (which is faster than using [61, Chap. IV, Prop. 2.2 and 2.3]).

Let  $A \subseteq \mathbb{Z}_p^\times$  be the image of  $\text{Gal}(\overline{\mathbb{Q}}/F)$  in  $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$ . Then there are identifications

$$K_3(F)_p^{\text{ind}} \simeq H^0(\text{Gal}(\overline{\mathbb{Q}}/F), \mathbb{Q}_p(2)/\mathbb{Z}_p(2)) \simeq \bigcap_{a \in A} \ker(\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{a^2-1} \mathbb{Q}_p/\mathbb{Z}_p),$$

where the first follows from [41, Cor. 4.6] and the second is clear.

We assume for the moment that  $p \neq 2$ . Then  $r = 0$  is equivalent with  $A \not\subseteq \{\pm 1\} \cdot (1 + p\mathbb{Z}_p)$ , so some  $a^2 - 1$  is in  $\mathbb{Z}_p^\times$  and the resulting kernel is trivial. For  $r \geq 1$ , we have  $A \subseteq \{\pm 1\} \cdot (1 + p^r\mathbb{Z}_p)$  but  $A \not\subseteq \{\pm 1\} \cdot (1 + p^{r+1}\mathbb{Z}_p)$ . Then  $1 + p^r\mathbb{Z}_p \subseteq A$  because  $p$  is odd; hence,  $A^2 = 1 + p^r\mathbb{Z}_p$  and the statement is clear.

To deal with the case  $p = 2$ , we note that  $A$  is the image of  $\text{Gal}(\overline{\mathbb{Q}}/F)$  in  $\text{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_2^\times = \{\pm 1\} \cdot (1 + 4\mathbb{Z}_2)$ . Then  $A \subseteq \{\pm 1\} \cdot (1 + 2^r\mathbb{Z}_2)$  but  $A \not\subseteq \{\pm 1\} \cdot (1 + 2^{r+1}\mathbb{Z}_2)$ , for some  $r \geq 2$ . In this case,  $A$  contains an element of  $\{\pm 1\} \cdot (1 + 2^r\mathbb{Z}_2^\times)$ , and  $A^2 = 1 + 2^{r+1}\mathbb{Z}_2$ , from which the statement follows.

We always have  $r \geq s$ . If  $\zeta_{2p}$  is in  $F$ , then  $r = s$  because  $\mathbb{Q}(\zeta_{p^r})^+(\zeta_{2p}) = \mathbb{Q}(\zeta_{p^r})$ . If  $\zeta_{2p}$  is not in  $F$ , then  $s = 0$  for  $p \neq 2$ , and  $s = 1$  for  $p = 2$ . The entries for  $|B(F)_p|$  are now immediate from (3.22). From this, we recover that  $B(\mathbb{Q})$  has order 6 and so is generated by  $c_{\mathbb{Q}}$ . Then we can compare the sequences (3.22) for the field  $\mathbb{Q}$  and for  $F$ . Using that  $K_3(F)_{\text{tor}}^{\text{ind}}$  is cyclic and that  $c_{\mathbb{Q}}$  maps to  $c_F$  under the injection  $K_3(\mathbb{Q})_{\text{tor}}^{\text{ind}} \rightarrow K_3(F)_{\text{tor}}^{\text{ind}}$ , it follows that  $3c_F$  has order 2 if and only if  $\zeta_4 \notin F$  and that  $2c_F$  has order 3 if and only if  $\zeta_3 \notin F$ : In order to have those orders, we must have  $|\text{Tor}(F^\times, F^\times)_p^\sim| = |\text{Tor}(\mathbb{Q}^\times, \mathbb{Q}^\times)_p^\sim|$  for  $p = 2$  or  $p = 3$ , respectively. (Cf. [55, Lem. 1.5].) This gives the entries for  $|\overline{B}(F)_p| = |(B(F)/\langle c_F \rangle)_p|$ .  $\square$

If  $F$  is a number field and  $p$  a prime number, then combining Theorem 3.23 and Proposition 3.27 gives a precise statement when  $\overline{B}(F)$  has no  $p$ -torsion.

**Theorem 3.28.** *Let  $F \subset \overline{\mathbb{Q}}$  be a number field. Then, for a given prime number  $p$ , the group  $\overline{B}(F)$  has no  $p$ -torsion if and only if the following condition is satisfied:*

- if  $p \geq 5$ , then either  $\mathbb{Q}(\zeta_p)^+ \not\subseteq F$  or  $\mathbb{Q}(\zeta_p) \subseteq F$ ;
- if  $p = 3$ , then either  $\mathbb{Q}(\zeta_9)^+ \not\subseteq F$  or  $\mathbb{Q}(\zeta_3) \subseteq F$ ;
- if  $p = 2$ , then either  $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\zeta_8)^+ \not\subseteq F$  or  $\mathbb{Q}(\zeta_4) \subseteq F$ .

*Proof.* Using Theorem 3.23, it is clear when  $\overline{B}(F)$  has nontrivial  $p$ -torsion. With notation as in Proposition 3.27, this is the case if and only if the following condition is satisfied:

- if  $p \geq 5$ , then  $r \geq 1$  and  $\zeta_p$  is not in  $F$ ;
- if  $p = 3$ , then  $r \geq 2$  and  $\zeta_3$  is not in  $F$ ;
- if  $p = 2$ , then  $r \geq 3$  and  $\zeta_4$  is not in  $F$ .

Negating this statement for each prime  $p$  gives the claimed result. □

**Corollary 3.29.** *We have that*

- (i)  $\overline{B}(\mathbb{Q})$  is trivial;
- (ii)  $\overline{B}(F) \simeq \mathbb{Z}^{[F:\mathbb{Q}]/2}$  if  $F = \mathbb{Q}(\zeta_N)$  with  $N \geq 3$ ;
- (iii)  $\overline{B}(F) \simeq \mathbb{Z}$  if  $F$  is an imaginary quadratic field.

*In addition, in those cases the composition of the two homomorphisms  $\psi_F: \overline{B}(F) \rightarrow K_3(F)_{\text{tf}}^{\text{ind}}$  and  $K_3(F)_{\text{tf}}^{\text{ind}} \rightarrow \prod_{\sigma} \mathbb{R}(1)$ , where  $\sigma$  runs through the places of  $F$ , is injective.*

*Proof.* We first let  $F$  be any number field. Then  $\overline{B}(F)$  is a finitely generated abelian group of the same rank as  $K_3(F)$  by (3.22) and Theorem 3.23; hence, by Theorem 3.4(ii), the kernel of  $\psi_F$  is the torsion subgroup of  $\overline{B}(F)$ . Because of the behaviour of the regulator with respect to complex conjugation, in Theorem 2.1(iii) we only have to consider all places of  $F$ , not all embeddings into  $\mathbb{C}$ .

It therefore suffices to check that, for every prime number  $p$ ,  $\overline{B}(F)$  has no  $p$ -torsion if  $F$  is  $\mathbb{Q}$ , a cyclotomic field or an imaginary quadratic field. But this follows from Theorem 3.28. □

### 3.4.2.

We conclude this subsection with a result on  $\overline{\mathfrak{p}}(\mathbb{Q})$  that we shall use in Sections 4 and 5.

**Proposition 3.30.**

- (i) *The torsion subgroup of  $\overline{\mathfrak{p}}(\mathbb{Q})$  has order 2 and is generated by [2].*
- (ii) *If  $k$  is an imaginary quadratic field, then the natural map  $\overline{\mathfrak{p}}(\mathbb{Q}) \rightarrow \overline{\mathfrak{p}}(k)$  and its composition with  $\partial_{2,k}$  are injective when  $k \neq \mathbb{Q}(\sqrt{-1})$ , but for  $k = \mathbb{Q}(\sqrt{-1})$  both kernels are generated by [2].*

*Proof.* To prove claim (i) we note that Corollary 3.29 implies that  $\overline{\mathfrak{p}}(\mathbb{Q})$  injects into  $\tilde{\lambda}^2 \mathbb{Q}^\times$ . We also know from Proposition 3.5 (or Corollary 3.8) that the natural map  $\langle -1 \rangle \otimes \mathbb{Q}^\times \rightarrow \tilde{\lambda}^2 \mathbb{Q}^\times$  gives an isomorphism with the torsion subgroup of the latter.

So we want to compute the kernel of the natural map  $\langle -1 \rangle \otimes \mathbb{Q}^\times \rightarrow K_2(\mathbb{Q})$ . Using the tame symbol and the fact that  $\{-1, -1\}$  is nontrivial in  $K_2(\mathbb{Q})$ , one sees that this kernel is cyclic of order 2, generated by  $(-1) \tilde{\lambda} 2 = \partial_{2,\mathbb{Q}}([2])$ . And  $0 = [\frac{1}{2}] + [1 - \frac{1}{2}] = 2[\frac{1}{2}] = -2[2]$ .

Turning to claim (ii), we note that the kernel of the composition by Corollary 3.29(i) under  $\partial_{2,\mathbb{Q}}$  must inject into the kernel of  $\tilde{\lambda}^2 \mathbb{Q}^\times \rightarrow \tilde{\lambda}^2 k^\times$ , which we computed in Lemma 3.10. In particular, because this kernel is a torsion group, we see from Corollary 3.29(ii) that the kernel of the composition and that of  $\overline{\mathfrak{p}}(\mathbb{Q}) \rightarrow \overline{\mathfrak{p}}(k)$  coincide as the torsion of  $\overline{\mathfrak{p}}(k)$  injects into  $\tilde{\lambda}^2 k^\times$  under  $\partial_{2,k}$ .

In addition, by claim (i), those kernels are either trivial or generated by [2]. They contain [2] if and only if  $(-1) \tilde{\lambda} 2$  is in the kernel of  $\tilde{\lambda}^2 \mathbb{Q}^\times \rightarrow \tilde{\lambda}^2 k^\times$ , which by Lemma 3.10 holds if and only if  $k = \mathbb{Q}(\sqrt{-1})$ .

This completes the proof. □

**Remark 3.31.** Note that  $[2] = 0$  in  $\overline{\mathfrak{p}}(\mathbb{Q}(\sqrt{-1}))$  follows explicitly from (3.18) with  $x = \sqrt{-1}, y = -\sqrt{-1}$ , which gives  $[-1] = 0$ , because  $[2] = -[-1]$  by (3.19).

### 3.5. A conjectural link between the groups of Bloch and Suslin

If  $F$  is an infinite field, then (3.22) gives an isomorphism  $K_3(F)_{\text{tf}}^{\text{ind}} \xrightarrow{\simeq} B(F)_{\text{tf}}$  and Theorem 3.23 gives an isomorphism  $B(F)_{\text{tf}} \xrightarrow{\simeq} \overline{B}(F)_{\text{tf}}$ . (We ignore the case of finite fields because then all of these groups are trivial.)

By Theorem 3.25, one also knows that the homomorphism  $\varphi_F$  in Theorem 3.4 induces a homomorphism of the form  $\psi_F: \overline{B}(F) \rightarrow K_3(F)_{\text{tf}}^{\text{ind}}$ . This in turn induces a homomorphism

$$\psi_{F,\text{tf}}: \overline{B}(F)_{\text{tf}} \rightarrow K_3(F)_{\text{tf}}^{\text{ind}},$$

thereby allowing us to form the composition

$$K_3(F)_{\text{tf}}^{\text{ind}} \xrightarrow{\cong} B(F)_{\text{tf}} \xrightarrow{\cong} \overline{B}(F)_{\text{tf}} \xrightarrow{\psi_{F,\text{tf}}} \text{im}(\psi_{F,\text{tf}}) \subseteq K_3(F)_{\text{tf}}^{\text{ind}}. \tag{3.32}$$

Both  $B(F)$  and  $\text{im}(\psi_{F,\text{tf}})$  are described as the kernel of a map from an abelian group that is generated by elements of the form  $[x]$  for  $x$  in  $F^b$ , sending each  $[x]$  to the class of  $(1-x) \otimes x$  in either  $(F^\times \otimes F^\times)_\sigma$  or a variant like  $\tilde{\lambda}^2 F^\times$ . As mentioned in Subsection 1.1, these groups are widely expected to be closely related even though there is no obvious map between them, the former being constructed using group homology of  $\text{GL}_2(F)$  and the latter using relative  $K$ -theory.

Our approach provides the first concrete evidence (in situations in which the groups are nontrivial) to suggest that  $B(F)$  and  $\text{im}(\psi_{F,\text{tf}})$  should be related in a very natural way and that the latter is all of  $K_3(F)_{\text{tf}}^{\text{ind}}$ . To be specific, if  $F$  is a number field, then  $\psi_{F,\text{tf}}$  is injective by the proof of Theorem 3.28, so by Proposition 3.4 the composite map (3.32) is an injection of a finitely generated free abelian group into itself. One can therefore determine the (finite) index of  $\text{im}(\psi_{F,\text{tf}})$  in  $K_3(F)_{\text{tf}}^{\text{ind}}$  by comparing the results of the regulator map on  $\text{im}(\psi_{F,\text{tf}})$  and on  $K_3(F)_{\text{tf}}^{\text{ind}}$ . Extensive evidence that we have obtained by computer calculations in the case that  $F$  is imaginary quadratic (cf. Section 6) motivates us to formulate the following conjecture.

**Conjecture 3.33.** *If  $F$  is a number field, then  $\psi_{F,\text{tf}}$  is an isomorphism.*

**Remark 3.34.** As mentioned above, the map  $\psi_{F,\text{tf}}$  is injective and so the main point of Conjecture 3.33 is that  $\text{im}(\psi_{F,\text{tf}}) = K_3(F)_{\text{tf}}^{\text{ind}}$ . However, for a general number field  $F$ , this equality would not itself resolve the problem of finding an explicit description of the resulting composite isomorphism in (3.32). Of course, for  $F$  imaginary quadratic all groups occurring in the composite are isomorphic to  $\mathbb{Z}$ , and so the conjecture would imply that (3.32) is multiplication by  $\pm 1$ . (Recall that  $\psi_{F,\text{tf}}$  is itself natural up to a universal choice of sign because this is true for  $\varphi_F$ .)

It would also seem reasonable to hope that (3.32) has a very simple description for any infinite field  $F$ , such as, perhaps, being given by multiplication by some integer that is independent of  $F$ . Assuming this to be the case, our numerical calculations would imply that this integer is  $\pm 1$ . If true, this would in turn imply that the isomorphism  $K_3(F)_{\text{tf}}^{\text{ind}} \rightarrow B(F)_{\text{tf}}$  constructed by Suslin in [55] could be given a more direct, and more directly  $K$ -theoretical, description, at least up to sign, as the inverse of the composite isomorphism  $B(F)_{\text{tf}} \rightarrow \overline{B}(F)_{\text{tf}} \rightarrow K_3(F)_{\text{tf}}^{\text{ind}}$  where the first map is induced by Theorem 3.23 and the second is  $\psi_{F,\text{tf}}$ .

#### 4. A geometric construction of elements in the modified Bloch group

Let  $k$  be an imaginary quadratic number field and  $\mathcal{O}$  its ring of algebraic integers. In this section, we shall use a geometric construction, the Voronoi theory of Hermitian forms, to construct a nontrivial element  $\beta_{\text{geo}}$  in  $\overline{B}(k) \simeq \mathbb{Z}$ .

To do this we shall invoke a tessellation of hyperbolic 3-space for  $k$ , based on perfect forms, to construct an element of the kernel of the homomorphism  $d: C_3(\mathcal{L}) \rightarrow C_2(\mathcal{L})$  that occurs in (3.11) with  $F = k$  and  $\nu' = \{1\}$ . By applying  $f_{3,k}$  to this element we shall then obtain  $\beta_{\text{geo}}$  by using the commutativity of the diagram (3.16).

Furthermore, we are able to explicitly determine the image of this element under the regulator map and compare it to the special value  $\zeta'_k(-1)$  by using a celebrated formula of Humbert. This will in particular show that the element  $\psi_k(\beta_{\text{geo}})$  of  $K_3(k)_{\text{tf}}^{\text{ind}}$  that is constructed in this geometric fashion generates a subgroup of index  $|K_2(\mathcal{O})|$  (cf. Corollary 4.10(i)).

##### 4.1. Voronoi theory of Hermitian forms

Our main tool is the polyhedral reduction theory for  $\text{GL}_2(\mathcal{O})$  developed by Ash [1, Chap. II] and Koecher [37], generalising work of Voronoi [56] on polyhedral reduction domains arising from the theory of

perfect forms (see [63, §3] and [22, §2, §6] for a description of the algorithms involved). We recall some details here to set notation.

We fix an embedding  $k \hookrightarrow \mathbb{C}$  and identify  $k$  with its image. We extend this identification to vectors and matrices as well. We use  $\bar{\phantom{x}}$  to denote complex conjugation on  $\mathbb{C}$ , which gives the nontrivial Galois automorphism on  $k$ . Let  $V = \mathcal{H}^2(\mathbb{C})$  be the 4-dimensional real vector space of  $2 \times 2$  complex Hermitian matrices with complex coefficients. Let  $C \subset V$  denote the codimension 0 open cone of positive definite matrices. Using the chosen complex embedding of  $k$ , we can view  $\mathcal{H}^2(k)$ , the  $2 \times 2$  Hermitian matrices with coefficients in  $k$ , as a subset of  $V$ . Define a map  $q: \mathcal{O}^2 \setminus \{0\} \rightarrow \mathcal{H}^2(k)$  by  $q(x) = x\bar{x}^t$ . For each  $x \in \mathcal{O}^2$ , we have that  $q(x)$  is on the boundary of  $C$ . Let  $C^*$  denote the union of  $C$  and the image of  $q$ .

The group  $\text{GL}_2(\mathbb{C})$  acts on  $V$  by  $g \cdot A = gA\bar{g}^t$ . The image of  $C$  in the quotient of  $V$  by positive homotheties can be identified with hyperbolic 3-space  $\mathbb{H}$ . The image of  $q$  in this quotient is identified with  $\mathbb{P}_k^1$ , the set of cusps. The action induces an action of  $\text{GL}_2(k)$  on  $\mathbb{H}$  and the cusps of  $\mathbb{H}$  that is compatible with other models of  $\mathbb{H}$  (see [24, Chap. 1] for descriptions of other models). We let  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}_k^1$ .

Each  $A \in V$  defines a Hermitian form  $A[x] = \bar{x}^t Ax$ , for  $x \in \mathbb{C}^2$ . Using the chosen complex embedding of  $k$ , we can view  $\mathcal{O}^2$  as a subset of  $\mathbb{C}^2$ .

**Definition 4.1.** For  $A \in C$ , we define the *minimum of  $A$*  as

$$\min(A) := \inf_{x \in \mathcal{O}^2 \setminus \{0\}} A[x].$$

Note that  $\min(A) > 0$  because  $A$  is positive definite. A vector  $v \in \mathcal{O}^2$  is called a *minimal vector of  $A$*  if  $A[v] = \min(A)$ . We let  $\text{Min}(A)$  denote the set of minimal vectors of  $A$ .

These notions depend on the fixed choice of the imaginary quadratic field  $k$ . Because  $\mathcal{O}^2$  is discrete in the topology of  $\mathbb{C}^2$ , a compact set  $\{z \mid A[z] \leq \text{bound}\}$  in  $\mathbb{C}^2$  gives a finite set in  $\mathcal{O}^2$ . Thus,  $\text{Min}(A)$  is finite.

**Definition 4.2.** We say a Hermitian form  $A \in C$  is a *perfect Hermitian form over  $k$*  if

$$\text{span}_{\mathbb{R}}\{q(v) \mid v \in \text{Min}(A)\} = V.$$

By a *polyhedral cone* in  $V$  we mean a subset  $\sigma$  of the form

$$\sigma = \left\{ \sum_{i=1}^n \lambda_i q(v_i) \mid \lambda_i \geq 0 \right\},$$

where  $v_1, \dots, v_n$  are nonzero vectors in  $\mathcal{O}^2$ . A set of polyhedral cones  $S$  forms a *fan* if the following two conditions hold. Note that a face here can be of codimension higher than 1.

1. If  $\sigma$  is in  $S$  and  $\tau$  is a face of  $\sigma$ , then  $\tau$  is in  $S$ .
2. If  $\sigma$  and  $\sigma'$  are in  $S$ , then  $\sigma \cap \sigma'$  is a common face of  $\sigma$  and  $\sigma'$ .

The reduction theory of Koecher [37] applied in this setting gives the following theorem.

**Theorem 4.3.** *There is a fan  $\tilde{\Sigma}$  in  $V$  with  $\text{GL}_2(\mathcal{O})$ -action such that the following hold.*

- (i) *There are only finitely many  $\text{GL}_2(\mathcal{O})$ -orbits in  $\tilde{\Sigma}$ .*
- (ii) *Every  $y \in C$  is contained in the interior of a unique cone in  $\tilde{\Sigma}$ .*
- (iii) *Any cone  $\sigma \in \tilde{\Sigma}$  with nontrivial intersection with  $C$  has finite stabiliser in  $\text{GL}_2(\mathcal{O})$ .*
- (iv) *The 4-dimensional cones in  $\tilde{\Sigma}$  are in bijection with the perfect forms over  $k$ .*

The bijection in claim (iv) of this result is explicit and allows one to compute the structure of  $\tilde{\Sigma}$  by using a modification of Voronoi’s algorithm [22, §2, §6]. Specifically,  $\sigma$  is a 4-dimensional cone in  $\tilde{\Sigma}$

if and only if there exists a perfect Hermitian form  $A$  such that

$$\sigma = \left\{ \sum_{v \in \text{Min}(A)} \lambda_v q(v) \mid \lambda_v \geq 0 \right\}.$$

Modulo positive homotheties, the fan  $\tilde{\Sigma}$  descends to a  $\text{GL}_2(\mathcal{O})$ -tessellation of  $\mathbb{H}$  by ideal polytopes. The output of the computation described above is a collection of finite sets  $\Sigma_n^*$ ,  $n = 1, 2, 3$ , of representatives of the  $\text{GL}_2(\mathcal{O})$ -orbits of the  $n$ -dimensional cells in  $\mathbb{H}^*$  that meet  $\mathbb{H}$ . The cells in each  $\Sigma_n^*$  have vertices described explicitly by finite sets of nonzero vectors in  $\mathcal{O}^2$ .

#### 4.2. Bloch elements from ideal tessellations of hyperbolic space

The collection of 3-cells

$$\Sigma_3^* = \{P_1, P_2, \dots, P_m\}$$

above gives rise, after choosing a triangulation of each, to an element in  $\overline{B}(k)$ , as follows.

We first establish a useful interpretation of a classical formula of Humbert in this setting. For the sake of brevity, we shall write  $\Gamma$  for  $\text{PGL}_2(\mathcal{O})$ .

**Lemma 4.4.** *Let  $\Gamma_{P_i}$  denote the stabiliser in  $\Gamma$  of  $P_i$ . Then one has*

$$\sum_{i=1}^m \frac{1}{|\Gamma_{P_i}|} \text{vol}(P_i) = -\pi \cdot \zeta'_k(-1).$$

*Proof.* One has

$$\sum_{i=1}^m \frac{1}{|\Gamma_{P_i}|} \text{vol}(P_i) = \text{vol}(\Gamma \backslash \mathbb{H}) = \frac{1}{8\pi^2} |D_k|^{\frac{3}{2}} \cdot \zeta_k(2). \tag{4.5}$$

Here the first equality is clear and the second is a celebrated result of Humbert (see [9], where the formula is given for general number fields). The claimed formula now follows because an analysis of the functional equation (2.9) shows that the final term in (4.5) is equal to  $-\pi \cdot \zeta'_k(-1)$ .  $\square$

We next subdivide each polytope  $P_i$  into ideal tetrahedra  $T_{i,j}$  with positive volume without introducing any new vertices,

$$P_i = T_{i,1} \cup T_{i,2} \cup \dots \cup T_{i,n_i}. \tag{4.6}$$

Here we assume that the subdivision is such that the faces of the tetrahedra that lie in the interior of the  $P_i$  match. An ideal tetrahedron  $T$  with vertices  $v_1, v_2, v_3, v_4$  has volume

$$\text{vol}(T) = D(\text{cr}_3(v_1, v_2, v_3, v_4)).$$

Here  $D$  denotes the Bloch-Wigner dilogarithm defined in Subsection 3.1.2 and  $\text{cr}_3$  denotes the cross-ratio discussed in Subsection 3.3.1. The ordering of vertices is chosen so that the right-hand side is positive.

To ease notation, we let  $r_{i,j}$  denote a resulting cross-ratio for  $T_{i,j}$ . We note that, though there is some ambiguity in choosing the order of the four vertices of  $T_{i,j}$  when defining this cross-ratio, the transformation rules in Remark 3.13 combine with the relations in (3.19) to imply that the induced element  $[r_{i,j}]$  of  $\overline{\mathfrak{p}}(k)$  is indeed independent of that choice.

We can now formulate the main result of this section (the proof of which will be given in Section 5).

By Corollary A.5 we know that each  $|\Gamma_{P_i}|$  divides 24, so the coefficients in the next theorem are integers. We also note that, by Proposition 3.30, the map  $\overline{\mathfrak{p}}(\mathbb{Q}) \rightarrow \overline{\mathfrak{p}}(k)$  is injective unless  $k = \mathbb{Q}(\sqrt{-1})$ , that the map  $2 \cdot \overline{\mathfrak{p}}(\mathbb{Q}) \rightarrow \overline{\mathfrak{p}}(k)$  is always injective and that  $2 \cdot \overline{\mathfrak{p}}(\mathbb{Q})$  is torsion free. Moreover, the

composition of the map  $\bar{p}(\mathbb{Q}) \rightarrow \bar{p}(k)$  with  $\partial_{2,k}: \bar{p}(k) \rightarrow \tilde{\Lambda}^2 k^\times$  is injective if  $k \neq \mathbb{Q}(\sqrt{-1})$ , and if  $k = \mathbb{Q}(\sqrt{-1})$  then this composition has the same kernel as the map  $\bar{p}(\mathbb{Q}) \rightarrow \bar{p}(k)$ . Therefore, the image of  $\bar{p}(\mathbb{Q})$  in  $\bar{p}(k)$  always injects into  $\tilde{\Lambda}^2 k^\times$  under  $\partial_{2,k}$ .

**Theorem 4.7.** *Let  $k$  be an imaginary quadratic number field, with the polytopes  $P_i$  and cross-ratios  $[r_{i,j}]$  chosen as above. Then the following hold.*

(i) *There exists a unique element  $\beta_{\mathbb{Q}}$  in the image of  $\bar{p}(\mathbb{Q})$  in  $\bar{p}(k)$ , such that the element*

$$\beta_{\text{geo}} = \beta_{\mathbb{Q}} + \sum_{i=1}^m \frac{24}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} [r_{i,j}]$$

*belongs to  $\bar{B}(k)$ . If  $k \neq \mathbb{Q}(\sqrt{-2})$ , then  $\beta_{\mathbb{Q}}$  belongs to the image of  $2 \cdot \bar{p}(\mathbb{Q})$ . In all cases the element  $\beta_{\text{geo}}$  is independent of the choice of representatives in  $\Sigma_3^*$  and the resulting subdivision (4.6) into tetrahedra.*

(ii) *If no stabiliser of an element in  $\Sigma_3^*$  or  $\Sigma_2^*$  has order divisible by 4, then there is a unique  $\tilde{\beta}_{\mathbb{Q}}$  in  $\bar{p}(\mathbb{Q})$ , which lies in  $2 \cdot \bar{p}(\mathbb{Q})$ , such that the element*

$$\tilde{\beta}_{\text{geo}} = \tilde{\beta}_{\mathbb{Q}} + \sum_{i=1}^m \frac{12}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} [r_{i,j}]$$

*belongs to  $\bar{B}(k)$ . Moreover, one has  $2 \cdot \tilde{\beta}_{\text{geo}} = \beta_{\text{geo}}$  and  $2 \cdot \tilde{\beta}_{\mathbb{Q}} = \beta_{\mathbb{Q}}$ .*

**Remark 4.8.** The situation for  $k$  equal to either  $\mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-2})$  is more complicated because the order of the stabiliser of the (in both cases unique) element of  $\Sigma_3^*$  has order 24. For  $k = \mathbb{Q}(\sqrt{-2})$  it can be subdivided in several different ways, resulting in the exception in Theorem 4.7(i). In fact, the subdivision in this case determines whether  $\beta_{\mathbb{Q}}$  either belongs or does not belong to  $2 \cdot \bar{p}(\mathbb{Q})$ , and both cases occur; see the argument in Section 5 for more details.

**Remark 4.9.** It is sometimes computationally convenient to avoid explicitly computing the element  $\beta_{\mathbb{Q}}$  in Theorem 4.7(i). In this regard it is useful to note that the injectivity in Corollary 3.29 combines with Theorem 3.4(iii) and the equality  $D(\bar{z}) = -D(z)$  in (3.3) to imply that

$$2 \cdot \beta_{\text{geo}} = \sum_{i=1}^m \frac{24}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} ([r_{i,j}] - [\bar{r}_{i,j}]).$$

### 4.3. Regulator maps and K-theory

As we fixed an injection of  $k$  into  $\mathbb{C}$ , by the behaviour of the regulator map  $\text{reg}_2$  with respect to complex conjugation (see (3.3)), we can compute regulators by considering only the composition

$$K_3(k) \rightarrow K_3(\mathbb{C}) \xrightarrow{\text{reg}_2} \mathbb{R}(1).$$

By slight abuse of notation, we shall denote this composition by  $\text{reg}_2$  as well.

**Corollary 4.10.** *Assume the notation and hypotheses of Theorem 4.7. Then the following hold.*

(i) *The element  $\psi_k(\beta_{\text{geo}})$  satisfies*

$$\frac{\text{reg}_2(\psi_k(\beta_{\text{geo}}))}{2\pi i} = -12 \cdot \zeta'_k(-1).$$

*It generates a subgroup of the infinite cyclic group  $K_3(k)_{\text{tf}}^{\text{ind}}$  of index  $|K_2(\mathcal{O})|$ .*

(ii) If no stabiliser of an element in  $\Sigma_3^*$  or  $\Sigma_2^*$  has order divisible by 4, then  $K_2(\mathcal{O})$  has even order and  $\psi_k(\tilde{\beta}_{\text{geo}})$  generates a subgroup of  $K_3(k)_{\text{tr}}^{\text{ind}}$  of index  $|K_2(\mathcal{O})|/2$ .

*Proof.* Before proving claim (i) we note that for each polytope  $P_i$  in  $\Sigma_3^*$  one has

$$\sqrt{-1} \cdot \text{vol}(P_i) = \sum_{j=1}^{n_i} \mathbb{D}([r_{i,j}]), \tag{4.11}$$

where  $\mathbb{D}$  is the homomorphism  $\bar{\mathfrak{p}}(k) \rightarrow \mathbb{R}(1)$  that is defined in Remark 3.20(ii) with respect to a fixed embedding  $k \rightarrow \mathbb{C}$ . This holds as  $\text{vol}(P_i) = \sum_{j=1}^{n_i} \text{vol}(T_{i,j})$  and for each  $i$  and  $j$  one has  $\sqrt{-1} \cdot \text{vol}(T_{i,j}) = \sqrt{-1} \cdot D(r_{i,j}) = \mathbb{D}([r_{i,j}])$ .

Turning now to the proof of claim (i), we observe that, because the element  $\beta_{\mathbb{Q}}$  that occurs in the definition of  $\beta_{\text{geo}}$  lies in the image of the map  $\bar{\mathfrak{p}}(\mathbb{Q}) \rightarrow \bar{\mathfrak{p}}(k)$ , it also lies in the kernel of the composite homomorphism  $\text{reg}_2 \circ \psi_k$ . One therefore computes that

$$\begin{aligned} \text{reg}_2(\psi_k(\beta_{\text{geo}})) &= \sum_{i=1}^m \frac{24}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} \text{reg}_2(\psi_k([r_{i,j}])) \\ &= 24 \cdot \sum_{i=1}^m \frac{1}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} \mathbb{D}([r_{i,j}]) \\ &= 24\sqrt{-1} \cdot \sum_{i=1}^m \frac{1}{|\Gamma_{P_i}|} \text{vol}(P_i) \\ &= -24\pi\sqrt{-1} \cdot \zeta'_k(-1), \end{aligned}$$

where the second equality follows from Theorem 3.4(iii) and Remark 3.20(ii), the third from (4.11) and the last from Lemma 4.4. This proves the first assertion of claim (i), and then the final assertion of claim (i) follows directly from Example (2.7).

For claim (ii) we note that, under the stated conditions, Theorem 4.7(ii) implies  $\beta_{\text{geo}} = 2 \cdot \tilde{\beta}_{\text{geo}}$ , so this follows from the final assertion of claim (i).  $\square$

**Remark 4.12.**

- (i) The condition in Theorem 4.7(ii) and Corollary 4.10(ii) holds for many fields  $\mathbb{Q}(\sqrt{-d})$ . Ordered by  $d$  the first five are  $\mathbb{Q}(\sqrt{-15})$ ,  $\mathbb{Q}(\sqrt{-30})$ ,  $\mathbb{Q}(\sqrt{-35})$ ,  $\mathbb{Q}(\sqrt{-39})$  and  $\mathbb{Q}(\sqrt{-42})$ . For the first, third and fifth of those  $\psi_k(\tilde{\beta}_{\text{geo}})$  generates  $K_3(k)_{\text{tr}}^{\text{ind}}$  as  $|K_2(\mathcal{O})| = 2$ .
- (ii) The example discussed in Subsection 5.3 shows that one cannot ignore the condition on the stabilisers of the elements of  $\Sigma_2^*$  in Theorem 4.7(ii) and Corollary 4.10(ii). Specifically, in this case the stabilisers of the elements of  $\Sigma_3^*$  have order 2 or 3 and one element of  $\Sigma_2^*$  has stabiliser of order 4, but  $\beta_{\text{geo}}$  generates  $\bar{B}(k)$  and so cannot be divided by 2.

**4.4. A cyclotomic description of  $\beta_{\text{geo}}$**

Let  $k$  be an imaginary quadratic field of conductor  $N$ . Fixing an injection  $k \rightarrow \mathbb{C}$ , the image is in the cyclotomic field  $F = \mathbb{Q}(\zeta_N)$  for  $\zeta_N := e^{2\pi i/N}$ . We shall identify  $\text{Gal}(F/\mathbb{Q})$  with  $(\mathbb{Z}/N\mathbb{Z})^\times$ .

The following result shows that the image of the element  $\beta_{\text{geo}}$  constructed in Theorem 4.7(i) under the induced map  $\bar{B}(k) \rightarrow \bar{B}(F)$  has a simple description in terms of elements constructed directly from roots of unity. (This result is, however, of very limited practical use because it is generally much more difficult to compute explicitly in  $\bar{\mathfrak{p}}(F)$  rather than in  $\bar{\mathfrak{p}}(k)$ .)

**Proposition 4.13.** *The image of  $\beta_{\text{geo}}$  in  $\bar{B}(F)$  is equal to  $N \sum_{\bar{a} \in \text{Gal}(F/k)} [\zeta_N^{\bar{a}}]$ .*

*Proof.* At the outset we note that there is a commutative diagram

$$\begin{array}{ccccccc}
 K_3(k)_{\text{tf}}^{\text{ind}} & \xrightarrow{\cong} & \overline{B}(k) & \xrightarrow{\psi_k} & K_3(k)_{\text{tf}}^{\text{ind}} & \xrightarrow{\text{reg}_2} & \mathbb{R}(1) \\
 \downarrow & & \downarrow & & \downarrow & & \parallel \\
 K_3(F)_{\text{tf}}^{\text{ind}} & \xrightarrow{\cong} & \overline{B}(F) & \xrightarrow{\psi_F} & K_3(F)_{\text{tf}}^{\text{ind}} & \xrightarrow{\text{reg}_2} & \mathbb{R}(1),
 \end{array}$$

where the isomorphisms are obtained from (3.22), as well as Theorems 3.23 and 3.28, and  $\text{reg}_2$  is the regulator map corresponding to our chosen embeddings of  $k$  and  $F$  into  $\mathbb{C}$ .

We further recall (from, for example, [54, Prop. 5.13] with  $X = Y' = \text{Spec}(F)$  and  $Y = \text{Spec}(k)$ ) that the composition  $K_3(F) \rightarrow K_3(k) \rightarrow K_3(F)$  of the norm and pullback is given by the trace and that the same is also true for the induced maps on  $K_3(F)_{\text{tf}}^{\text{ind}} = K_3(F)_{\text{tf}}^{\text{ind}}$  and  $K_3(k)_{\text{tf}}^{\text{ind}} = K_3(k)_{\text{tf}}^{\text{ind}}$ . By applying this fact to the element of  $K_3(F)_{\text{tf}}^{\text{ind}}$  corresponding to  $N[\zeta_N]$  in  $\overline{B}(F)$ , we deduce from the left-hand square in the above diagram that there exists an element  $\beta_{\text{cyc}}$  in  $\overline{B}(k)$  that maps to  $N \sum_{\overline{a} \in \text{Gal}(F/k)} [\zeta_N^{\overline{a}}]$  in  $\overline{B}(F)$ .

We now identify  $\text{Gal}(F/k)$  with a subgroup of index 2 of  $(\mathbb{Z}/N\mathbb{Z})^\times$ , which is the kernel of a primitive character  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  of order 2, corresponding to  $k$  (so  $\chi(-1) = -1$ ). Then from the above diagram and Theorem 3.4(iii), one finds that  $(2\pi i)^{-1} \cdot \text{reg}_2(\psi_k(\beta_{\text{cyc}}))$  is equal to

$$\begin{aligned}
 \frac{N}{4\pi i} \sum_{\overline{a} \in \text{Gal}(F/k)} \sum_{n \geq 1} \frac{\zeta_N^{na} - \zeta_N^{-na}}{n^2} &= \frac{N}{4\pi i} \sum_{\overline{a} \in \text{Gal}(F/\mathbb{Q})} \sum_{n \geq 1} \chi(\overline{a}) \frac{\zeta_N^{na}}{n^2} \\
 &= \frac{N^{3/2}}{4\pi} L(\mathbb{Q}, \chi, 2) \\
 &= -12\zeta'_k(-1)
 \end{aligned}$$

as  $\zeta_k(s) = \zeta_{\mathbb{Q}}(s)L(\mathbb{Q}, \chi, s)$ , with the Gauß sum  $\sum_{\overline{a} \in \text{Gal}(F/\mathbb{Q})} \chi(\overline{a})\zeta_N^{\overline{a}} = i\sqrt{N}$  (see [31, §58]). (Cf. the more general (and involved) calculation of [66, p. 421] or the calculation in the proof of [14, Th. 3.1] with  $r = -1$ ,  $\ell = 1$  and  $\mathcal{O} = \mathbb{Z}$ .)

According to Theorem 4.7 one has  $(2\pi i)^{-1} \cdot \text{reg}_2(\psi_k(\beta_{\text{geo}})) = -12 \cdot \zeta'_k(-1)$  as well; hence  $\psi_k(\beta_{\text{geo}}) = \psi_k(\beta_{\text{cyc}})$  by the injectivity of  $\text{reg}_2$  on  $K_3(k)_{\text{tf}}^{\text{ind}}$  (cf. Corollary 3.29). It then follows that  $\beta_{\text{geo}} = \beta_{\text{cyc}}$  because  $\psi_k$  is injective. □

### 5. The proof of Theorem 4.7

Throughout this section we fix an imaginary quadratic field  $k$  with ring of integers  $\mathcal{O}$ , as in Section 4. In Subsection 5.2 we also use the embedding of  $k$  into  $\mathbb{C}$  chosen there.

#### 5.1. A preliminary result concerning orbits

We start by proving a technical result that will play an important role in later arguments.

We set  $V = k^2 \setminus \{(0, 0)\}$  and let  $\Gamma$  denote either  $\text{SL}_2(\mathcal{O})$  or  $\text{GL}_2(\mathcal{O})$ .

##### Lemma 5.1.

- (i) For  $v$  in  $V$ ,  $\mathcal{O}^\times$  acts on the orbit  $\Gamma v$ , and the natural map  $V \rightarrow \mathbb{P}_k^1$  induces an injection of  $\Gamma v / \mathcal{O}^\times$  into  $\mathbb{P}_k^1$ , compatible with the action of  $\Gamma$ .
- (ii) For  $v_1$  and  $v_2$  in  $V$ , the images of  $\Gamma v_1 / \mathcal{O}^\times$  and  $\Gamma v_2 / \mathcal{O}^\times$  are either disjoint or coincide.

*Proof.* That  $\mathcal{O}^\times$  acts on  $\Gamma v$  is clear if  $k \neq \mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-3})$  because  $\mathcal{O}^\times = \{\pm 1\}$  and  $\Gamma$  contains  $\pm \text{id}_2$ . For the two remaining cases,  $\mathcal{O}$  is Euclidean, and based on iterated division with remainder in  $\mathcal{O}$  it is easy to find  $g$  in  $\text{SL}_2(\mathcal{O})$  with  $gv = \begin{pmatrix} c \\ 0 \end{pmatrix}$  for some  $c$  in  $k^\times$ , so if  $u$  is in  $\mathcal{O}^\times$ , then  $uv$  is in the orbit of

$v$  because  $g^{-1} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} g$  maps  $v$  to  $uv$ . Alternatively, this follows immediately from [24, Chap. 7, Lem. 2.1] because if  $v = (\alpha, \beta)$  and  $u$  is in  $\mathcal{O}^\times$ , then  $(\alpha, \beta) = (u\alpha, u\beta)$ .

Now assume that  $g_1v = cg_2v$  with  $c$  in  $k^\times$  and the  $g_i$  in  $\Gamma$ . Then  $v$  is an eigenvector with eigenvalue  $c$  of the element  $g_2^{-1}g_1$  in  $\Gamma$ , which has determinant in  $\mathcal{O}^\times$ . Hence,  $c$  is in  $\mathcal{O}^\times$ , and  $g_1v$  and  $g_2v$  give the same element in  $\Gamma v / \mathcal{O}^\times$ . So we do get the claimed injection, and it is clearly compatible with the action of  $\Gamma$ .

For the last part, suppose  $g_1v_1 = cg_2v_2$  for some  $c$  in  $k^\times$ ,  $v_i$  in  $V$  and  $g_i$  in  $\Gamma$ . Then  $\Gamma v_1 = c\Gamma v_2$  and the result is clear. □

**Proposition 5.2.** *If  $h$  is the class number of  $k$ , then we can find  $v_1, \dots, v_h$  in  $V$  such that  $\mathbb{P}_k^1$  is the disjoint union of the images of the  $\Gamma v_i / \mathcal{O}^\times$ . In particular, every element in  $\mathbb{P}_k^1$  lifts uniquely to some  $\Gamma v_i / \mathcal{O}^\times$ , and this lifting is compatible with the action of  $\Gamma$ .*

*Proof.* By [24, Chap. 7, Lem. 2.1] we may identify  $\Gamma \backslash V$  with the set of fractional ideals of  $\mathcal{O}$  and hence  $\Gamma \backslash V / k^\times = \Gamma \backslash \mathbb{P}_k^1$  with the ideal class group of  $k$ . We can then apply Lemma 5.1. □

## 5.2. The proof of Theorem 4.7

### 5.2.1.

We first establish some convenient notation and conventions.

For a 2-cell or, more generally, any flat polytope with vertices  $v_1, \dots, v_n$  in that order along its boundary, we indicate an orientation by  $[v_1, \dots, v_n]$  up to cyclic rotation. The inverse orientation corresponds to reversing the order of the vertices. If we want to denote the face with either orientation, we write  $(v_1, \dots, v_n)$ . In particular, an orientated triangle is the same as a 3-tuple  $[v_1, v_2, v_3]$  of vertices up to the action (with sign) of  $S_3$ . Similarly, an orientated tetrahedron is the same as a 4-tuple  $[v_1, v_2, v_3, v_4]$  up to the action (with sign) of  $S_4$ . Recall that we defined maps  $f_{3,k}$  and  $f_{2,k}$  just after (3.12). As mentioned above, the map  $f_{3,k}$  is compatible with the action of  $S_4$  by Remark 3.13 and (3.19). By the properties of  $cr_2$  mentioned just before (3.12), the map  $f_{2,k}$  is also compatible with the action of  $S_3$  on orientated triangles if we lift them to elements of  $C_3(\mathcal{L})$  for some suitable  $\mathcal{L}$ .

### 5.2.2.

By our discussion before the statement of the theorem, the uniqueness of elements  $\beta_Q$  and  $\tilde{\beta}_Q$  with the stated properties is clear. It is also clear that for any element  $\beta_Q$  in  $\bar{\mathfrak{p}}(k)$  the explicit sum  $\tilde{\beta}_{\text{geo}}$  belongs to  $\bar{\mathfrak{p}}(k)$ . In addition, the uniqueness of  $\beta_Q$  combines with the explicit expression for  $\tilde{\beta}_{\text{geo}}$  to imply that  $2\tilde{\beta}_Q = \beta_Q$  and, hence, that  $2\tilde{\beta}_{\text{geo}} = \beta_{\text{geo}}$ .

The fact that  $\beta_{\text{geo}}$  is independent of the subdivision (4.6) and of the choice of representatives in  $\Sigma_3^*$  also follows directly from the equality in Corollary 4.10(i) and the injectivity assertions in Corollary 3.29, once  $\beta_{\text{geo}}$  is known to be in  $\bar{B}(k)$ .

Hence, to prove Theorem 4.7, it suffices to prove the existence of  $\beta_Q$  and  $\tilde{\beta}_Q$  in the stated groups such that the sums  $\beta_{\text{geo}}$  and  $\tilde{\beta}_{\text{geo}}$  belong to  $\bar{B}(k)$ , and to do this we shall use the tessellation.

This argument is given in the next subsection. The basic idea is that, for  $k$  different from  $\mathbb{Q}(\sqrt{-1})$  and  $\mathbb{Q}(\sqrt{-2})$ , the sum  $\sum_{i=1}^m 12 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}]$  has integer coefficients and belongs to the kernel of  $\partial_{2,k}$  because the faces of the polytopes  $P_i$  with those multiplicities can be matched under the action of  $\Gamma$ . This argument uses that  $f_{2,k}$  is invariant under the action of  $\text{GL}_2(k)$  and behaves compatibly with respect to permutations, just as  $f_{3,k}$ .

The precise argument is complicated slightly by the fact that the subdivision (4.6) induces triangulations of the faces of the  $P_i$  that may not correspond, necessitating the introduction of ‘flat tetrahedra’, which give rise to the term  $\beta_Q$ . Also, the faces themselves may have orientation reversing elements in their stabilisers. But the resulting matching of faces does imply that the explicit sum  $\beta_{\text{geo}}$  lies in the kernel  $\bar{B}(k)$  of  $\partial_{2,k}$ .

For the special cases  $k = \mathbb{Q}(\sqrt{-1})$ ,  $k = \mathbb{Q}(\sqrt{-2})$  and  $k = \mathbb{Q}(\sqrt{-3})$ , we have to compute more explicitly for the single polytope involved in each case.

5.2.3.

We note first that each polytope  $P$  in the tessellation of  $\mathbb{H}$  comes with an orientation corresponding to it having positive volume. For a face (2-cell)  $F$  in the tessellation, we fix an orientation and consider the group  $\oplus_F \mathbb{Z}[F]$ , where we identify  $[F^\dagger]$  with  $-[F]$  if  $F^\dagger$  denotes  $F$  with the opposite orientation.

To  $P$  we associate its boundary  $\partial P$  in this group, where each face has the induced orientation. Because the action of  $\Gamma$  on  $\mathbb{H}$  preserves the orientation, it commutes with the boundary map.

We now need to do some counting. For a face  $[F]$ , we let  $\Gamma_F$  denote the stabiliser of the (nonoriented) face  $F$  and  $\Gamma_F^+$  the subgroup that preserves the orientation  $[F]$ . We note that the index of  $\Gamma_F^+$  in  $\Gamma_F$  is either 1 or 2.

Let  $P$  and  $P'$  be the polytopes in the tessellation that have  $F$  in their boundaries. If  $g$  is in  $\Gamma_F$  then  $gP = P$  or  $P'$ , and  $gP = P$  precisely when  $g$  is in  $\Gamma_F^+$ . Therefore  $\Gamma_F^+ = \Gamma_F \cap \Gamma_P$ .

It is convenient to distinguish between the following two cases for the  $\Gamma$ -orbits of  $F$ .

- $\Gamma_F = \Gamma_F^+$ . If  $P$  and  $P'$  in  $\Sigma_3^*$  are such that their boundaries each contain an element in the  $\Gamma$ -orbit of  $[F]$ , then  $P$  and  $P'$  are in the same  $\Gamma$ -orbit and hence are the same. Therefore, there is exactly one  $P$  in  $\Sigma_3^*$  that contains faces in the  $\Gamma$ -orbit of  $[F]$ . If two faces of  $P$  are in the  $\Gamma$ -orbit of  $[F]$ , then they are transformed into each other already by  $\Gamma_P$ . Hence, the number of elements in the  $\Gamma$ -orbit of  $F$  in  $\partial P$  is  $[\Gamma_P : \Gamma_F] = [\Gamma_P : \Gamma_F^+]$ . If  $P'$  is the element in  $\Sigma_3^*$  that has an element in the  $\Gamma$ -orbit of  $[F^\dagger]$  in its boundary (with  $P = P'$  and  $P \neq P'$  both possible), then there are  $[\Gamma_{P'} : \Gamma_F] = [\Gamma_{P'} : \Gamma_F^+]$  elements in the  $\Gamma$ -orbit of  $[F^\dagger]$  in the boundary of  $P'$ .
- $\Gamma_F \neq \Gamma_F^+$ . Note that in this case  $[\Gamma_F : \Gamma_F^+] = 2$ . Here  $[F]$  and  $[F^\dagger]$  are in the same  $\Gamma$ -orbit and, as above, one sees that there is only one element  $P$  of  $\Sigma_3^*$  that has elements in this  $\Gamma$ -orbit in its boundary. Any two such elements can be transformed into each other using elements of  $\Gamma_P$ , so there are  $[\Gamma_P : \Gamma_F^+]$  of those in the boundary of  $P$ .

5.2.4.

In this subsection we prove Theorem 4.7 for  $k$  not equal to  $\mathbb{Q}(\sqrt{-1})$ ,  $\mathbb{Q}(\sqrt{-2})$  or  $\mathbb{Q}(\sqrt{-3})$ .

In this case Corollary A.5 implies that the order of each group  $\Gamma_{P_i}$  divides 12, and so both the formal sum of elements in  $\Sigma_3^*$  given by

$$\pi_P = \sum_{i=1}^m \frac{12}{|\Gamma_{P_i}|} [P_i],$$

and the formal sum of tetrahedra resulting from the subdivision (4.6),

$$\pi_T = \sum_{i=1}^m \frac{12}{|\Gamma_{P_i}|} \sum_{j=1}^{n_i} [T_{i,j}],$$

have integral coefficients.

We extend the boundary map  $\partial$  to such formal sums, where for  $\pi_T$  the boundary is a formal sum of ideal triangles, contained in the original faces of the polytopes  $P_i$ . (Note that the subdivision (4.6) may introduce ‘internal faces’ inside each polytope, but by construction the parts of the boundaries of the tetrahedra here cancel exactly. This also holds after lifting all vertices to  $\mathcal{O}^2$  because there is no group action involved in order to match them. So we may, and shall, ignore those internal faces.)

The subdivision (4.6) induces a triangulation of each face  $F$  of each  $P_i$  in  $\Sigma_3^*$ . We let  $[\Delta_F]$  denote the induced triangulation. But if  $F$  is a face of such a  $P_i$  with  $[F] \neq [F^\dagger]$ , then the induced triangulations  $[\Delta_F]$  and  $[\Delta_{F^\dagger}]$  (which may come from different elements in  $\Sigma_3^*$ ) may not match. Similarly, if  $gF$  and  $F$  are both faces of  $P_i$  with  $g$  in  $\Gamma_{P_i}$ , then  $g[\Delta_F]$  and  $[\Delta_{gF}]$  may not match.

A typical example of nonmatching triangulations is that of a ‘square’ face  $F = [v_1, v_2, v_3, v_4]$  that is cut into two triangles using either diagonal, resulting in the triangulations  $[v_1, v_2, v_4] + [v_2, v_3, v_4]$  and  $[v_1, v_2, v_3] + [v_1, v_3, v_4]$ . But the boundary of the orientated tetrahedron  $[v_1, v_2, v_3, v_4]$  gives exactly the former minus the latter. Using induction on the number of vertices of a face  $F$  it is easily seen that any

two triangulations of  $[F]$  with the same orientation differ by the boundary of a formal sum of tetrahedra contained in  $F$ . Such tetrahedra have no volume, and the cross-ratio of its four cusps is in  $\mathbb{Q}^b$ . We refer to them as ‘flat tetrahedra’, and if  $[\Delta_F^1]$  and  $[\Delta_F^2]$  are two triangulations (with the same orientation) of an orientated face  $[F]$ , we shall write

$$[\Delta_F^1] \equiv [\Delta_F^2] \text{ modulo } \partial(\text{flat tetrahedra})'$$

if  $[\Delta_F^1] - [\Delta_F^2]$  is the boundary of a formal sum of such flat tetrahedra.

In particular, if  $[\Delta_F]$  and  $[\Delta_{F^\dagger}]$  are any triangulations of the faces  $[F]$  and  $[F^\dagger]$  (so with opposite orientation), then  $[\Delta_F] + [\Delta_{F^\dagger}] \equiv 0$  modulo  $\partial(\text{flat tetrahedra})$ .

We extend the boundary map to the free abelian group  $\oplus_{P \in \Sigma_3^*} \mathbb{Z}[P]$  by linearity. For a given  $[F]$ , in  $\partial(\pi_P)$  we find  $12 \cdot |\Gamma_F^+|^{-1}$  copies of  $[F]$  (up to the action of  $\Gamma$ ). If  $[F] \neq [F^\dagger]$  (i.e., if  $\Gamma_F = \Gamma_F^+$ ), then this equals the number of copies of  $[F^\dagger]$  and we combine  $[F]$  and  $[F^\dagger]$ .

We consider four cases, based on the exponents of 2 in  $|\Gamma_F|$  and  $|\Gamma_F^+|$ . Note that  $\Gamma_F^+$  is cyclic, so by Lemma A.2 and our assumptions on  $k$  we can write its order as  $2^s m$  with  $m = 1$  or 3 and  $s = 0$  or 1, with the case  $m = 3$  and  $s = 1$  not occurring. Then  $|\Gamma_F| = 2^t |\Gamma_F^+|$  with  $t = 0$  or 1.

- (1)  $s = t = 0$ . Here  $F^\dagger$  is not in the same  $\Gamma$ -orbit as  $F$ , and in  $\partial(\pi_P)$  the contribution of their  $\Gamma$ -orbits is  $\frac{12}{m} [F] + \frac{12}{m} [F^\dagger]$ , modulo the action of  $\Gamma$ . Then for  $\partial(\pi_T)$  they contribute  $\frac{12}{m} [\Delta_F] + \frac{12}{m} [\Delta_{F^\dagger}]$  modulo  $\partial(\text{flat tetrahedra})$  and modulo the action of  $\Gamma$ .
- (2)  $s = 0$  and  $t = 1$ . Here  $F$  and  $F^\dagger$  are in the same  $\Gamma$ -orbit, and in the boundary of  $\pi_P$ , up to the action of  $\Gamma$ , we have  $\frac{12}{m} [F] = \frac{6}{m} [F] + \frac{6}{m} [F^\dagger]$ . Then in  $\partial(\pi_T)$  we obtain  $\frac{6}{m} [\Delta_F] + \frac{6}{m} [\Delta_{F^\dagger}]$  modulo  $\partial(\text{flat tetrahedra})$  and modulo the action of  $\Gamma$ .
- (3)  $s = 1$  and  $t = 0$ . This is similar to case (1) but now  $[F]$  and  $[F^\dagger]$  both occur with coefficient 6 in  $\partial(\pi_P)$  because  $m = 1$ . In  $\partial(\pi_T)$  we obtain  $6[\Delta_F] + 6[\Delta_{F^\dagger}]$  modulo  $\partial(\text{flat tetrahedra})$  and modulo the action of  $\Gamma$ .
- (4)  $s = t = 1$ . This is similar to case (2) but now in  $\partial(\pi_P)$  we find  $6[F] = 3[F] + 3[F^\dagger]$ , again because  $m = 1$ ; hence in  $\partial(\pi_T)$  this gives  $3[\Delta_F] + 3[\Delta_{F^\dagger}]$  modulo  $\partial(\text{flat tetrahedra})$  and modulo the action of  $\Gamma$ .

We see that there exists some  $\alpha$ , a formal sum of flat tetrahedra, such that  $\pi_T + \alpha$  has boundary, up to the action of  $\Gamma$ , a formal sum with terms  $[t] + [t^\dagger]$  with  $t$  an ideal triangle. Lifting all cusps to  $\mathcal{L} = \Gamma v_1 / \mathcal{O}^\times \amalg \dots \amalg \Gamma v_h / \mathcal{O}^\times$  as in Proposition 5.2 and applying  $f_{3,k}$  as in (3.16), we see from the  $\Gamma$ -equivariance of  $f_{2,k}$  and the fact that this map is alternating, and so kills elements of the form  $[t] + [t^\dagger]$ , that  $\sum_{i=1}^m 12 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}] + \beta'$  is in the kernel of  $\partial_{2,k}^{\mathcal{O}^\times}$ , where  $\beta'$  is the image of  $\alpha$ .

Note that  $\beta'$  lies in the image of  $\bar{\mathfrak{p}}(\mathbb{Q})$  in  $\bar{\mathfrak{p}}(k)$ . Multiplying by  $|\mathcal{O}^\times| = 2$  and setting  $\beta_{\mathbb{Q}} := 2\beta'$  we complete the proof of Theorem 4.7(i) in this case.

The proof of Theorem 4.7(ii) is similar, starting with  $\sum_{i=1}^m 6 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}]$  (which has integer coefficients under the stated assumptions). In this case the coefficients in the above cases (1), (2) and (3) are divided by 2, and case (4) is ruled out by the assumptions.

### 5.2.5.

We now consider the special fields  $\mathbb{Q}(\sqrt{-1})$ ,  $\mathbb{Q}(\sqrt{-2})$  and  $\mathbb{Q}(\sqrt{-3})$ .

In each of these cases either  $|\Gamma_{P_i}|$  does not divide 12 or  $|\mathcal{O}^\times|$  is larger than 2. However, one also knows that  $\Sigma_3^*$  has only one element and its stabiliser has order 12 or 24 and so the result of Theorem 4.7(ii) does not apply. It is therefore enough to prove Theorem 4.7(i) for these fields.

If  $k = \mathbb{Q}(\sqrt{-1})$ , then  $\Sigma_3^*$  is an octahedron, with stabiliser isomorphic to  $S_4$ . Using that an ideal tetrahedron with positive volume in this octahedron must contain exactly two antipodal points, it is easy to see that the subdivision is unique up to the action of the stabiliser. Hence the resulting element under  $f_{3,k}$  is well defined. Computing it explicitly as  $\sum_{j=1}^4 [r_{1,j}] = 4[\omega]$  one finds that it is in the kernel of  $\partial_{2,k}$  as  $\omega^2 = -1$ , so we can simply take  $\beta_{\mathbb{Q}} = 0$ .

If  $k = \mathbb{Q}(\sqrt{-3})$ , then the polytope is a tetrahedron, with stabiliser isomorphic to  $A_4$ . Computing its image  $[r_{1,1}]$  under  $f_{3,k}$  explicitly one finds  $[\omega]$  with  $\omega^2 = \omega - 1$  and  $\partial_{2,k}([r_{1,1}]) = \omega \tilde{\lambda} (1 - \omega) = (-1) \tilde{\lambda} (-1)$  in  $\tilde{\lambda}^2 k^\times$ , which has order 2 by Remark 3.7. So we can again take  $\beta_{\mathbb{Q}} = 0$ .

If  $k = \mathbb{Q}(\sqrt{-2})$ , then the polytope is a rectified cube (i.e., a cuboctahedron), with stabiliser isomorphic to  $S_4$ , so it has six 4-gons and eight triangles as faces. By the commutativity of (3.16), for  $v' = \{1\}$ , we can compute  $\partial_{2,k}(\sum_j [r_{1,j}])$  by choosing lifts of all vertices involved and applying  $f_{2,k}$  to each of the lifted triangles (with correct orientation) of the induced triangulation of the faces of  $P_1$ . (This provides an alternative approach for  $\mathbb{Q}(\sqrt{-1})$  and  $\mathbb{Q}(\sqrt{-3})$  as well.) Note that any triangulation of the faces occurs for some subdivision: fix a vertex  $V$  and use the cones on all of the triangles that do not have  $V$  as a vertex.

So we must consider all triangulations. Giving the six 4-gons, one of the two possible triangulations at random resulted in  $(-1) \tilde{\lambda} (-1) = (-1) \tilde{\lambda} 2 = \partial_{2,k}([2])$  in  $\tilde{\lambda}^2 k^\times$ . The other triangulations we obtain from this one by triangulating one or more of the 4-gons differently. For each 4-gon, this adds the image under  $\partial_{2,k}$  of the cross-ratio of the corresponding flat tetrahedron. Because the 4-gons are equivalent under  $\Gamma$ , one easily computes this equals  $\partial_{2,k}([2])$ . So by Proposition 3.30 we find that  $\partial_{2,k}(\sum_j [r_{1,j}])$  equals either 0 or  $\partial_{2,k}([2])$ , and both occur. By Corollary 3.29, we must take  $\beta_{\mathbb{Q}} = 0$  or  $[2]$ , and by Proposition 3.30 the latter is not in  $2\bar{\mathfrak{p}}(\mathbb{Q})$ .

This completes the proof of Theorem 4.7.

### 5.2.6.

We make several observations concerning the above argument.

**Remark 5.3.** Let  $k$  not be equal to  $\mathbb{Q}(\sqrt{-1})$ ,  $\mathbb{Q}(\sqrt{-2})$  or  $\mathbb{Q}(\sqrt{-3})$ . One can try to find a better element than  $\beta_{\text{geo}}$  by going through the calculations in the proof of Theorem 4.7 after replacing  $\pi_P$  by an element of the form  $\sum_{i=1}^m M \cdot |\Gamma_{P_i}|^{-1} [P_i]$  for some positive integer  $M$  that is divisible by the orders of the stabilisers  $\Gamma_{P_i}$ . If we start with  $M$  equal to the least common multiple of the orders  $|\Gamma_{P_i}|$ , then we may have to multiply this element by 2 perhaps twice in the proof in order to ensure that the resulting element in  $\bar{\mathfrak{p}}(k)$  belongs to  $\bar{B}(k)$ :

- (1) in order to ensure that the boundary  $\partial$  of the resulting analogue of  $\pi_T$  is trivial up to the action of  $\Gamma$ , which is not automatic if some  $P_i$  has a face with reversible orientation under  $\Gamma$  and  $M \cdot |\Gamma_{P_i}|^{-1}$  is odd;
- (2) in order to ensure that  $\sum_{i=1}^m M \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}] + \beta'$  is in the kernel of  $\partial_{2,k}$  and not just  $\partial_{2,k}^{\otimes \times}$ , where  $M$  results from (1), and  $\beta'$  (coming from flat tetrahedra) is in  $\bar{\mathfrak{p}}(\mathbb{Q})$ , which we view as inside  $\bar{\mathfrak{p}}(k)$  by Proposition 3.30.

Note that in the second statement here we use  $|\otimes^\times| = 2$ , which excludes  $k = \mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-3})$ .

For our  $k$ , the Hermitian form  $(x, y) \mapsto \text{Nm}(x) + \text{Nm}(y) + \text{Nm}(x - y)$  on  $\mathbb{C}^2$ , with  $\text{Nm}: k \rightarrow \mathbb{Q}$  the norm, has minimal vectors  $\{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$ . By [22, Th. 2.7], this means that the triangle with vertices 0, 1 and  $\infty$  is a 2-cell of the tessellation. The element  $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$  in  $\Gamma$  of order 3 stabilises this triangle while preserving its orientation. Therefore, the 3-cells that share this triangle as faces have stabilisers with orders divisible by 3, and  $M$  is divisible by 3.

Also, the elements  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  have order 2 and generate a subgroup of  $\Gamma$  of order 4. The first has as axis of rotation the 1-cell connecting 0 and  $\infty$ , so the axis of rotation of the second, which meets this 1-cell, must meet either a 3-cell, or a 2-cell with vertices 0,  $\infty$  and purely imaginary numbers. In the first case we start with  $M$  divisible by 6. In the second case, (2) above ensures that  $M$  is even because the 2-cell reverses orientation under  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Because in this remark we are also assuming that  $k \neq \mathbb{Q}(\sqrt{-2})$ , we know from Corollary A.5 that the greatest common divisor of the orders of the  $\Gamma_{P_i}$  divides 12. So this method could lead to an element  $\beta_{\text{geo}}$  as in Theorem 4.7(i) but with 24 replaced by either 6, 12 or 24.

**Remark 5.4.** In our calculations, we find the following for all fields  $k$  that differ from  $\mathbb{Q}(\sqrt{-1})$ ,  $\mathbb{Q}(\sqrt{-2})$  and  $\mathbb{Q}(\sqrt{-3})$ :

- the gcd of the orders of the stabilisers  $\Gamma_{P_i}$  is 6 or 12; that is, 3 does not occur;
- the sum  $\sum_{i=1}^m 12 \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} ([r_{i,j}] - [\bar{r}_{i,j}])$  belongs to  $\bar{B}(k)$ , so that by Remark 4.9 and Corollary 3.29(iii), this must be another expression for  $\beta_{\text{geo}}$  in  $\bar{B}(k)$ .

Unfortunately, we have not been able to prove either of these statements in general.

### 5.3. An explicit example

With the same notation as before Theorem 4.7, we consider the element  $\beta' = \sum_{i=1}^m M \cdot |\Gamma_{P_i}|^{-1} \cdot \sum_{j=1}^{n_i} [r_{i,j}]$  of  $\bar{p}(k)$ , where  $M$  is the greatest common divisor of the orders  $|\Gamma_{P_i}|$ . As in Remark 5.3, the proof of Theorem 4.7 shows that there is a positive divisor  $e$  of  $2|\mathcal{O}^\times|$  such that  $\partial_{2,k}(e\beta')$  is in the image of the composition  $\bar{p}(\mathbb{Q}) \rightarrow \tilde{\lambda}^2\mathbb{Q}^\times \rightarrow \tilde{\lambda}^2k^\times$ , and it gives a way of computing  $e$  from the tessellation. But this depends on choices – for example, on how one pairs the faces of the  $P_i$  under the action of  $\Gamma$  – so the  $e$  found may not be optimal.

But one can also do this algebraically, by computing  $\beta'$  and determining a (minimal) positive integer  $e$  with  $e\partial_{2,k}(\beta')$  in the image of  $\bar{p}(\mathbb{Q})$ . For this we can use Corollary 3.8 and Remark 3.9: if  $S$  is a finite set of finite places of  $k$  such that  $\tilde{\lambda}^2\mathcal{O}_S^\times \subset \tilde{\lambda}^2k^\times$  contains  $\delta = \partial_{2,k}(\beta)$ , and  $A = \mathbb{Q}^\times \cap \mathcal{O}_S^\times$ , then one can compute whether  $e\delta$  is in the image of  $\tilde{\lambda}^2A$  or not. If this is the case then one can find its preimage in  $\tilde{\lambda}^2\mathbb{Q}^\times$  from Lemma 3.10, algorithmically determine whether an element in there gives the trivial element in  $K_2(\mathbb{Q})$  and, if so, express it in terms of  $\partial_{2,\mathbb{Q}}([x])$  with  $x$  in  $\mathbb{Q}^b$ .

Note that a different subdivision (4.6) might a priori give rise to a different  $\beta'$  and a different  $e$ , but the other choices are irrelevant in this algebraic approach.

For the reader’s convenience we illustrate this, and the methods of the proof of Theorem 4.7, in the special case that  $k = \mathbb{Q}(\sqrt{-5})$ . In particular, in this case we find that both methods give the same element, which generates  $\bar{B}(k)$ .

The lifts to  $\mathcal{O}^2$  (up to scaling by  $\mathcal{O}^\times$ ) of the vertices  $v_1, \dots, v_8$  in the two elements  $P_1$  and  $P_2$  of  $\Sigma_3^*$  are the columns of the matrix

$$\begin{pmatrix} \omega + 1 & 1 & 2 & 2 & 0 & -\omega & 1 & -\omega + 1 \\ -2 & 0 & \omega - 1 & \omega & 1 & 2 & -1 & \omega + 1 \end{pmatrix}. \tag{5.5}$$

Both polytopes are triangular prisms, which we write as  $[a, b, c; A, B, C]$ , for  $[a, b, c]$  and  $[A, B, C]$  triangles, with  $A$  above  $a$ , etc. Such a prism can be subdivided into orientated tetrahedra as  $[a, A, B, C] - [a, b, B, C] + [a, b, c, C]$ , and the resulting subdivision of its orientated boundary is

$$[A, B, C] + [a, A, C] - [a, c, C] + [a, b, B] - [a, A, B] + [b, c, C] - [b, B, C] - [a, b, c]. \tag{5.6}$$

Here the first and last terms correspond to the triangular faces, and the middle terms are grouped as pairs of triangles in the rectangular faces.

Then  $P_1 = [v_3, v_5, v_4; v_1, v_2, v_6]$  with  $\Gamma_{P_1}$  of order 2, generated by  $g_1 = (v_1v_3)(v_2v_4)(v_5v_6)$  in cycle notation on the vertices of  $P_1$  ( $v_7$  and  $v_8$  are mapped elsewhere). It interchanges the orientated faces  $[v_1, v_3, v_4, v_6]$  and  $[v_3, v_1, v_2, v_5]$  of  $P_1$ , and those faces have trivial stabilisers. The orientated face  $[v_2, v_6, v_4, v_5]$  is mapped to itself by  $g_1$  but its stabiliser is noncyclic of order 4, with one of the two orientation reversing elements acting as  $h_1 = (v_2v_5)(v_4v_6)$ . The two triangles  $[v_1, v_6, v_2]$  and  $[v_3, v_5, v_4]$  are interchanged by  $g_1$ , and both have stabilisers of order 2, with the one for  $[v_1, v_6, v_2]$  generated by  $(v_2v_6)$ .

We have  $P_2 = [v_1, v_8, v_3; v_2, v_7, v_5]$  with  $\Gamma_{P_2}$  of order 3, generated by  $g_2 = (v_1v_3v_8)(v_2v_5v_7)$ . The three orientated faces  $[v_2, v_7, v_8, v_1]$ ,  $[v_7, v_5, v_3, v_8]$  and  $[v_5, v_2, v_1, v_3]$  are all in the same  $\Gamma$ -orbit and have trivial stabilisers. The two triangles  $[v_1, v_8, v_3]$  and  $[v_2, v_5, v_7]$  are necessarily nonconjugate (even ignoring orientation) because  $\Gamma_{P_2}$  has order 3, but both have (orientation nonpreserving) stabiliser of order 6, which acts as the full permutation group on their vertices.

We now subdivide both  $P_1$  and  $P_2$  as stated just before (5.6). This gives an element

$$\pi'_T := \frac{1}{2}\pi_T = 3[v_3, v_1, v_2, v_6] - 3[v_3, v_5, v_2, v_6] + 3[v_3, v_5, v_4, v_6] + 2[v_1, v_2, v_7, v_5] - 2[v_1, v_8, v_7, v_5] + 2[v_1, v_8, v_3, v_5].$$

Applying  $f_{3,k}$  to this element results in

$$\beta' = 7[\frac{1}{3}\omega + \frac{2}{3}] - 3[-\frac{2}{3}\omega + \frac{5}{3}] + 3[\frac{1}{6}\omega + \frac{5}{6}] - 2[-\frac{1}{6}\omega + \frac{7}{6}]$$

in  $\bar{p}(k)$ . Using (5.6) one can easily compute the boundary  $\partial\pi'_T$ . Because

$$\Sigma_2^* = \{(v_1, v_3, v_4, v_6), (v_2, v_6, v_4, v_5), (v_3, v_4, v_5), (v_1, v_8, v_3), (v_7, v_2, v_5)\},$$

under the action of  $\Gamma$  we can move the resulting triangles to the eight triangles that result from the elements of  $\Sigma_2^*$ , with one of the ‘squares’ giving rise to four inequivalent triangles due to the two ways of triangulating a ‘square’, the other to only one inequivalent triangle.

In  $(\oplus_t \mathbb{Z}[t])_\Gamma$ , where  $t$  runs through the triangles, the triangular faces in  $\partial(\pi'_T)$  coming from those of  $P_1$  and  $P_2$  cancel under the action of  $\Gamma$  (this uses that the coefficient of  $[P_2]$  in  $\pi'_T$  is even and the triangular faces of  $P_2$  have orientation reversing elements in their stabilisers). Of course, the ‘internal’ triangles created by the subdivision into tetrahedra always cancel. Using  $g_1, g_2, h_1, h_2$  one moves the triangles coming from the ‘square’ faces to the five inequivalent triangles coming from  $[v_1, v_3, v_4, v_6]$  and  $[v_2, v_6, v_4, v_5]$ . This yields the sum of the elements

$$\partial[v_4, v_3, v_1, v_6] = 3([v_3, v_1, v_6] - [v_3, v_4, v_6] + [v_1, v_6, v_4] - [v_1, v_3, v_4]) + 2([v_3, v_4, v_6] - [v_3, v_1, v_6] + 2([v_1, v_3, v_4] - [v_1, v_6, v_4]))$$

and

$$3[v_5, v_4, v_6] - 3[v_5, v_2, v_6] = [v_5, v_4, v_6] - [v_5, v_2, v_6] - \partial[v_5, v_2, v_4, v_6],$$

where we used  $h_1$ .

So  $\partial(\pi_T - 2\partial[v_4, v_3, v_1, v_6] + 3\partial[v_5, v_2, v_4, v_6]) = 0$  modulo the action of  $\Gamma$  for  $\pi_T = 2\pi'_T = 6[P_1] + 4[P_2]$ . After multiplying by 2 in order to deal with the ambiguity of the lifts of the cusps to  $\mathcal{O}^2$ , we then find the element

$$\beta_{\text{geo}} = 4\beta' - 4[3] + 6[4/5] \in \bar{B}(k),$$

because  $\text{cr}_3([v_4, v_3, v_1, v_6]) = 3$  and  $\text{cr}_3([v_5, v_2, v_4, v_6]) = 4/5$ .

To see if one could do better, as discussed just before this example, we instead compute  $\partial_{2,k}$  of  $\beta'$ . This can be done easily using the matching of triangles under the action of  $\Gamma$  as before, using the commutativity of (3.16) for  $\partial_{2,k}^{\mathcal{O}^\times}$ , but because for this we have to lift the vertices to the column vectors in  $\mathcal{O}^2$  in (5.5) (and not up to scaling by  $\mathcal{O}^\times$ ) we pick up some additional torsion along the way. Alternatively, one can choose a finite set of finite primes  $S$  for  $k$  such that, for every  $[z]$  occurring in  $\beta'$ , both  $z$  and  $1 - z$  are  $S$ -units and compute in  $\tilde{\Lambda}^2 k^\times$  as in Remark 3.9 and Corollary 3.8. The result is

$$-(-4) \tilde{\Lambda} 5 + (-2) \tilde{\Lambda} 3 + (-1) \tilde{\Lambda} 2 + (-1) \tilde{\Lambda} (-\omega) - 2 \tilde{\Lambda} (-5)$$

in  $\tilde{\Lambda}^2 k^\times$ . Here we used that  $[v_5, v_4, v_6] - [v_5, v_2, v_6]$  under  $f_{2,k}$  is mapped to  $(-\frac{1}{2}) \tilde{\Lambda} (-\frac{\omega}{2}) - 2 \tilde{\Lambda} (-\omega) = (-1) \tilde{\Lambda} (-\omega) - 2 \tilde{\Lambda} (-5)$ . The first three terms are in the image of  $\bar{p}(\mathbb{Q})$ , and if we multiply the last by 2 then we obtain  $-4 \tilde{\Lambda} (-5) = -(-4) \tilde{\Lambda} (-5) = -(-4) \tilde{\Lambda} 5 + (-1) \tilde{\Lambda} (-1)$ , with the first again in the image of  $\bar{p}(\mathbb{Q})$ . If  $(-1) \tilde{\Lambda} (-1)$  would come from  $\bar{p}(\mathbb{Q})$  then it would come from its torsion by Proposition 3.30(ii), which is generated by  $[2]$ . By Lemma 3.10, the kernel of  $\tilde{\Lambda}^2 \mathcal{O}^\times \rightarrow \tilde{\Lambda}^2 k^\times$  has order 2

and is generated by  $(-1) \tilde{\lambda} (-5)$ , and one easily checks using Corollary 3.8 that both  $(-1) \tilde{\lambda} (-1)$  and  $(-1) \tilde{\lambda} (-1) + (-1) \tilde{\lambda} (-5)$  in  $\tilde{\lambda}^2 \mathbb{Q}^\times$  are neither trivial nor equal to  $\partial_{2,\mathbb{Q}}([2]) = (-1) \tilde{\lambda} 2$ . Therefore,  $(-1) \tilde{\lambda} (-1)$  in  $\tilde{\lambda}^2 k^\times$  is not in the image of  $\bar{p}(\mathbb{Q})$ . Multiplying by 2 again kills the term  $(-1) \tilde{\lambda} (-1)$ ; hence,  $4\beta - 4[3] + 6[5]$  is in  $\bar{B}(k)$  and is best possible for our choice of subdivision. (Note that this element equals  $\beta_{\text{geo}}$  above as  $[\frac{4}{5}] = -[\frac{1}{5}] = [5]$ . Also note that these calculations show that  $2\beta' - 2\bar{\beta}'$  is in  $\bar{B}(k)$ , in line with Remark 5.4, and that this element must also equal  $\beta_{\text{geo}}$ .)

In fact,  $K_2(\mathcal{O})$  is trivial by [2, §7], so by Corollary 4.10(i),  $\psi_k(\beta_{\text{geo}})$  is a generator of the infinite cyclic group  $K_3(k)_{\text{tf}}^{\text{ind}} = K_3(k)_{\text{tf}}$ ,  $\beta_{\text{geo}}$  is a generator of the infinite cyclic group  $\bar{B}(k)$  and the map  $\psi_k : \bar{B}(k) \rightarrow K_3(k)_{\text{tf}}^{\text{ind}} = K_3(k)_{\text{tf}}$  is an isomorphism.

Hence a different triangulation cannot give a better result. Also, both factors 2 in Remark 5.3 are necessary, so the obstruction of ‘incompatible lifts’ under the action of  $\Gamma$  is nontrivial.

### 6. Finding a generator of $K_3(k)$ and computing $|K_2(\mathcal{O}_k)|$

In this section, we again restrict to the case of an imaginary quadratic field  $k$  and set  $\mathcal{O} := \mathcal{O}_k$ . As in Section 4, we fix an embedding  $\sigma : k \rightarrow \mathbb{C}$  and regard  $k$  as a subfield of  $\mathbb{C}$ .

We explain how to combine an implementation of an algorithm of Tate’s, which produces a natural number that is known to be divisible by the order of  $K_2(\mathcal{O})$ , with either the result of Corollary 4.10(i) or just the known validity of the precise form of Lichtenbaum’s conjecture for  $k$  and  $m = 2$  (cf. Example 2.7), to deduce our main computational results.

At this stage we have successfully applied the first of these approaches to about 20 fields and the second to hundreds of fields. In this way, for example, we have, for all imaginary quadratic fields of discriminant bigger than  $-1000$  determined the order of  $K_2(\mathcal{O})$ , where not yet known, and a generator of the infinite cyclic group  $K_3(k)_{\text{tf}}^{\text{ind}}$  that lies in the image of the injective homomorphism  $\psi_k$  constructed in Theorem 3.25 (thereby verifying that  $k$  validates Conjecture 3.33) and hence also the Beilinson regulator value  $R_2(k)$ .

The results are available online [62]. In particular, for each of the listed imaginary quadratic number fields  $k$ , the element  $\beta_{\text{alg}}$  is such that its image  $\psi_k(\beta_{\text{alg}})$  generates  $K_3(k)_{\text{tf}}^{\text{ind}}$ , thus verifying Conjecture 3.33 for all of those fields. The element  $\beta_{\text{geo}}$  is the element of Theorem 4.7, obtained in the way described in Remark 5.4.

#### 6.1. Dividing $\beta_{\text{geo}}$ by $|K_2(\mathcal{O}_k)|$

##### 6.1.1.

The basic approach is as follows. An implementation by Belabas and Gangl [2] of (a refinement of) an algorithm of Tate gives an explicit natural number  $M$  divisible by  $|K_2(\mathcal{O})|$ . Because typically  $M = |K_2(\mathcal{O})|$ , for any element  $\alpha_{\text{geo}}$  of  $\ker(\delta_{2,k})$  that lifts  $\beta_{\text{geo}}$  we try to find an element  $\alpha$  in this kernel for which the difference  $M \cdot \alpha - \alpha_{\text{geo}}$  lies in the subgroup generated by (3.18) and (3.19). If one finds such an  $\alpha$ , then its class  $\beta$  in  $\bar{B}(k)$  satisfies  $\beta_{\text{geo}} = M \cdot \beta$ . From the result of Corollary 4.10(i) it then follows that  $|K_2(\mathcal{O})| = M$ , that  $\beta$  generates  $\bar{B}(k)$ , that  $\psi_k(\beta)$  generates  $K_3(k)_{\text{tf}}^{\text{ind}}$  and hence, by Theorem 3.4(iii), that  $R_2(k) = |\text{reg}_2(\psi_k(\beta))|$ .

##### 6.1.2.

To find a candidate element  $\alpha$  as above we first use the methods described in Subsection 6.3 to identify an element  $\alpha$  for which one can verify numerically that  $M \cdot \mathbb{D}_\sigma(\alpha) = \mathbb{D}_\sigma(\alpha_{\text{geo}})$ , with  $\mathbb{D}_\sigma$  the homomorphism from Remark 3.20(ii). We then aim to prove algebraically that  $M \cdot \beta = \beta_{\text{geo}}$  by writing the difference  $M \cdot \alpha - \alpha_{\text{geo}}$  as a sum of explicit relations of the form (3.18) and (3.19).

To complete this last step we use a strategy that can be used to investigate whether any element of the form  $\sum_i n_i [x_i]$ , where the  $n_i$  are integers and the  $x_i$  are in  $k^b$ , can be written as a sum of such relations, using suitable finite subsets  $U$  of  $k^b$ .

We let  $U$  consist of all  $x_i$  and their images under the 6-fold symmetry implied by the relations (3.19); that is, for  $u$  in  $U$  we also adjoin  $1 - u, u^{-1}, 1 - u^{-1}, (1 - u^{-1})^{-1} = \frac{-u}{1-u}$  and  $(1 - u)^{-1}$ .

Next, for  $u \neq v$  in  $U$ , we consider the element in  $\mathbb{Z}[k^b]$  obtained by putting  $x = u$  and  $y = v$  in (3.18). We use the result only if all five terms are in  $U$ .

We then form a matrix  $A$  of width  $|U|$ , as follows.

- o For the first row we write  $\sum_i n_i [x_i]$  in terms of the  $\mathbb{Z}$ -basis  $\{[u]$  with  $u$  in  $U\}$  of the subgroup  $\mathbb{Z}[U]$  of  $\mathbb{Z}[k^b]$ .
- o For each of the,  $n$ , say, 5-term relations that we have just generated, we add a row writing it in terms of the basis.
- o For each  $u$  in  $U$  we add rows corresponding to the relations  $[u] + [1 - u], [u] + [u^{-1}], [u] - [1 - u^{-1}], [u] + [\frac{-u}{1-u}]$  and  $[u] - [(1 - u)^{-1}]$ , resulting in, say,  $m$  rows in total.

Then the kernel of the right-multiplication by  $A$  on  $\mathbb{Z}^{1+n+m}$  (as row vectors) gives the relations among the various elements that we put into the rows of  $A$ . An element in this kernel with 1 as its first entry encodes a rewriting of  $\sum_i n_i [x_i]$  as the sum of elements as in (3.18) and (3.19).

Unfortunately, this straightforward method is rarely successful. Instead, we may have to enlarge  $U$ , and the computation can simply become too large. It was, however, done successfully, to some extent by trial and error, for several imaginary quadratic number fields.

**Example 6.1.** The most notable example among those is  $k = \mathbb{Q}(\sqrt{-303})$ , for which it is known from [2] that  $|K_2(\mathcal{O})| = 22$ . The results for this case are described in Appendix B.

**Remark 6.2.** We note that the method described above for verifying identities in  $\bar{B}(k)$  only depends on the definition of  $\bar{B}(k)$  in terms of the boundary map  $\delta_{2,k}$  and the relations (3.18) and (3.19) on  $\mathbb{Z}[k^b]$  that are used to define  $\bar{p}(k)$ . In particular, it does not rely on knowing the validity of Lichtenbaum’s conjecture and so, in principle, the same approach could be used to show that an element is trivial in  $\bar{p}(F)$  for any number field  $F$  (although, in practice, the computations are likely to quickly become unfeasibly large).

### 6.2. Finding a generator of $K_3(k)_{\text{tf}}^{\text{ind}}$ directly

This approach relies on the effective bounds on  $|K_2(\mathcal{O})|$  that are discussed above, the known validity of Lichtenbaum’s conjecture as in Example 2.7 and an implementation of the ‘exceptional  $S$ -unit’ algorithm (see Subsection 6.3) that produces elements in  $\bar{B}(k)$ . In particular, the reliance on Lichtenbaum’s conjecture means that the general applicability of this type of approach is currently restricted to abelian fields.

To describe the basic idea, we assume to be given an element  $\gamma$  that equals  $N_\gamma$  times a generator of the (infinite cyclic) group  $K_3(k)_{\text{tf}}^{\text{ind}}$  for some nonnegative integer  $N_\gamma$ .

Then one has  $|\text{reg}_2(\gamma)| = N_\gamma \cdot R_2(k)$  and so Example 2.7 implies that

$$-\frac{\text{reg}_2(\gamma)}{12 \cdot \zeta'_k(-1)} = \frac{N_\gamma}{|K_2(\mathcal{O})|}.$$

If one also has an explicit natural number  $M$  that is known to be divisible by  $|K_2(\mathcal{O})|$ , then

$$-M \frac{\text{reg}_2(\gamma)}{12 \cdot \zeta'_k(-1)} = N_\gamma \frac{M}{|K_2(\mathcal{O})|} \tag{6.3}$$

is a product of a nonnegative and a positive integer. Hence, if the left-hand side of this equality is numerically close to a natural number  $d_\gamma$ , then  $N_\gamma$  and  $M/|K_2(\mathcal{O})|$  are both divisors of  $d_\gamma$ . In particular, if one has an element  $\gamma$  with  $d_\gamma = 1$ , then one concludes both that  $N_\gamma = 1$  (so that  $\gamma$  generates  $K_3(k)_{\text{tf}}^{\text{ind}}$ ) and that  $|K_2(\mathcal{O})| = M$ . We would therefore have identified a generator of  $K_3(k)_{\text{tf}}^{\text{ind}}$  and determined  $|K_2(\mathcal{O})|$ .

To find suitable elements  $\gamma$  we proceed as described in Subsection 6.3 to generate elements  $\alpha$  in the subgroup  $\ker(\delta_{2,k})$  of  $\mathbb{Z}[k^b]$ . We then let  $\gamma$  be the image under  $\psi_k$  of the image of some such  $\alpha$  in  $\bar{B}(k)$ .

Note that it is not a priori guaranteed that a generator of  $K_3(k)_{\text{tr}}^{\text{ind}}$  is contained in  $\text{im}(\psi_k)$ . However, if this is the case (as it was in all of the examples we tested), then  $\psi_k$  is surjective and so one has verified that  $k$  validates Conjecture 3.33.

### 6.3. Constructing elements in $\ker(\delta_{2,k})$ via exceptional $S$ -units

#### 6.3.1.

In order to find enough elements in  $\ker(\delta_{2,k})$ , we fix a finite set  $S$  of finite places of  $k$  and consider ‘exceptional  $S$ -units’, where an  $S$ -unit  $x$  is exceptional if  $1 - x$  is also an  $S$ -unit.

To compute with such elements it is convenient to fix a basis of the  $S$ -units of  $k$ ; that is, a set of  $S$ -units that gives a  $\mathbb{Z}$ -basis of the  $S$ -units modulo torsion. (This is implemented in GP/PARI [48] as ‘bnfsunits’.) For each exceptional  $S$ -unit  $x$  we encode  $x$  and  $1 - x$  using the exponents that arise when they are expressed in terms of the basis and a suitable root of unity. Corollary 3.8 and Remark 3.9 then enable us to compute effectively in  $\tilde{\Lambda}^2 k^\times$  with the elements  $(1 - x) \tilde{\Lambda} x$ .

**Example 6.4.** In the case  $k = \mathbb{Q}(\sqrt{-11})$  and  $S = \{\wp_2, \wp_3, \bar{\wp}_3\}$  where  $\wp_2 = (2)$  is the unique prime ideal of norm 4 and  $\wp_3$  and  $\bar{\wp}_3$  denote the two prime ideals of norm 3 in  $\mathcal{O}$ , PARI provides the  $S$ -unit basis  $\mathcal{B} = \{b_1, b_2, b_3\}$  with  $b_1 = 2$ ,  $b_2 = \frac{-1+\sqrt{-11}}{2}$  and  $b_3 = \frac{-1-\sqrt{-11}}{2}$ . We find the exceptional  $S$ -unit  $x = \frac{5}{36} - \frac{\sqrt{-11}}{36}$  of norm  $\frac{1}{36}$ , for which  $1 - x$  has norm  $\frac{3}{4}$ , and write

$$x = -b_1^{-1} b_2^{-2}, \quad 1 - x = -b_1^{-1} b_2^{-2} b_3^3.$$

It follows that

$$(1 - x) \tilde{\Lambda} x = (-1) \tilde{\Lambda} (-1) + (-1) \tilde{\Lambda} b_1 + (-1) \tilde{\Lambda} b_3 + 3(b_1 \tilde{\Lambda} b_3) + 6(b_2 \tilde{\Lambda} b_3),$$

which corresponds to the element  $(\bar{1}, \bar{1}, \bar{0}, \bar{1}, 0, 3, 6)$  under the isomorphism in Corollary 3.8.

This approach effectively reduces the problem of finding elements in  $\ker(\delta_{2,k})$  to a concrete problem in linear algebra. Of course, one wants to choose a finite set  $S$  of finite places for which one can find sufficiently many exceptional  $S$ -units in  $k$  such that some linear combination of them in  $\ker(\delta_{2,k})$  gives a nontrivial element in (and preferably a generator of) the quotient  $\bar{B}(k)$ .

Note that, though one can check for nontriviality of an element  $\beta$  in  $\bar{B}(k)$  by simply verifying that its image under the map  $\mathbb{D}$  is numerically nontrivial, in order to conclude that  $\beta$  is trivial we need to know an explicit natural number  $M$  that is divisible by  $|K_2(\mathcal{O})|$ . Then the quantity on the left-hand side of (6.3) is numerically close to zero if and only if  $\gamma = \psi_k(\beta)$  is trivial. If that is the case, then the injectivity of  $\psi_k$  implies that the element  $\beta$  is itself trivial.

#### 6.3.2.

Because  $\bar{B}(k)$  is cyclic of infinite order, there exists a finite set  $S$  for which the above procedure can lead to a nontrivial element. By Remark 4.9 one can take it to comprise all of the places that divide any of the principal ideals  $\mathcal{O} \cdot r_{i,j}$ ,  $\mathcal{O} \cdot \bar{r}_{i,j}$ ,  $\mathcal{O} \cdot (1 - r_{i,j})$  and  $\mathcal{O} \cdot (1 - \bar{r}_{i,j})$  for the elements  $r_{i,j}$  in Theorem 4.7. In general, this set is far too large to be practical for the exceptional  $S$ -unit approach. Fortunately, however, in all of the cases investigated in this article we found that a much smaller set suffices. In fact, it is often enough to take  $S$  to comprise all places that divide either 2 or 3 or any of the first 10 (say) primes that split in  $k$ .

**Example 6.5.** In the case  $k = \mathbb{Q}(\sqrt{-303})$  it suffices to take  $S$  to be the set of places that divide either of 2, 11 and 13 (all of which split in  $k$ ) or 3 (which ramifies in  $k$ ). Imposing small bounds on the exponents with respect to a chosen basis, we already find 683 exceptional  $S$ -units in  $k$ . Setting  $\omega := (1 + \sqrt{-303})/2$ , GP/PARI’s [48] ‘bnfsunits’ gives as a basis of the  $S$ -units the set

$$\{-20 + 3\omega, 2, -4 - \omega, -36 - \omega, 4 - \omega, 28 - \omega, -12 + \omega\}$$

of norms  $2^{10}$ ,  $2^2$ ,  $2^5 \cdot 3$ ,  $2^7 \cdot 11$ ,  $2^3 \cdot 11$ ,  $2^5 \cdot 13$ , and  $2^4 \cdot 13$ , respectively.

Computing the kernel of  $\delta_{2,k}$  on the corresponding subgroup of  $\mathbb{Z}[k^b]$  gives a free  $\mathbb{Z}$ -module of rank several hundreds, but most of the elements of a  $\mathbb{Z}$ -basis for this kernel turn out to result in the trivial element of  $\bar{\rho}(k)$ ; that is, they correspond to relations of the type (3.18) and (3.19). In this case, with a set of exceptional units that differed from the one used in Appendix B, but again using that  $|K_2(\mathcal{O})| = 22$ , we found using the approach described in Subsection 6.2 that

$$\begin{aligned}
 & -46\left[\frac{-\omega-27}{64}\right] + 36\left[\frac{-\omega+15}{16}\right] - 14\left[\frac{-\omega+41}{16}\right] - 48\left[\frac{-\omega+8}{4}\right] - 62\left[\frac{-\omega+41}{4}\right] + 18\left[\frac{-\omega+41}{48}\right] + 34\left[\frac{-11}{2}\right] + 42[-143] - 16\left[\frac{-253\omega+2321}{6144}\right] + 28\left[\frac{-253\omega-495}{3328}\right] \\
 & + 158\left[\frac{-3\omega+23}{32}\right] + 4\left[\frac{-39\omega-221}{512}\right] + 120\left[\frac{-5\omega+73}{64}\right] + 18\left[\frac{-5\omega+49}{416}\right] + 44\left[\frac{\omega+91}{64}\right] + 70\left[\frac{\omega+27}{64}\right] - 12\left[\frac{\omega+91}{128}\right] - 36\left[\frac{\omega+1}{16}\right] + 82\left[\frac{\omega-2}{2}\right] + 92\left[\frac{\omega+7}{32}\right] - 36\left[\frac{\omega+40}{4}\right] \\
 & - 22\left[\frac{\omega-8}{4}\right] - 116\left[\frac{11}{2}\right] + 58\left[\frac{11}{8}\right] - 34\left[\frac{13}{2}\right] - 84\left[\frac{13}{4}\right] + 34\left[\frac{3\omega+20}{8}\right] + 14\left[\frac{9\omega-17}{352}\right]
 \end{aligned}$$

in  $\mathbb{Z}[k^b]$  belongs to  $\ker(\delta_{2,k})$  and that its image in  $\bar{B}(k)$  is sent by  $\psi_k$  to a generator of  $K_3(k)_{\text{lf}}^{\text{ind}}$ .

**Example 6.6.** We now set  $k = \mathbb{Q}(\sqrt{-4547})$ , so that  $\mathcal{O} = \mathbb{Z}[\omega]$  with  $\omega = \frac{1+\sqrt{-4547}}{2}$ . We recall that in [12] it was conjectured that  $|K_2(\mathcal{O})| = 233$ . In fact, though the program developed in loc. cit. showed that  $|K_2(\mathcal{O})|$  divides 233, the authors were unable to verify their conjecture because this would have required them to work in a cyclotomic extension of too high a degree.

By using the approach described in Subsection 6.2, we were now able to verify that  $|K_2(\mathcal{O})|$  is indeed equal to 233 and, in addition, that the element

$$\begin{aligned}
 & 132\left[\frac{-2\omega+5}{117}\right] - 2\left[\frac{-1}{12}\right] + 8\left[\frac{-2\omega-3}{1404}\right] - 6\left[\frac{-2\omega+5}{18}\right] - 14\left[\frac{-2\omega+1752}{19683}\right] - 2\left[\frac{-1}{2}\right] + 8\left[\frac{-2\omega-3}{27}\right] + 2\left[\frac{-1}{288}\right] + 2\left[\frac{-1}{3}\right] - 24\left[\frac{-2\omega+1752}{3159}\right] - 2\left[\frac{-1}{36}\right] + 128\left[\frac{-2\omega+5}{39}\right] \\
 & + 24\left[\frac{-2\omega-3}{4212}\right] + 74\left[\frac{-2\omega+5}{52}\right] + 12\left[\frac{-2\omega+5}{54}\right] - 12\left[\frac{-2\omega-840}{6591}\right] - 54\left[\frac{-\omega+421}{351}\right] + 40\left[\frac{-2\omega-3}{72}\right] - 16\left[\frac{-2\omega-3}{78}\right] + 12\left[\frac{-2\omega+5}{8}\right] - 10\left[\frac{-\omega+421}{468}\right] + 2\left[\frac{-13}{16}\right] + 4\left[\frac{-13}{18}\right] \\
 & - 6\left[\frac{-13}{24}\right] + 2\left[\frac{-13}{33}\right] - 2\left[\frac{-13}{33}\right] - 2\left[\frac{-13}{4}\right] - 8\left[\frac{-169}{324}\right] - 2\left[\frac{-4\omega-1680}{177957}\right] - 6\left[\frac{-208}{81}\right] - 2[-26] - 4\left[\frac{-26}{3}\right] + 2\left[\frac{-27}{3}\right] + 42\left[\frac{-3\omega+1263}{2197}\right] - 12\left[\frac{-31\omega+162}{13182}\right] \\
 & + 24\left[\frac{-31\omega-131}{2197}\right] - 4\left[\frac{-5\omega+6}{64}\right] - 8\left[\frac{-16\omega+6736}{2197}\right] + 2\left[\frac{-16\omega-4523}{2197}\right] - 26\left[\frac{2\omega-5}{1404}\right] + 2\left[\frac{2\omega+3}{18}\right] + 2\left[\frac{1}{18}\right] - 4\left[\frac{\omega+875}{1053}\right] - 4\left[\frac{\omega+875}{1296}\right] - 24\left[\frac{2\omega-5}{27}\right] + 12\left[\frac{2\omega+1750}{3159}\right] \\
 & - 2\left[\frac{1}{32}\right] + 14\left[\frac{2\omega-5}{351}\right] - 14\left[\frac{2\omega-5}{4912}\right] - 50\left[\frac{2\omega+3}{52}\right] - 78\left[\frac{2\omega+3}{54}\right] + 4\left[\frac{2\omega-842}{6591}\right] + 38\left[\frac{\omega+420}{351}\right] - 30\left[\frac{2\omega-5}{72}\right] - 2\left[\frac{2\omega-5}{78}\right] - 14\left[\frac{2\omega+3}{8}\right] + 14\left[\frac{\omega+420}{4056}\right] - 6\left[\frac{\omega+420}{468}\right] \\
 & - 6[117] - 2\left[\frac{13}{243}\right] - 2\left[\frac{13}{81}\right] - 4\left[\frac{169}{16}\right] - 14\left[\frac{169}{243}\right] + 4\left[\frac{169}{256}\right] - 10\left[\frac{3494\omega-13298}{2197}\right] - 16\left[\frac{4\omega-1684}{177957}\right] - 42\left[\frac{16\omega+4523}{8788}\right] + 2\left[\frac{243}{256}\right] + 2\left[\frac{26}{27}\right] - 2\left[\frac{26}{9}\right] + 4\left[\frac{3}{32}\right] \\
 & - 16\left[\frac{3\omega+1260}{2197}\right] + 10\left[\frac{31\omega+131}{13182}\right] - 24\left[\frac{31\omega-162}{2197}\right] - 6\left[\frac{39}{2}\right] + 6\left[\frac{39}{8}\right] + 8\left[\frac{8\omega+3360}{351}\right] + 20\left[\frac{8\omega+3360}{9477}\right] - 30\left[\frac{10\omega+2}{1053}\right] - 38\left[\frac{5\omega+1}{54}\right] + 6\left[\frac{5\omega+1}{64}\right] - 12\left[\frac{5\omega-6}{78}\right] \\
 & - 4\left[\frac{5\omega+1}{1053}\right] + 24\left[\frac{5\omega+1}{27}\right] + 8\left[\frac{10\omega-12}{729}\right] - 6[52] - 4\left[\frac{52}{81}\right] - 2\left[\frac{64}{81}\right] - 14\left[\frac{16\omega+4523}{4563}\right] + 4\left[\frac{841\omega-176104}{177957}\right]
 \end{aligned}$$

of  $\mathbb{Z}[k^b]$  belongs to  $\ker(\delta_{2,k})$  and that its image in  $\bar{B}(k)$  is sent by  $\psi_k$  to a generator of  $K_3(k)_{\text{lf}}^{\text{ind}}$ .

**Appendix A. Orders of finite subgroups**

In this appendix we again consider a fixed imaginary quadratic field  $k$ , embedded into  $\mathbb{C}$ . The main aim of this subsection is to prove, in Corollary A.5, that the least common multiple of the orders of finite subgroups of  $\text{PGL}_2(\mathcal{O})$  is either 12 or 24, with the latter being the case only if  $k = \mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-2})$ . The authors are not aware of a suitable reference for this in the literature or, in fact, of an explicit classification of types of finite subgroups of  $\text{PGL}_2(\mathcal{O})$  that do not lie in  $\text{PSL}_2(\mathcal{O})$ . Because this is not difficult, we include it for the sake of completeness.

Using the inclusions  $\text{PSL}_2(\mathcal{O}) \subset \text{PGL}_2(\mathcal{O}) \subset \text{PGL}_2(\mathbb{C}) = \text{PSL}_2(\mathbb{C})$ , our arguments are based on the following classical result [24, Chap. 2, Th. 1.6] that goes back to Klein [35].

**Proposition A.1.** *A finite subgroup of  $\text{PSL}_2(\mathbb{C})$  is isomorphic to a cyclic group of order  $m \geq 1$ , a dihedral group of order  $2m$  with  $m \geq 2$ ,  $A_4$ ,  $S_4$  or  $A_5$ . Further, all of these possibilities occur.*

It seems that the finite subgroups of  $\text{PSL}_2(\mathcal{O})$  have been studied more than those of  $\text{PGL}_2(\mathcal{O})$  even though it is harder to determine them.

An element  $\gamma$  in  $\text{SL}_2(\mathcal{O})$  of finite order has characteristic polynomial of the form  $x^2 + ax + 1$  with  $a$  in  $[-2, 2] \cap \mathcal{O}$  because the two roots must be conjugate roots of unity. Hence,  $a = \pm 2, \pm 1$  or  $0$ , from which it follows readily that the image  $\bar{\gamma}$  in  $\text{PSL}_2(\mathcal{O})$  of  $\gamma$  has order 1, 2 or 3. In view of Proposition A.1, this limits the possibilities of a finite subgroup of  $\text{PSL}_2(\mathcal{O})$  to the cyclic groups of orders 1, 2 or 3, the dihedral groups of order 4 or 6 and  $A_4$ . Cyclic groups of order 2 or 3 can be obtained already in the

subgroup  $\text{PSL}_2(\mathbb{Z})$ , generated by  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ , respectively. The occurrence of the dihedral groups of order 4 and 6, and of  $A_4$ , in  $\text{PSL}_2(\mathcal{O})$  for  $k = \mathbb{Q}(\sqrt{-d})$  with  $d$  a positive square-free integer, depends on the prime factorisation of  $d$ ; see [39, Satz 6.8].

We now consider finite subgroups of  $\text{PGL}_2(\mathcal{O})$ . An element in  $\text{PGL}_2(\mathcal{O})$  of odd order is contained in  $\text{PSL}_2(\mathcal{O})$ : if an odd power of a  $2 \times 2$ -determinant is a square in  $\mathcal{O}^\times$ , then so is the determinant. It follows that the only finite subgroups that can occur in  $\text{PGL}_2(\mathcal{O})$  but are not necessarily contained in  $\text{PSL}_2(\mathcal{O})$  are the cyclic groups of even order, the dihedral groups and  $S_4$ . In order to obtain a complete answer, we first look at elements of finite order in  $\text{PGL}_2(k)$ .

**Lemma A.2.** *Let  $\gamma$  be an element of finite order in  $\text{GL}_2(k)$  and  $\bar{\gamma}$  its image in  $\text{PGL}_2(k)$ . Write  $\text{ord}(\gamma)$  and  $\text{ord}(\bar{\gamma})$  for the respective orders of these elements.*

- (i) For  $k = \mathbb{Q}(\sqrt{-1})$ , one has  $\text{ord}(\gamma) \in \{1, 2, 3, 4, 6, 8, 12\}$  and  $\text{ord}(\bar{\gamma}) \in \{1, 2, 3, 4\}$ .
- (ii) For  $k = \mathbb{Q}(\sqrt{-2})$ , one has  $\text{ord}(\gamma) \in \{1, 2, 3, 4, 6, 8\}$  and  $\text{ord}(\bar{\gamma}) \in \{1, 2, 3, 4\}$ .
- (iii) For  $k = \mathbb{Q}(\sqrt{-3})$ , one has  $\text{ord}(\gamma) \in \{1, 2, 3, 4, 6, 12\}$  and  $\text{ord}(\bar{\gamma}) \in \{1, 2, 3, 6\}$ .
- (iv) For all other  $k$ , one has  $\text{ord}(\gamma) \in \{1, 2, 3, 4, 6\}$  and  $\text{ord}(\bar{\gamma}) \in \{1, 2, 3\}$ .

*Proof.* Assume that  $\gamma$  has order  $n$ , and let  $a(x)$  in  $k[x]$  be its minimal polynomial, so that  $a(x)$  divides  $x^n - 1$ . The statement is clear if  $a(x)$  splits into linear factors, so we may assume that  $a(x)$  is irreducible in  $k[x]$  and of degree 2. If  $a(x)$  is in  $\mathbb{Q}[x]$  then it is an irreducible factor in  $\mathbb{Q}[x]$  of  $x^n - 1$ , necessarily the  $n$ th cyclotomic polynomial because the  $m$ th cyclotomic polynomial divides  $x^m - 1$  if  $m$  divides  $n$ . So  $\varphi(n) = 2$ , and  $n = 3, 4$  or  $6$ . If  $a(x)$  is not in  $\mathbb{Q}[x]$ , then  $a(x)\overline{a(x)}$  is irreducible in  $\mathbb{Q}[x]$ , it must be the  $n$ th cyclotomic polynomial and  $k$  must be a subfield of  $\mathbb{Q}(\zeta_n)$ . Then  $\varphi(n) = 4$ , so  $n = 8$  and  $k = \mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-2})$ , or  $n = 12$  and  $k = \mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-3})$ . (Note that  $n = 5$  is excluded because  $\mathbb{Q}(\zeta_5)$  contains no imaginary quadratic field.) The statement about the order of  $\bar{\gamma}$  follows by taking into account the factorisation of  $x^n - 1$  over  $k[x]$ . In general, the  $2m$ th cyclotomic polynomial divides  $x^m + 1$ . But for  $k = \mathbb{Q}(\sqrt{-1})$  and  $n = 12$ , so  $m = 6$ , we also have  $x^6 + 1 = (x^3 - i)(x^3 + i)$  in  $k[x]$ , and  $a(x)$  divides one of those factors. □

**Remark A.3.** In the cell stabiliser calculation for  $k = \mathbb{Q}(\sqrt{-3})$  in [44] the symbol  $A_4$  should be a dihedral group of order 12 in  $\text{PGL}_2(\mathcal{O})$ , where the subgroup in  $\text{PSL}_2(\mathcal{O})$  is dihedral of order 6.

We can now determine the types of finite subgroups in  $\text{PGL}_2(\mathcal{O})$  that do not lie in  $\text{PSL}_2(\mathcal{O})$ .

**Proposition A.4.** *Let  $G$  be a finite subgroup of  $\text{PGL}_2(\mathcal{O})$  that is not contained in  $\text{PSL}_2(\mathcal{O})$ .*

- (i) For  $k$  not equal to  $\mathbb{Q}(\sqrt{-m})$  with  $m = 1, 2$  or  $3$ ,  $G$  is isomorphic to a cyclic group of order 2 or a dihedral group of order 4 or 6. For  $k = \mathbb{Q}(\sqrt{-1})$  and  $\mathbb{Q}(\sqrt{-2})$ ,  $G$  can also be isomorphic to a cyclic group of order 4, a dihedral group of order 8 or  $S_4$ . For  $k = \mathbb{Q}(\sqrt{-3})$ ,  $G$  can also be isomorphic to a cyclic group of order 6 or a dihedral group of order 12.
- (ii) All of the groups listed in claim (i) occur.

*Proof.* We already observed that an element of finite odd order in  $\text{PGL}_2(\mathcal{O})$  is contained in  $\text{PSL}_2(\mathcal{O})$ , so the groups listed in (i) are those not ruled out by combining Proposition A.1 with Lemma A.2.

It remains to show that all such groups occur. Various examples may, of course, exist in the literature, but for the sake of completeness we give some here. In fact, for  $S_4$  we use the stabiliser of the single element in  $\Sigma_3^*$  in our calculations for  $\mathbb{Q}(\sqrt{-1})$  respectively  $\mathbb{Q}(\sqrt{-2})$  (see Subsection 5.2.5).

*Cyclic examples.* If  $u$  in  $\mathcal{O}^\times$  is not a square, then  $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$  is not in  $\text{PSL}_2(\mathcal{O})$  and its order equals the order of  $u$ . This gives the required subgroups except for those of order 2 for  $\mathbb{Q}(\sqrt{-1})$  and of order 4 for  $\mathbb{Q}(\sqrt{-2})$ . The former can be obtained by using  $\begin{pmatrix} 0 & \sqrt{-1} \\ 1 & 0 \end{pmatrix}$ , which is not in  $\text{PSL}_2(\mathcal{O})$  and has order 2, and the latter by using  $\begin{pmatrix} 0 & 1 \\ 1 & \sqrt{-2} \end{pmatrix}$ , which is not in  $\text{PSL}_2(\mathcal{O})$  and has order 4.

*Dihedral examples.* If  $u$  in  $\mathcal{O}^\times$  is not a square and has order  $m = 2, 4$  or  $6$ , then  $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  generate a dihedral group of order  $2m$ . With  $u = -1$  this constructs a copy of  $D_4$ , except for  $\mathbb{Q}(\sqrt{-1})$ ,

but for this field we can use generators  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ \sqrt{-1} & 0 \end{pmatrix}$ . Taking  $u$  a generator of  $\mathcal{O}^\times$  gives a copy of  $D_8$  for  $\mathbb{Q}(\sqrt{-1})$  and a copy of  $D_{12}$  for  $\mathbb{Q}(\sqrt{-3})$ .

For  $k$  not equal to  $\mathbb{Q}(\sqrt{-1})$ , a suitable copy of  $D_6$  is generated by  $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , whereas for  $\mathbb{Q}(\sqrt{-1})$  we can use  $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & \sqrt{-1} \\ 1+\sqrt{-1} & -1 \end{pmatrix}$ . Finally, a copy of  $D_8$  for  $\mathbb{Q}(\sqrt{-2})$  is generated by  $\begin{pmatrix} 0 & 1 \\ 1 & \sqrt{-2} \end{pmatrix}$  and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

*S<sub>4</sub>-examples.* For  $\mathbb{Q}(\sqrt{-1})$ , the orders of  $\begin{pmatrix} 1 & \sqrt{-1}+1 \\ 0 & -1 \end{pmatrix}$ ,  $\begin{pmatrix} \sqrt{-1} & -1 \\ -1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & \sqrt{-1} \\ 0 & -\sqrt{-1} \end{pmatrix}$  are 2, 3 and 4, respectively, and they generate a subgroup isomorphic to  $S_4$ .

For  $\mathbb{Q}(\sqrt{-2})$ , one finds  $\begin{pmatrix} -2 & -\sqrt{-2}-1 \\ -\sqrt{-2}+1 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} -\sqrt{-2}-1 & -\sqrt{-2}+1 \\ 1 & \sqrt{-2} \end{pmatrix}$  and  $\begin{pmatrix} 2 & \sqrt{-2}+1 \\ \sqrt{-2}-1 & -2 \end{pmatrix}$  have orders 3, 3 and 2, respectively, and they also generate a subgroup isomorphic to  $S_4$ . □

**Corollary A.5.** *The least common multiple of the orders of the finite subgroups of  $\text{PGL}_2(\mathcal{O})$  is 24 if  $k$  is either  $\mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-2})$  and is 12 in all other cases.*

*Proof.* This is true for the groups listed in Proposition A.4, and the possible finite subgroups of  $\text{PSL}_2(\mathcal{O})$  (discussed before Lemma A.2) have order dividing 12. □

**Appendix B. A generator of  $K_3(k)_{\text{ff}}^{\text{ind}}$  for  $k = \mathbb{Q}(\sqrt{-303})$**

For an imaginary quadratic field  $k = \mathbb{Q}(\sqrt{-d})$ , with ring of integers  $\mathcal{O}$ , Browkin [11] has identified conditions under which the order  $|K_2(\mathcal{O})|$  is divisible by either 2 or 3 (for example, he shows that  $|K_2(\mathcal{O})|$  is divisible by 3 if  $d \equiv 3 \pmod{9}$ ).

Moreover, all of the coefficients in the linear combination  $\beta_{\text{geo}}$  that occurs in Theorem 4.7(i) are divisible by 2 if  $k$  is not equal to either  $\mathbb{Q}(\sqrt{-1})$  or  $\mathbb{Q}(\sqrt{-2})$ . In addition, if  $k$  is also not equal to  $\mathbb{Q}(\sqrt{-3})$ , then although Remark 5.3 shows that at least one of the coefficients in  $\beta_{\text{geo}}$  is not divisible by 3, one finds in practice that most of these coefficients are divisible by 3.

For these reasons, it can be relatively easy to divide  $\beta_{\text{geo}}$  by either 2 or 3. But no such arguments work for division by primes larger than 3, and this requires considerably more work.

It follows that if one uses the approach of Subsection 6.1, then any attempt to obtain a solution  $\beta$  in  $\bar{\mathfrak{p}}(k)$  to the equation  $|K_2(\mathcal{O})| \cdot \beta = \beta_{\text{geo}}$  or, equivalently (taking advantage of Remark 4.9), to the equation  $2|K_2(\mathcal{O})| \cdot \beta = 2\beta_{\text{geo}}$ , in order to find a generator of  $\bar{B}(k)$ , is likely to be much more difficult when  $|K_2(\mathcal{O})|$  is divisible by a prime larger than 3.

This observation motivates us to discuss the field  $k := \mathbb{Q}(\sqrt{-303})$ , for which  $\mathcal{O} = \mathbb{Z}[\omega]$  with  $\omega = (1 + \sqrt{-303})/2$ . We recall that  $k$  was conjectured in [12] and verified in [2] to be the imaginary quadratic field of largest discriminant for which  $|K_2(\mathcal{O})|$  is divisible by a prime larger than 3. More precisely, this order was first conjectured and later determined to equal 22.

We apply the technique described in Section 4. The quotient  $\text{PGL}_2(\mathcal{O}) \backslash \mathbb{H}$  has volume

$$\text{vol}(\text{PGL}_2(\mathcal{O}) \backslash \mathbb{H}) = -\pi \cdot \zeta'_k(-1) \approx 140.1729768601914879815382141215 \dots$$

The tessellation of  $\mathbb{H}$  consists of 132 distinct  $\text{PGL}_2(\mathcal{O})$ -orbits of 3-dimensional polytopes:

- 87 tetrahedra
- 29 square pyramids
- 13 triangular prisms
- 1 octahedron
- 2 hexagonal caps – a polytope with a hexagonal base, 4 triangular faces and 3 quadrilateral faces as shown in Figure B.1.

The stabiliser  $\Gamma_P$  in  $\text{PGL}_2(\mathcal{O})$  of each of these polytopes  $P$  is trivial except for eight polytopes  $P$ . It has order 2 for four triangular prisms and order 3 for one triangular prism, the octahedron and both hexagonal caps. By Theorem 4.7, the tessellation and stabiliser data give rise to an explicit element  $\beta_{\text{geo}}$ .

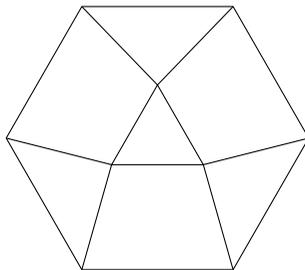


Figure B.1. Hexagonal cap

which we compute by using the ‘conjugation trick’ of Remark 4.9 as  $\frac{1}{2}(2\beta_{\text{geo}})$  (see Remark 5.4 for why we are allowed to divide by 2). The latter can be written as a sum of 188 terms  $a_i[z_i]$ , where  $a_i$  is in  $2\mathbb{Z}$  and  $z_i$  is in  $k^b$ . By Theorem 4.7(i) and (4.5) we have  $\sum a_i D(z_i) = 24 \text{vol}(\text{PGL}_2(\mathcal{O}) \backslash \mathbb{H})$  with  $D$  the Bloch-Wigner dilogarithm.

Using the algebraic approach described in Subsection 6.1, we can find  $\beta_{\text{alg}} = \sum b_j[w_j]$  in  $\overline{B}(k)$  with image under the Bloch-Wigner function bounded away from 0, and it turns out that it suffices to restrict the search to exceptional  $S$ -units where  $S$  consists of the prime ideals above  $\{2, 3, 11, 13, 19\}$ . Here 3 ramifies in  $\mathcal{O}$ , and the other primes are the first four primes that split in  $\mathcal{O}$ . One of the  $\beta_{\text{alg}}$  found with smallest positive dilogarithm value has 110 terms and all coefficients  $\pm 2$ .

By comparing  $\sum b_j D(w_j)$  with  $\sum a_i D(z_i)$  above, we expect  $\beta_{\text{geo}} - 22 \cdot \beta_{\text{alg}}$  to be trivial in  $\overline{B}(k)$ . We can prove this by writing a lift to  $\mathbb{Z}[k^b]$  explicitly as a sum of the elements specified in (3.18) and (3.19). A linear algebra calculation in Magma [10] shows that this can be done as an integral linear combination of 1,648 5-term relations, plus a good number of 2-term relations, so that, indeed,  $\beta_{\text{geo}} = 22 \cdot \beta_{\text{alg}}$ . (Note that  $\overline{B}(k)$  by Corollary 3.29 injects into  $\mathbb{R}$  under the dilogarithm, whereas one has to contend with torsion if attempting this calculation in  $B(k)$  instead.) The elements  $\beta_{\text{alg}}$  and  $\beta_{\text{geo}}$  are given below. The 5-term combinations are available online [62].

It follows from Corollary 4.10(i) that the resulting element  $\psi_k(\beta_{\text{alg}})$  generates  $K_3(k)_{\text{ff}}^{\text{ind}}$ . This also implies that  $\psi_k : \overline{B}(k) \rightarrow K_3(k)_{\text{ff}}^{\text{ind}}$  is bijective (as predicted by Conjecture 3.33).

$$\begin{aligned} \beta_{\text{geo}} = & 108\left[\frac{\omega+4}{6}\right] + 36\left[\frac{\omega+3}{5}\right] + 108\left[\frac{\omega+11}{13}\right] + 36\left[\frac{\omega+1}{5}\right] + 64\left[\frac{\omega+3}{4}\right] + 24\left[\frac{\omega+14}{26}\right] + 12\left[\frac{3\omega-3}{35}\right] + 36\left[\frac{3\omega}{38}\right] + 64\left[\frac{\omega}{4}\right] + 180\left[\frac{\omega+3}{8}\right] + 12\left[\frac{\omega-4}{22}\right] + 24\left[\frac{\omega+36}{32}\right] \\ & + 88\left[\frac{\omega+5}{10}\right] + 36\left[\frac{21\omega+192}{689}\right] + 24\left[\frac{\omega-4}{8}\right] + 136\left[\frac{3\omega+17}{32}\right] + 180\left[\frac{\omega+3}{11}\right] + 12\left[\frac{5\omega+44}{104}\right] + 30\left[\frac{9\omega+45}{106}\right] + 12\left[\frac{\omega-6}{2}\right] + 88\left[\frac{5\omega+23}{48}\right] + 20\left[\frac{5\omega-25}{3}\right] + 12\left[\frac{9\omega+38}{38}\right] \\ & + 12\left[\frac{\omega+19}{20}\right] + 48\left[\frac{\omega+4}{24}\right] + 24\left[\frac{5\omega-25}{128}\right] + 24\left[\frac{5\omega+148}{128}\right] + 60\left[\frac{\omega+24}{26}\right] + 24\left[\frac{42\omega+665}{1121}\right] + 24\left[\frac{6\omega+95}{77}\right] + 12\left[\frac{3\omega+17}{33}\right] + 12\left[\frac{\omega-1}{3}\right] + 48\left[\frac{\omega-1}{19}\right] + 24\left[\frac{7\omega-28}{55}\right] \\ & + 48\left[\frac{35\omega+644}{984}\right] + 24\left[\frac{4\omega+24}{59}\right] + 12\left[\frac{4\omega-28}{7}\right] + 24\left[\frac{7\omega+76}{55}\right] + 48\left[\frac{7\omega-28}{40}\right] + 60\left[\frac{\omega+3}{7}\right] + 12\left[\frac{\omega-5}{7}\right] + 24\left[\frac{5\omega-25}{56}\right] + 24\left[\frac{\omega+11}{9}\right] + 24\left[\frac{9\omega+99}{208}\right] + 60\left[\frac{4\omega+8}{41}\right] \\ & + 24\left[\frac{15\omega-4}{26}\right] + 36\left[\frac{\omega+27}{26}\right] + 36\left[\frac{\omega+27}{32}\right] + 12\left[\frac{2\omega+10}{53}\right] + 12\left[\frac{2\omega+41}{45}\right] + 12\left[\frac{\omega+75}{76}\right] + 6\left[\frac{\omega+5}{5}\right] + 6\left[\frac{5\omega}{81}\right] + 12\left[\frac{25\omega+1123}{1298}\right] + 12\left[\frac{11\omega+66}{118}\right] + 60\left[\frac{4\omega+29}{41}\right] \\ & + 24\left[\frac{15\omega+15}{26}\right] + 28\left[\frac{5\omega-28}{1272}\right] + 12\left[\frac{25\omega+1175}{1166}\right] + 12\left[\frac{\omega+5}{3}\right] + 72\left[\frac{3\omega+6}{41}\right] + 24\left[\frac{35\omega+440}{1007}\right] + 24\left[\frac{3\omega+20}{104}\right] + 24\left[\frac{\omega+58}{54}\right] + 24\left[\frac{9\omega+522}{583}\right] + 24\left[\frac{3\omega+20}{11}\right] \\ & + 12\left[\frac{\omega+37}{39}\right] + 12\left[\frac{6\omega+19}{247}\right] + 12\left[\frac{15\omega+980}{902}\right] + 12\left[\frac{\omega+7}{10}\right] + 24\left[\frac{3\omega-6}{13}\right] + 24\left[\frac{3\omega+57}{76}\right] + 24\left[\frac{\omega+36}{44}\right] + 12\left[\frac{\omega+76}{78}\right] + 12\left[\frac{27\omega+27}{150}\right] + 16\left[\frac{3\omega-21}{20}\right] + 28\left[\frac{3\omega+19}{58}\right] \\ & + 48\left[\frac{5\omega+67}{82}\right] + 28\left[\frac{15\omega+1370}{1298}\right] + 28\left[\frac{\omega+6}{10}\right] + 72\left[\frac{3\omega+32}{41}\right] + 12\left[\frac{8\omega+57}{209}\right] + 12\left[\frac{\omega+2}{3}\right] + 24\left[\frac{\omega+2}{7}\right] + 12\left[\frac{7\omega+64}{78}\right] + 12\left[\frac{16\omega+133}{53}\right] + 36\left[\frac{3\omega+15}{53}\right] \\ & + 24\left[\frac{\omega+4}{7}\right] + 48\left[\frac{2\omega+2}{39}\right] + 12\left[\frac{\omega-1}{2}\right] + 12\left[\frac{18\omega+684}{779}\right] + 12\left[\frac{9\omega-72}{308}\right] + 12\left[\frac{\omega+7}{7}\right] + 48\left[\frac{\omega+1}{4}\right] + 24\left[\frac{175\omega+1600}{4134}\right] + 30\left[\frac{9\omega+52}{106}\right] + 6\left[\frac{25\omega+867}{792}\right] + 24\left[\frac{7\omega+64}{50}\right] \\ & + 24\left[\frac{36\omega+1876}{2173}\right] + 24\left[\frac{\omega+40}{44}\right] + 24\left[\frac{3\omega+20}{14}\right] + 24\left[\frac{\omega+3}{4}\right] + 24\left[\frac{7\omega+137}{2173}\right] + 6\left[\frac{27\omega-28}{123}\right] + 6\left[\frac{\omega+79}{82}\right] + 12\left[\frac{9\omega+55}{118}\right] + 12\left[\frac{16\omega+155}{779}\right] + 28\left[\frac{\omega+6}{6}\right] \\ & + 4\left[\frac{3\omega+17}{5}\right] + 16\left[\frac{30\omega+627}{1007}\right] + 24\left[\frac{4\omega-24}{9}\right] + 12\left[\frac{84\omega+1480}{2173}\right] + 24\left[\frac{9\omega-36}{28}\right] + 12\left[\frac{27\omega+988}{826}\right] + 12\left[\frac{7\omega+69}{76}\right] + 24\left[\frac{7\omega-28}{198}\right] + 12\left[\frac{21\omega+151}{130}\right] + 12\left[\frac{\omega+84}{88}\right] \\ & + 6\left[\frac{81\omega+331}{574}\right] + 12\left[\frac{9\omega+113}{104}\right] + 24\left[\frac{4\omega+31}{59}\right] + 16\left[\frac{11\omega+40}{106}\right] + 8\left[\frac{165\omega-765}{11236}\right] + 8\left[\frac{11\omega-51}{15}\right] + 36\left[\frac{21\omega+476}{689}\right] + 28\left[\frac{3\omega+274}{295}\right] + 28\left[\frac{5\omega+30}{59}\right] + 12\left[\frac{9\omega-54}{7}\right] \\ & + 8\left[\frac{5\omega-20}{9}\right] + 8\left[\frac{9\omega-36}{440}\right] + 4\left[\frac{\omega-4}{15}\right] + 4\left[\frac{5\omega+95}{152}\right] + 4\left[\frac{3\omega+35}{50}\right] + 4\left[\frac{\omega+19}{25}\right] + 16\left[\frac{10\omega+209}{159}\right] + 20\left[\frac{\omega-5}{160}\right] + 28\left[\frac{\omega+3}{265}\right] - 12\left[\frac{-3\omega}{35}\right] - 12\left[\frac{-\omega-3}{22}\right] \\ & - 12\left[\frac{-5\omega+49}{104}\right] - 12\left[\frac{-\omega-5}{2}\right] - 12\left[\frac{-9\omega+47}{38}\right] - 12\left[\frac{-\omega+20}{20}\right] - 12\left[\frac{-2\omega+19}{33}\right] - 12\left[\frac{-\omega}{3}\right] - 12\left[\frac{-4\omega-24}{7}\right] - 12\left[\frac{-\omega-4}{7}\right] - 12\left[\frac{-2\omega+12}{53}\right] - 12\left[\frac{-2\omega+43}{45}\right] \\ & - 6\left[\frac{-\omega+6}{5}\right] - 6\left[\frac{-5\omega+81}{5}\right] - 12\left[\frac{-25\omega+1148}{1298}\right] - 12\left[\frac{-11\omega+77}{118}\right] - 12\left[\frac{-25\omega+1200}{1166}\right] - 12\left[\frac{-\omega+6}{3}\right] - 12\left[\frac{-\omega+38}{39}\right] - 12\left[\frac{-6\omega+25}{247}\right] - 12\left[\frac{-15\omega+995}{902}\right] \\ & - 12\left[\frac{-\omega+8}{10}\right] - 12\left[\frac{-\omega+77}{78}\right] - 12\left[\frac{-27\omega+54}{130}\right] - 16\left[\frac{-3\omega-18}{20}\right] - 12\left[\frac{-8\omega+65}{209}\right] - 12\left[\frac{-\omega+3}{3}\right] - 12\left[\frac{-7\omega+71}{78}\right] - 12\left[\frac{-16\omega+149}{53}\right] - 12\left[\frac{-\omega}{2}\right] - 12\left[\frac{-18\omega+702}{779}\right] \end{aligned}$$

$$\begin{aligned}
 & -12\left[\frac{-9\omega-63}{308}\right] - 12\left[\frac{\omega+8}{7}\right] - 6\left[\frac{-25\omega+892}{792}\right] - 6\left[\frac{-27\omega-1}{26}\right] - 6\left[\frac{\omega+80}{82}\right] - 12\left[\frac{-9\omega+64}{118}\right] - 12\left[\frac{-16\omega+171}{779}\right] - 4\left[\frac{-3\omega+20}{5}\right] - 12\left[\frac{-84\omega+1564}{2173}\right] \\
 & - 12\left[\frac{-27\omega+1015}{826}\right] - 12\left[\frac{-7\omega+76}{76}\right] - 12\left[\frac{-21\omega+172}{130}\right] - 12\left[\frac{-\omega+85}{88}\right] - 6\left[\frac{-81\omega+412}{574}\right] - 12\left[\frac{-9\omega+122}{104}\right] - 12\left[\frac{-9\omega-45}{7}\right] - 4\left[\frac{\omega-3}{15}\right] - 4\left[\frac{-5\omega+100}{152}\right] \\
 & - 4\left[\frac{-3\omega+38}{50}\right] - 4\left[\frac{-\omega+20}{25}\right].
 \end{aligned}$$

$$\begin{aligned}
 \beta_{\text{alg}} = & -2\left[\frac{-\omega+41}{52}\right] - 2\left[\frac{-\omega+2}{6}\right] - 2\left[\frac{-\omega-1}{6}\right] - 2\left[\frac{-\omega+92}{64}\right] - 2\left[\frac{-\omega+701}{676}\right] - 2\left[\frac{-\omega+12}{8}\right] - 2\left[\frac{-\omega+4}{8}\right] - 2\left[\frac{-\omega-3}{8}\right] - 2\left[\frac{-\omega+8}{12}\right] - 2\left[\frac{-\omega+12}{16}\right] - 2\left[\frac{-\omega+15}{16}\right] \\
 & - 2\left[\frac{-\omega+2}{16}\right] - 2\left[\frac{-\omega-11}{16}\right] - 2\left[\frac{-\omega-25}{22}\right] - 2\left[\frac{-\omega+15}{22}\right] - 2\left[\frac{-\omega+26}{22}\right] - 2\left[\frac{-3\omega+46}{66}\right] - 2\left[\frac{-\omega-14}{22}\right] - 2\left[\frac{-\omega+26}{24}\right] - 2\left[\frac{-\omega+2}{26}\right] - 2\left[\frac{-\omega+8}{32}\right] - 2\left[\frac{-\omega+25}{32}\right] - 2\left[\frac{-\omega+28}{32}\right] \\
 & - 2\left[\frac{-\omega-4}{32}\right] - 2\left[\frac{-\omega+389}{352}\right] - 2\left[\frac{-\omega-36}{352}\right] - 2\left[\frac{-\omega+8}{4}\right] - 2\left[\frac{-\omega-7}{4}\right] - 2\left[\frac{-\omega+41}{44}\right] - 2\left[\frac{-\omega+8}{48}\right] - 2\left[\frac{-15\omega+147}{104}\right] - 2\left[\frac{-4\omega+21}{13}\right] - 2\left[\frac{-4\omega+21}{33}\right] - 2\left[\frac{-21\omega+172}{352}\right] \\
 & - 2\left[\frac{-21\omega+201}{352}\right] - 2\left[\frac{-23\omega+211}{256}\right] - 2\left[\frac{-23\omega+124}{312}\right] - 2\left[\frac{-2457\omega-611}{22528}\right] - 2\left[\frac{-253\omega+2321}{6144}\right] - 2\left[\frac{-253\omega-495}{3328}\right] - 2\left[\frac{-27\omega+535}{676}\right] - 2\left[\frac{-27\omega+168}{676}\right] \\
 & - 2\left[\frac{-27\omega+535}{832}\right] - 2\left[\frac{-29\omega+1332}{1331}\right] - 2\left[\frac{-29\omega+149}{484}\right] - 2\left[\frac{-3\omega+84}{64}\right] - 2\left[\frac{-3\omega+45}{88}\right] - 2\left[\frac{-3\omega+46}{88}\right] - 2\left[\frac{-3\omega-9}{11}\right] - 2\left[\frac{-3\omega+36}{16}\right] - 2\left[\frac{-3\omega+45}{22}\right] - 2\left[\frac{-3\omega+46}{22}\right] \\
 & - 2\left[\frac{-3\omega-20}{22}\right] - 2\left[\frac{-3\omega-21}{22}\right] - 2\left[\frac{-3\omega-17}{26}\right] - 2\left[\frac{-3\omega+15}{32}\right] - 2\left[\frac{-3\omega+24}{44}\right] - 2\left[\frac{-39\omega-221}{512}\right] - 2\left[\frac{-8\omega+31}{39}\right] - 2\left[\frac{-51\omega+3807}{3328}\right] - 2\left[\frac{-12\omega+63}{143}\right] - 2\left[\frac{-9\omega+17}{26}\right] \\
 & + 2\left[\frac{\omega+11}{8}\right] + 2\left[\frac{\omega+3}{8}\right] + 2\left[\frac{\omega-4}{8}\right] + 2\left[\frac{\omega+25}{11}\right] + 2\left[\frac{\omega+14}{11}\right] + 2\left[\frac{\omega+7}{12}\right] + 2\left[\frac{\omega+11}{16}\right] + 2\left[\frac{\omega+14}{16}\right] + 2\left[\frac{\omega+1}{16}\right] + 2\left[\frac{\omega+4}{16}\right] + 2\left[\frac{\omega-25}{16}\right] + 2\left[\frac{\omega+14}{22}\right] + 2\left[\frac{\omega+25}{24}\right] + 2\left[\frac{\omega-2}{24}\right] \\
 & + 2\left[\frac{\omega+14}{26}\right] + 2\left[\frac{\omega-2}{3}\right] + 2\left[\frac{\omega+4}{32}\right] + 2\left[\frac{\omega+7}{4}\right] + 2\left[\frac{\omega-4}{4}\right] + 2\left[\frac{\omega+11}{48}\right] + 2\left[\frac{\omega+7}{48}\right] + 2\left[\frac{15\omega+132}{104}\right] + 2\left[\frac{15\omega+204}{176}\right] + 2\left[\frac{21\omega-3}{169}\right] + 2\left[\frac{21\omega+151}{352}\right] + 2\left[\frac{21\omega+180}{352}\right] \\
 & + 2\left[\frac{23\omega+45}{256}\right] + 2\left[\frac{27\omega-51}{484}\right] + 2\left[\frac{29\omega+335}{512}\right] + 2\left[\frac{29\omega-1}{176}\right] + 2\left[\frac{3\omega-123}{121}\right] + 2\left[\frac{3\omega+33}{13}\right] + 2\left[\frac{3\omega+33}{16}\right] + 2\left[\frac{3\omega+43}{22}\right] + 2\left[\frac{3\omega+3}{26}\right] + 2\left[\frac{3\omega+12}{32}\right] + 2\left[\frac{3\omega+17}{32}\right] \\
 & + 2\left[\frac{3\omega+20}{32}\right] + 2\left[\frac{3\omega+20}{44}\right] + 2\left[\frac{33\omega+2651}{5408}\right] + 2\left[\frac{8\omega+23}{39}\right] + 2\left[\frac{6399\omega+16348}{42592}\right] + 2\left[\frac{7\omega-67}{64}\right] + 2\left[\frac{7\omega-1}{66}\right] + 2\left[\frac{7\omega+181}{312}\right] + 2\left[\frac{9\omega+60}{52}\right] + 2\left[\frac{9\omega+8}{26}\right] + 2\left[\frac{9\omega+8}{44}\right].
 \end{aligned}$$

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