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EQUATIONALLY DEFINED RADICAL CLASSES

N.R. MCCONNELL AND TIMOTHY STOKES

We consider universal classes of multioperator groups, and give a sufficient condition for a subclass defined by algebraic elementwise rules to be a radical class.

Consider a universal class $\mathcal{U}$ of multioperator groups. We obtain a sufficient condition for algebraic element-wise definitions of classes of algebras in $\mathcal{U}$ to give rise to radical classes. Some work of this sort has been done by Gardner [1] and Wiegandt [5]. As our work is largely motivated by rings, we refer to normal subobjects as ideals throughout. We direct the reader to [2] for terminology and background.

DEFINITION 1: Let $\mathcal{U}$ be a universal class in a variety $\mathcal{V}$ of multioperator groups, and let F be a set of elements in the free algebra with countable generators $\{x_1, x_2, \dots\}$ in $\mathcal{V}$. Let $\mathcal{R}F$ be the class in $\mathcal{U}$ defined as follows: R is in $\mathcal{R}F$ providing that for every $r \in R$ there exist $f(x_1, x_2, \dots, x_n) \in F$ and $r_2, r_3, \dots, r_n$ in R, such that $f(r, r_2, r_3, \dots, r_n) = 0$. For any algebra R, define $\mathcal{R}F'(R) = \{r \in R \mid \text{ there exists } f \in F, \text{ and } r_2, \dots, r_n \in R \text{ with } f(r, r_2, \dots, r_n) = 0\}$.

It is obvious from these definitions that R is in $\mathcal{R}F$ if and only if $\mathcal{R}F'(R) = R$.

LEMMA 2. *For all F, $\mathcal{R}F$ is homomorphically closed.*

PROOF: Suppose $R \in \mathcal{R}F$, $I \triangleleft R$. If $r + I \in R/I$, then by assumption there exist $f(x_1, x_2, \dots, x_n) \in F$ and $r_2, r_3, \dots, r_n \in R$, such that $f(r, r_2, r_3, \dots, r_n) = 0$, whence $f(r + I, r_2 + I, r_3 + I, \dots, r_n + I) = 0 + I$, and so R/I is in $\mathcal{R}F$. ∎

It is shown in [5] that the class of rings whose elements satisfy some property P is a radical class if P satisfies the following conditions (where I is an ideal of some ring A):

(a) If a is a P–element of A, then the coset $a + I$ is a P–element of A/I;
(d) If a is a P–element in I, then a is a P–element in A too;
and
(e) If the coset $a + I$ is a P–element of A/I and I consists of P–elements, then a is a P–element of A.

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We can use this to get a criterion for $\mathcal{R}F$ to be a radical class.

DEFINITION 3: F is $\mathcal{U}$-associating (or just associating when no confusion arises) if and only if for all $R \in \mathcal{U}$, whenever there are $g, h \in F$, $r, a_i \in R$ and $b_j \in I$ where I is some ideal of R with $\mathcal{R}F'(I) = I$ for which $g(h(r, a_i), b_j) = 0$, there are also $f \in F$ and $c_k \in R$ such that $f(r, c_k) = 0$. Note that a_i here refers to $n-1$ elements of R where n is the number of generators involved in h, et cetera.

THEOREM 4. *If F is $\mathcal{U}$-associating then $\mathcal{R}F$ is a radical class.*

PROOF: Suppose F is associating, and let r be a P-element of a ring R if $f(r, a_i) = 0$ for some $f \in F$, $a_i \in R$. (a) is satisfied by Lemma 2, and (d) is trivial. Suppose that $r + I$ is a P-element of R/I, where I consists of P-elements. Then for some $f \in F$ and $a_i \in R$, $f(r + I, a_i + I) = 0$, that is $f(r, a_i) \in I$. Since I consists of P-elements, $\mathcal{R}F'(I) = I$, so there exist $g \in F$ and $b_j \in I$ with $g(f(r, a_i), b_j) = 0$; hence $h(r, c_k) = 0$ for some $h \in F$ as F is associating. Thus r is a P-element of R, (e) holds, and $\mathcal{R}F$ is a radical class. ∎

COROLLARY 5. *Suppose F is such that $\mathcal{R}F'(R)$ is an ideal of R for all $R \in \mathcal{U}$, and further that $\mathcal{R}F'(\mathcal{R}F'(R)) = \mathcal{R}F'(R)$ for all $R \in \mathcal{U}$. Then $\mathcal{R}F$ is a radical class if and only if F is $\mathcal{U}$-associating.*

PROOF: We show that if $\mathcal{R}F$ is a radical class, then F is associating, the result then following by Theorem 4. Let $R \in \mathcal{U}$. We observe that $\mathcal{R}F'(R)$ is the radical of R with respect to $\mathcal{R}F$, since $\mathcal{R}F'(R)$ is an ideal of R which obviously contains all other $\mathcal{R}F$-ideals of R; moreover, $\mathcal{R}F'(R)$ is itself an $\mathcal{R}F$-ideal of R since $\mathcal{R}F'(\mathcal{R}F'(R)) = \mathcal{R}F'(R)$ by assumption.

Suppose there are $g, h \in F$ and $r, a_i \in R$, $b_j \in I$ where I is an ideal of R with $\mathcal{R}F'(I) = I$, for which $g(h(r, a_i), b_j) = 0$. Then certainly $h(r, a_i)$ is in $\mathcal{R}F'(R)$, so that $h(r + \mathcal{R}F'(R), a_i + \mathcal{R}F'(R)) = 0 + \mathcal{R}F'(R)$. Thus $r + \mathcal{R}F'(R) \in \mathcal{R}F'(R/\mathcal{R}F'(R)) = 0$ whence $r \in \mathcal{R}F(R)$ so that there are $f \in F, c_k \in R$ such that $f(r, c_k) = 0$ and the result follows. ∎

In this case, A is $\mathcal{R}F$-semisimple if and only if $\mathcal{R}F'(A) = \mathcal{R}F(A) = 0$ if and only if whenever $f(r, a_i) = 0, r = 0$.

COROLLARY 6. *Suppose that for all $R \in \mathcal{U}$, $\mathcal{R}F'(R)$ is an ideal of R, and that whenever $f(r, r_2, r_3, \ldots, r_n) = 0$ for some $r, r_2, \ldots, r_n \in R, f \in F$, then all of $r, r_2, r_3, \ldots, r_n$ are in $\mathcal{R}F'(R)$. Then $\mathcal{R}F$ is a radical class in $\mathcal{U}$ if and only if F is associating.*

PROOF: Let $R \in \mathcal{U}$. Evidently, for all $r \in \mathcal{R}F'(R)$, there exist $r_2, r_3, \ldots, r_n \in \mathcal{R}F'(R)$ for which $f(r, r_2, \ldots, r_n) = 0$. Hence $\mathcal{R}F'(\mathcal{R}F'(R)) = \mathcal{R}F'(R)$, and the result follows. ∎

Note that if every $f \in F$ involves only one generator, the second condition of Corollary 6 is satisfied immediately.

EXAMPLES. In all the examples below, $\mathcal{U}$ is the class of all associative rings (unless otherwise stated).

(i) Let $F = \{f\}$ where $f(x,y) = x + y + xy$; then $\mathcal{R}F$ is the class of quasiregular rings. Since $f(f(x,y),z) = f(x, y + z + yz)$ for all x, y, z in any ring R, F is associating and by Theorem 4 $\mathcal{R}F$ is a radical class (the Jacobson radical class). In fact, F is *strongly associating* in the sense that for every $f, g \in F$ we have $f(g(x, y_i), z_j)$ actually equal to $h(x, p_k(x, y_i, z_j))$ for some $h \in F$ and polynomials p_k in the free algebra. This is also true of all subsequent examples.

(ii) Let $F = \{x, x^2, \dots\}$; then $\mathcal{R}F$ is the class of nil rings, which is a radical class (the nil radical) because $(x^n)^m = x^{nm}$, and so F is associating. Similarly, letting $F = \{mx^n \mid m, n \text{ positive integers}\}$ gives the Veldsman radical. Note that if $\mathcal{U}$ is the class of commutative rings, then $\mathcal{R}F'(R)$ is an ideal of R for every $R \in \mathcal{U}$, and the remarks after Corollaries 5 and 6 apply. These two are also examples of sets of polynomials closed under substitution, which makes them 1–radicals, where an n–radical is a radical class for which a ring R is radical if and only if every n–generated subring of R is radical [1, Proposition 4.2].

(iii) In a similar way, all the examples 1–8 from [5, Section 3] arise from strongly associating F. It appears possible that all radical classes of the form

$$\{R \mid \text{ every element of } R \text{ is a } P\text{–element }\}$$

arise in this way.

(iv) Let $F_p = \{x - py\}$, where p is prime; then $\mathcal{R}F_p$ is the class of p–divisible rings, which is a radical class because $(x - py) - pz = x - p(y + z)$, and so F_p is associating. Note that the divisible radical is the intersection of all the $\mathcal{R}F_p$'s.

(v) Let p and q be integer polynomials and let $F = \{x - p(x)yq(x)\}$; then $\mathcal{R}F$ is the class of (p, q)–regular rings. By an argument similar to that in [3, Lemma 1], F can be shown to be associating.

(vi) Let $\mathcal{U}$ be the class of commutative rings, and let $F = \{x^{2m} + {y_1}^2 + \dots + {y_n}^2 \mid m, n \text{ positive integers}\}$; then it is fairly easy to see that for any $f_1, f_2 \in F$ there will be $f_3 \in F$ with $f_1(f_2(x, y_i), v_j) = f_3(x, z_k)$, so F is associating. Note that in this case the conditions of Corollaries 5 and 6 hold. This radical class arises from the real Nullstellensatz of Stengle [4] in the same way that the nil radical arises from the Hilbert Nullstellensatz.

(vii) Let $F = \{x^2 - x\}$; then $\mathcal{R}F$ is the class of Boolean rings. $\mathcal{R}F$ is a radical class, but F is not associating. It follows from Corollary 6 that $\mathcal{R}F'(R)$ is not always an ideal of R; that is, the idempotent elements of a ring do not always form an ideal.

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Department of Mathematics and Computing
University of Central Queensland
Rockhampton MC Qld 4701
Australia

Department of Mathematics
University of Tasmania
Hobart, Tas 7000
Australia