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ON THE SET OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS OF RICCATI TYPE

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1. Introduction

The purpose of this paper is to expand upon the results obtained in [**4**].

We consider the set H of differential equations

$$\dot{z}=z^2+p(t)z+r(t) \qquad (z\in\mathbb{C}, t\in\mathbb{R}), \tag{1.1}$$

where p and r are continuous real-valued functions of period ω (ω being fixed throughout). The equation (1.1) is denoted by P or (p,r), and we regard H as the set of pairs of continuous functions of period ω. On H we define a norm:

$$\|P\|=\max\{|p(t)|, \quad |r(t)|; \quad 0\leqq t\leqq\omega\};$$

then

$$(H, \|.\|) \text{ is a Banach space.}$$

We introduce some notation and recall some of the preparatory results from [**2**] and [**4**]. The solution of (1.1) satisfying $z(t_0)=c$ is written $z_{\mathsf{P}}(t;t_0,c)$. The periodic solutions of (1.1) are determined by the zeros of

$$q_{\mathsf{P}}: c\mapsto z_{\mathsf{P}}(\omega;0,c)-c.$$

The domain of definition of q_{P} is an open set $Q_{\mathsf{P}}\subset\mathbb{C}$. We also define

$$q: H\times\mathbb{C}\to\mathbb{C}; (\mathsf{P},c)\mapsto q_{\mathsf{P}}(c).$$

The domain of definition of q *is an open set* Q of $H\times\mathbb{C}$; on Q, q is holomorphic in c and continuous in P. If $\mathsf{P}_n\to\mathsf{P}$ in H, $c_n\to c$ in $\mathbb{C}$, and $q(\mathsf{P}_n,c_n)=0$, then either $q(\mathsf{P},c)=0$ or the solution $z_{\mathsf{P}}(t;0,c)$ is not defined for all $t\in[0,\omega]$.

The set of zeros of q_{P} is denoted by B_{P}. In [**2**] and [**4**] the *multiplicity of a* periodic solution ϕ of (1.1) is defined as the multiplicity of $\phi(0)$ as a zero of q_{P}. Also (1.1) is said to have a *singular periodic solution* if there are sequences (P_n) and $c_n)$ in H and $\mathbb{C}$, respectively, such that $q(\mathsf{P}_n,c_n)=0$ but either $\mathsf{P}_n\to\mathsf{P}$, $c_n\to c$ and $z_{\mathsf{P}}(t;0,c)$ is not defined for $0\leqq t\leqq\omega$, or $\mathsf{P}_n\to\mathsf{P}$ and $c_n\to\infty$. The set of $\mathsf{P}\in H$ with no singular periodic solutions is denoted by $\mathscr{A}$. We quote some results from [**4**].

Theorem 1.1 *If $\mathsf{P} \notin \mathscr{A}$, then P has a solution which is unbounded both as t increases and as t decreases, and is defined for a t-interval of length less than ω.*

It was shown in [2] that $\mathscr{A}$ is an open set: by a component of $\mathscr{A}$ we mean a maximal connected subset of $\mathscr{A}$.

Theorem 1.2. *If P_1 and P_2 are in the same component of $\mathscr{A}$, they have the same number of periodic solutions.*

Remark. In Theorem 1.2 the multiplicity of solutions is taken into account. This we do throughout the paper.

It is shown in [4] that a member of H either has no periodic solutions, two periodic solutions or infinitely many. We make the following definition.

Definition 1.3. $H_1 = \{\mathsf{P};\ \mathsf{P}$ has exactly two periodic solutions, both real$\}$,
$H_2 = \{\mathsf{P};\ \mathsf{P}$ has exactly two periodic solutions neither real$\}$,
$H_3 = \{\mathsf{P};$ every non-real solution is periodic; no real solution is periodic$\}$,
$H_4 = \{\mathsf{P};\ \mathsf{P}$ has no periodic solution$\}$.

The results summarised in the following theorems were also proved in [4].

Theorem 1.4. *H is the disjoint union of H_1, H_2, H_3 and H_4.*

Theorem 1.5. (1) *$H_1 \cup H_2$ is a component of $\mathscr{A}$,*
(2) *H_4 contains infinitely many components of $\mathscr{A}$.*
(3) *$H_3 \cap \mathscr{A} = \varnothing$.*

In [2] equation (1.1) was investigated by considering the related linear equation P^*:

$$\ddot{u} - p(t)\dot{u} + r(t)u = 0. \tag{1.2}$$

Since we shall also use this technique, we briefly describe the necessary background.

Equation (1.2) is obtained from (1.1) by the transformation $z = -\dot{u}u^{-1}$. Since $u \equiv 0$ is a solution of P^*, a solution of P^* cannot vanish together with its derivative; consequently, every non-trivial solution of P^* yields a solution of $\mathsf{P} = (p, r)$. Conversely every solution of P can be written as $-\dot{u}u^{-1}$, where u is a solution of P^*. The period solutions of P can be studied by choosing a suitable base for the solutions of P^*.

We take a Floquet base (u_1, u_2) for P^* and use this to examine the periodic solutions of P. A Floquet base is either of the form

$$(\alpha_1(t)\, e^{\lambda_1 t}, \alpha_2(t)\, e^{\lambda_2 t})$$

or

$$(\alpha_1(t)\, e^{\lambda_1 t}, (t\alpha_1(t) + \alpha_2(t))\, e^{\lambda_1 t}),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\alpha_1, \alpha_2 : \mathbb{R} \mapsto \mathbb{C}$ are ω-periodic. (For details of the theory leading to the existence of such basis see Coddington and Levinson [1]).

We can so choose u_1 and u_2 that either both are real or u_1 and u_2 have independent real and imaginary parts and $u_1=\bar{u}_2$. The form of the *basis* depends only on the nature of the characteristic multipliers. It is always of the first form except when P* has equal multipliers (necessarily real).

We adopt the convention that

$$-\frac{\pi}{\omega}<\operatorname{Im}\lambda_1,\operatorname{Im}\lambda_2\leqq\pi/\omega.$$

We shall need formulae for p,r in terms of a base (u_1,u_2) of P*:

$$p(t)=\frac{\ddot{u}_1u_2-\ddot{u}_2u_1}{\dot{u}_1u_2-\dot{u}_2u_1}, \tag{1.3}$$

$$r(t)=\frac{\ddot{u}_1\dot{u}_2-\ddot{u}_2\dot{u}_1}{\dot{u}_1u_2-\dot{u}_2u_1}. \tag{1.4}$$

This paper is concerned with the topological properties of H_i $(i=1,2,3,4)$. It will be shown that the boundary between H_1 and H_2 is a manifold and $H_3\subset\bar{H}_4\backslash H_4$.

I wish to express my gratitude to Dr N. G. Lloyd for his valuable guidance and encouragement during the preparation of this paper.

2. Two periodic solutions

In this section we shall study some of the properties of H_1 and H_2.

In [2], Lloyd proved the following result which we quote without proof.

Theorem 2.1. *If $r(t)<0$ for all t, then* P *has exactly two ω-periodic solutions, counting multiplicity.*

Theorem 2.1 was proved by showing that an equation with $r<0$ is in the component of the origin in the set $\mathscr{A}$ of equations with no singular periodic solution. From this it follows that (p,r) must have the same number of periodic solutions as the equation $\dot{x}=x^2$ which is two.

Theorem 2.1 simply states that, if $r(t)<0$ for all t, then $\mathsf{P}\in H_1\cup H_2$. We shall prove that if P satisfies the hypothesis of Theorem 2.1, then it does not belong to H_2. In order to prove this result we need to recall that if $x(t)$ is a non-real solution of P, then there exist differentiable real-valued functions $s(t)$ and $\phi(t)$ such that $x(t)=s(t)e^{i\phi(t)}$ and consequently

$$\dot{s}(t)=(s^2(t)+r(t))\cos\phi(t)+s(t)p(t), \tag{2.1}$$

$$s(t)\dot{\phi}(t)=(s^2(t)-r(t))\sin\phi(t) \tag{2.2}$$

for all t.

Theorem 2.2 *If p and r not both constants and $r<0$ for all t, then* $\mathsf{P}=(p,r)\in H_1$.

Proof. Suppose, if possible, that for some $\xi\in\mathbb{C}$, where Im $\xi\neq 0$, $x_{\mathsf{P}}(t;0,\xi)\equiv s(t)\,e^{i\phi(t)}$ is ω-periodic. So that $x_{\mathsf{P}}(t;0,\xi)$, $(0\leqq t\leqq\omega)$, forms a closed curve either in the upper half-plane or in the lower half-plane. Hence there exists $t_0\in[0,\omega]$ such that $\dot\phi(t_0)=0$.

Since $r(t)<0$ and $\sin\phi(t)\neq 0$, (2.2) gives $\dot\phi(t)\neq 0$ for all t, which contradicts the existence of t_0.

We see in the next lemma that if $x(t)$ is a real ω-periodic solution, then

$$\|x\|=\sup_{0\leqq t\leqq\omega}|x(t)|$$

depends only on $\|\mathsf{P}\|$.

Lemma 2.3. *Suppose that $x(t)$ is a real ω-periodic solution of* P. *Then*

$$\|x\|\leqq\|\mathsf{P}\|+\sqrt{\|\mathsf{P}\|^2+2\|\mathsf{P}\|}$$

Proof. Suppose, if possible, that $x(t)$ is a real ω-periodic solution of P and

$$|x(t_0)|>\|\mathsf{P}\|+\sqrt{\|\mathsf{P}\|^2+2\|\mathsf{P}\|}$$

for some $t_0\in[0,\omega]$. We have two cases to consider: (*i*) $x(t_0)>0$ and (ii) $x(t_0)<0$.

Case (*i*). Suppose that $x(t_0)>0$. Since $x(t)$ is differentiable and periodic, there exists $t_1\in[0,\omega]$ such that $x(t_1)=\max\limits_{0\leqq t\leqq\omega}x(t)$. Hence $\dot x(t_1)=0$ and moreover

$$x(t_1)>\|\mathsf{P}\|+\sqrt{\|\mathsf{P}\|^2+2\|\mathsf{P}\|}.$$

But $x(t)$ is a solution of equation P, so for all t

$$\dot x(t)=x^2(t)+p(t)x(t)+r(t)\geqq x^2(t)-\|\mathsf{P}\|(x(t)+1). \tag{2.3}$$

It is easily seen that

$$x^2(t_1)>\|\mathsf{P}\|(x(t_1)+1),$$

hence,

$$\dot x(t_1)>0,\quad \text{a contradiction.}$$

Case (*ii*). Suppose that $x(t_0)<0$. Consider the transformations $t\to -t$, $x\to -x$, then $-x(t)$ is a periodic solution of

$$\dot x=x^2-p(t)x+r(t).$$

Hence we have case (i) and the lemma is proved.

Theorem 2.4. *H_1 is a closed subset of H.*

Proof. Suppose that (P_n) is a convergent sequence in H_1 and $\mathsf{P}_n \to \mathsf{P}$ as $n \to \infty$. Suppose, if possible, that $\mathsf{P} \notin H_1$; that is, B_{P} has no real members. Let $M_n = \|\mathsf{P}_n\|$. Since $\mathsf{P}_n \to \mathsf{P}$ as $n \to \infty$, $\{M_n : n \in \mathbb{Z}^+\}$ is bounded. Let $M^* = \sup_n M_n$. Choose $c_n \in B_{\mathsf{P}_n}$ $(n = 1, 2, \ldots)$. By Lemma 2.3,

$$|x_{\mathsf{P}_n}(t; 0, c_n)| < M^* + \sqrt{(M^*)^2 + 2M^*}, \qquad (0 \leqq t \leqq \omega). \tag{2.4}$$

Hence we can assume, without loss of generality, that $c_n \to c_0$ as $n \to \infty$ for some $c_0 \in \mathbb{R}$. So either $q(\mathsf{P}, c_0) = 0$ or $x_{\mathsf{P}}(t; 0, c_0)$ is not defined in $[0, \omega]$. Since, by hypothesis, B_{P} contains no real member, $q(\mathsf{P}, c_0) = 0$ is excluded. Hence there exists $\tau \in (0, \omega]$ such that $|x_{\mathsf{P}}(t; 0, c_0)| \to \infty$ as $t \uparrow \tau$. But $x_{\mathsf{P}_n}(t; 0, c_n) \to x_{\mathsf{P}}(t; 0, c_0)$ as $n \to \infty$ for all $t \in [0, \omega)$. It follows that (2.4) is satisfied for $x_{\mathsf{P}}(t; 0, c_0)$, a contradiction. Therefore $q(\mathsf{P}, c_0) = 0$, hence $\mathsf{P} \in H_1$ and the theorem is proved.

Corollary 2.5. *If* p and r *are not constant and* $r(t) \leqq 0$ *for* $0 \leqq t \leqq \omega$, *then* $\mathsf{P} \in H_1$.

Proof. Let $\mathsf{P}_n = (p, r_n)$ where $r_n = r - \frac{1}{n}$ $(n = 1, 2, \ldots)$. Hence, by Theorem 2.2, for all n $\mathsf{P}_n \in H_1$ and $\mathsf{P}_n \to \mathsf{P}$ as $n \to \infty$. By Theorem 2.4, $\mathsf{P} \in H_1$. Whence the result is proved.

Corollary 2.6. *H_2 is open in H.*

Proof. We know from [4] that $H_1 \cup H_2$ is open in H and we have proved that H_1 is closed in H. Since $H_1 \cap H_2$ is empty, H_2 is open in H and the corollary is proved.

Directly from Corollary 2.5, we have if $\mathsf{P}_1 = (p, 0)$, then $\mathsf{P}_1 \in H_1$. We shall use this result to show that under certain conditions $\mathsf{P} \in H_1$ $(\mathsf{P} = (p, r))$ if $|r|$ is small enough, irrespective of the size of P.

The following lemma can be deduced from Proposition 3.1 of Lloyd [**2**].

Lemma 2.7. *If $\int_0^\omega p(t)\,dt \neq 0$, then $x = 0$ is an ω-periodic solution of* $\mathsf{P} = (p, 0)$ *with multiplicity* 1.

Theorem 2.8. *If $\int_0^\omega p(t)\,dt \neq 0$, and $|r|$ is small enough, then* $\mathsf{P} \in H_1$.

Proof. Let $\mathsf{P}_1 = (p, 0)$. By Corollary 2.5, $\mathsf{P}_1 \in H_1$. By Lemma 2.7 the zero solution is a periodic solution of P_1 of multiplicity 1; there is therefore another periodic solution $x_{\mathsf{P}}(t; 0, c)$, say, with c real and non-zero. We know that $H_1 \cup H_2$ is open. Suppose then, if possible, that there exists a sequence (P_n) in H_2 convergent to P_1. Let c_1^n, c_2^n be the two starting points of the two ω-periodic solutions of P_n. Then, $c_1^n \to 0$ and $c_2^n \to c$ as $n \to \infty$. But c_1^n, c_2^n are complex conjugates; it follows that $c = 0$, a contradiction. Therefore, $\mathsf{P}_1 \in \operatorname{int} H_1$ and the theorem is proved.

3. The boundary between $\mathbf{H_1}$ and $\mathbf{H_2}$

In this section we study the characteristics of the boundary between H_1 and H_2 and show that it is a manifold.

Definition. Let $H_{11}=\{\mathsf{P}; B_{\mathsf{P}}$ contains exactly one point$\}$.

Remark. It is clear that

$$H_{11}\cap H_2=H_{11}\cap H_3=H_{11}\cap H_4=\varnothing.$$

So $H_{11}\subset H_1$ and if $x\in B_{\mathsf{P}}$ $(\mathsf{P}\in H_{11})$, then $i(q(\mathsf{P},.),x,0)=2$, (where $i(q(\mathsf{P},.),x,0)$ is the index of $q(\mathsf{P},.)$ at the 0-point x, for more details see [5]).

We prove that H_{11} is the boundary between H_1 and H_2.

Theorem 3.1. *H_1 is a perfect subset of H.*

Proof. Let $\mathsf{P}=(p,r)\in H_1$. Suppose that ϕ is one of the ω-periodic solutions of P. It can be checked that $\phi_n=\phi+1/n$ $(n\in\mathbb{Z}^+)$ is a real ω-periodic solution of

$$\mathsf{P}_n=\left(p-\frac{2}{n},r-\frac{p}{n}+\frac{1}{n^2}\right),$$

and $\mathsf{P}_n\to\mathsf{P}$ as $n\to\infty$. Since B_{P_n} contains at least one real element, namely $\phi_n(0)$, $\mathsf{P}_n\in H_1$. Hence P is an accumulation point of H_1. Since H_1 is closed, H_1 is perfect and the result is proved.

Recall that P^* was the related linear equation (1.2).

Lemma 3.2. *If $\mathsf{P}\in H_{11}$, then the Floquet base of P^* is of form*

$$(e^{\lambda t}\alpha_1(t),e^{\lambda t}(t\alpha_1(t)+\alpha_2(t)).$$

Proof. Suppose that $(e^{\lambda_1 t}\alpha_1(t),e^{\lambda_2 t}\alpha_2(t))$ is a base of P^*. Hence if

$$u(t)=c_1\,e^{\lambda_1 t}\alpha_1(t)+c_2\,e^{\lambda_2 t}\alpha_2(t)\qquad (c_1,c_2\in\mathbb{C}),$$

then

$$\phi(t)=-\frac{c_1\,e^{\lambda_1 t}\beta_1(t)+c_2\,e^{\lambda_2 t}\beta_2(t)}{c_1\,e^{\lambda_1 t}\alpha_1(t)+c_2\,e^{\lambda_2 t}\alpha_2(t)}\tag{3.1}$$

is a solution of P, where $\beta_i=\dot{\alpha}_i+\lambda_i\alpha_i$ $(i=1,2)$.

Hence ϕ is ω-periodic if and only if

$$c_1c_2(e^{\lambda_1\omega}-e^{\lambda_2\omega})=0\tag{3.2}$$

and $c_1\,e^{\lambda_1 t}\alpha_1(t)+c_2\,e^{\lambda_2 t}\alpha_2(t)$ does not vanish.

We have one of the following two cases:

(i) $\lambda_1\neq\lambda_2$,

(ii) $\lambda_1=\lambda_2$.

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Case (*i*). By (3.2) ϕ is ω-periodic if and only if $c_1=0$ or $c_2=0$. So, by (3.1), P has only two ω-periodic solutions, namely

$$\phi_1(t)=-\frac{\beta_1(t)}{\alpha_1(t)} \quad \text{and} \quad \phi_2(t)=-\frac{\beta_2(t)}{\alpha_2(t)}$$

provided that $\alpha_1(t)$, $\alpha_2(t)\neq 0$. Since $u_1=e^{\lambda_1 t}\alpha_1(t)$ and $u_2=e^{\lambda_2 t}\alpha_2(t)$ are linearly independent, $\phi_1\neq\phi_2$. If both u_1 and u_2 are real, then their zeros, according to the Sturm separation theorem, interlace; both ϕ_1 and ϕ_2 are then defined for all t or neither is. Hence if P has a real ω-periodic solution, then it has two distinct such solutions; so $\mathsf{P}\notin H_{11}$.

Case (*ii*). By (3.2) every solution of P is either ω-periodic or is defined for a time less than ω. Hence if P has an ω-periodic solution, then it has infinitely many. Hence $\mathsf{P}\notin H_{11}$.

We conclude that the base of P* is of the form

$$(e^{\lambda t}\alpha_1(t), e^{\lambda t}(t\alpha_1(t)+\alpha_2(t))).$$

Remark. We note that $\lambda_1-\lambda_2\neq 2n\pi i/\omega$ for all $n\in\mathbb{Z}^+$, (because $|\text{Im}\,\lambda_1|<\pi/\omega$ and $|\text{Im}\,\lambda_2|\leqq\pi/\omega$).

Theorem 3.3. $H_{11}\subset\partial H_2$.

Proof. Suppose that $\mathsf{P}=(p,r)\in H_{11}$. Then by Lemma 3.2 P* has a base $u_1=e^{\lambda t}\alpha_1(t)$, $u_2=e^{\lambda t}(t\alpha_1(t)+\alpha_2(t))$. Since u_1,u_2 are independent solutions of P*, computing the Wronskian gives

$$\dot\alpha_1(t)\alpha_2(t)-\alpha_1(t)\dot\alpha_2(t)-\alpha_1^2(t)\neq 0$$

for all t.

Substituting u_1 and u_2 in (1.3) and (1.4) gives us

$$p=2\lambda+\frac{\ddot\alpha_1\alpha_2-\alpha_1\ddot\alpha_2-2\alpha_1\dot\alpha_2}{\dot\alpha_1\alpha_2-\alpha_1\dot\alpha_2-\alpha_1^2}$$

$$r=\lambda^2+\lambda(p-2\lambda)-\frac{\dot\alpha_1\ddot\alpha_2-\ddot\alpha_1\alpha_2-\alpha_1\ddot\alpha_2+2\dot\alpha_1^2}{\dot\alpha_1\alpha_2-\alpha_1\dot\alpha_2-\alpha_1^2}.$$

We construct a sequence P_n in H_2 converging to P.

For $n\in\mathbb{Z}^+$ let

$$V_n=\left(\alpha_1+\frac{\mathrm{i}}{n}\alpha_2\right)\exp\left(\lambda+\frac{i}{n}\right)t$$

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and

$$W_n = \bar{V}_n.$$

Substituting V_n and W_n in (1.3) and (1.4) gives us

$$p_n = 2\lambda + \frac{\alpha_1\ddot{\alpha}_2 - \ddot{\alpha}_1\alpha_2 + 2\alpha_1\dot{\alpha}_1 + 2\alpha_2\dot{\alpha}_2/n^2}{\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2 + \alpha_1^2 + \alpha_2^2/n^2}$$

and

$$r_n = \frac{\dot{\alpha}_1\ddot{\alpha}_2 - \ddot{\alpha}_1\dot{\alpha}_2 + 2\dot{\alpha}_1 - \alpha_1\ddot{\alpha}_1 - \lambda(\alpha_1\ddot{\alpha}_2 + \ddot{\alpha}_1\alpha_2 + 2\alpha_1\dot{\alpha}_1) + \lambda^2(\alpha_1\dot{\alpha}_1 - \dot{\alpha}_1\alpha_2 + \alpha_1) + E/n^2 + \alpha_2^2/n^4}{\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2 + \alpha_1^2 + \alpha_2^2/n^2}$$

where $E = (\dot{\alpha}_2 + \alpha_1 + \lambda\alpha_2)(\alpha_1 + 2\dot{\alpha}_2 + 2\lambda\alpha_2) - \alpha_2(\dot{\alpha}_1 + \lambda\alpha_1)$

$$-\alpha_2(\ddot{\alpha}_2 + 2\dot{\alpha}_1 + 2\lambda(\alpha_1 + \dot{\alpha}_2) + \lambda^2\alpha_2).$$

It can be checked that $\mathsf{P}_n = (p_n, r_n) \in H$ and (V_n, W_n) is a base of P_n^*. Since for large n, $(\lambda + i/n) \neq (\lambda - i/n)$ then $\mathsf{P}_n \in H_1 \cup H_2$. It can be checked that

$$\phi_n = -\left[\lambda + \frac{i}{n} + \left(\dot{\alpha}_1 + \frac{i}{n}\dot{\alpha}_2\right) \Big/ \left(\alpha_1 + \frac{i}{n}\alpha_2\right)\right]$$

is an ω-periodic solution of P_n. Therefore for large n, $\mathsf{P}_n \in H_2$. Since $\mathsf{P}_n \to \mathsf{P}$ as $n \to \infty$, $\mathsf{P} \in \partial H_2$ and the theorem is proved.

Theorem 3.4 *H_{11} is the boundary "between" H_1 and H_2; that is, $H_{11} = \bar{H}_1 \cap \bar{H}_2$.*

Proof. Suppose that $\mathsf{P} \in \bar{H}_1 \cap \bar{H}_2$. Since H_1 is a closed subset of H, $\mathsf{P} \in H_1$. Suppose that $\mathsf{P} \notin H_{11}$; then B_{P} contains exactly two real elements, x_1, x_2 say. Since $\mathsf{P} \in \partial H_2$ there exists a sequence (P_n) in H_2 convergent to P. If $c_n \in B_{\mathsf{P}_n}$, then $\bar{c}_n \in B_{\mathsf{P}_n}$. Hence we can assume without loss of generality that $c_n \to x_1$ and $\bar{c}_n \to x_2$ as $n \to \infty$. But $x_1 \neq x_2$ are real, a contradiction. Therefore $\mathsf{P} \in H_{11}$.

Conversely, if $\mathsf{P} \in H_{11}$, then by Theorem 3.3 $\mathsf{P} \in \bar{H}_2$ and Theorem 3.1 $\mathsf{P} \in \bar{H}_1$ and the theorem is proved.

Lloyd in [2] proved the following result which we quote without proof.

(Recall that an ω-periodic solution ϕ is simple if it has multiplicity 1).

Lemma 3.5. *Let $\mathsf{P} = (p, r) \in H_1$ and suppose that ϕ is ω-periodic solution of P. Then $\mathsf{P} \in H_{11}$ if and only if*

$$\int_0^\omega (2\phi(t) + p(t))\,dt = 0.$$

Let $H^1 = \mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the set of all differentiable functions in $\mathbb{P}$, and $\mathbb{P}$ is the set of all functions $P: \mathbb{R} \to \mathbb{R}$ continuous and ω-periodic, H^1 is a subspace of H. On H^1

define the norm

$$\|(r_1,r_2)\|_1=\max_{0\leq t\leq\omega}(|r_1(t)|,|r_2(t)|,|\dot r_2(t)|).$$

Note that this norm is not the norm induced by the norm on H.
Let

$$H^{11}=\{(\alpha_1,\alpha_2)\in H^1;\ \int_0^\omega(\alpha_1(t)+2\alpha_2(t))dt=0\}$$

Lemma 3.6. *H^{11} is a hyperplane in H^1.*

Proof. Let $\mathscr{T}:H^1\to\mathbb{R}$ be defined by

$$\mathscr{T}(\alpha_{11},\alpha_2)=\int_0^\omega(\alpha_1(t)+2\alpha_2(t))\,dt.$$

It is clear that $\mathscr{T}$ is a non-zero linear functional and

$$H^{11}=\{(\alpha_1,\alpha_2)\in H^1;\quad \mathscr{T}(\alpha_1,\alpha_2)=0\}.$$

Hence H^{11} is a hyperplane.

Theorem 3.7. *H_{11} is a manifold (modelled on a Banach space).*

Proof. Let $L:H_{11}\to H^{11}$ be defined by

$$L(p,r)=(p,\phi),$$

where ϕ is the unique ω-periodic solution of (p,r). It is clear that L is bijective, for if $(p,\phi)\in H^{11}$, let $r=\dot\phi-\phi^2-p\phi$. Then by Lemma 3.5 $(p,r)\in H_{11}$ and $L(p,r)=(p,\phi)$, and if $L(p,r)=L(p',r')=(r,\phi)$, then $p'=p$ and $r=\dot\phi-\phi^2-p\phi=r'$.

We shall prove that L is continuous. If $\varepsilon>0$ is given, then there exists $\delta_1>0$ with $0<\delta_1<\varepsilon$ such that if

$$(p,r),(p',r')\in H_{11}\quad\text{and}\quad \|(p-p',r-r)\|<\delta,$$

then

$$\max_{0\leq t\leq\omega}|\phi(t)-\phi_1(t)|<\delta,$$

implies

$$\max_{0\leq t\leq\omega}|\phi^2(t)+p(t)\phi(t)+r(t)-\phi_1^2(t)-p'(t)\phi_1(t)-r'(t)|\leq\varepsilon \tag{3.3}$$

(ϕ,ϕ_1 are the unique periodic solutions of (p,r), (p',r') respectively).

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By Theorem 2 in [3] there exists δ_2 with $0<\delta_2<\delta_1$ such that if (p,r), $(p',r')\in H_{11}$ and $\|(p-p',r-r')\|<\delta_2$, then

$$\max_{0\leqq t\leqq \omega} |\phi(t)-\phi_1(t)|<\delta_1. \tag{3.4}$$

Hence by (3.3) and (3.4),

$$\max_{0\leqq t\leqq \omega} |\phi(t)-\phi_1(t)|<\varepsilon \tag{3.5}$$

for all (p,r), $(p',r')\in H_{11}$ for which $\|(p-p',r-r')\|<\delta_2$. Therefore by (3.5) and (3.4),

$$\begin{aligned}\|L(p,r)-L(p',r')\|_1 &= \|(p,\phi)-(p',\phi_1)\|_1\\ &= \max_{0\leqq t\leqq \omega} (|p(t)-p'(t)|, |\phi(t)-\phi_1(t)|,\\ &\qquad |\dot{\phi}(t)-\dot{\phi}_1(t)|)\\ &<\varepsilon,\end{aligned}$$

for all (p,r), $(p',r')\in H_{11}$, for which

$$\|(p-p',r-r')\|<\delta_2.$$

Next we show that L^{-1} is also continuous. For if $\varepsilon>0$ is given, then there exists δ such that $0<\delta<\varepsilon$ and if (p,ϕ), $(p',\phi_1)\in H^{11}$ and $\|(p-p',\phi-\phi_1)\|_1<\delta$, then

$$\begin{aligned}\max_{0\leqq t\leqq \omega} |r(t)-r'(t)| &\leqq \max_{0\leqq t\leqq \omega} |\dot{\phi}(t)-\phi^2(t)-p(t)\phi(t)-\dot{\phi}_1(t)+\phi_1^2(t)+p'(t)\phi_1(t)|\\ &<\varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}/L^{-1}(p,\phi)-L^{-1}(p',\phi_1)\| &= \|(p-p',r-r')\|\\ &\leqq \max_{0\leqq t\leqq \omega} (|p(t)-p'(t)|, |r(t)-r'(t)|)\\ &<\varepsilon,\end{aligned}$$

for all (p,ϕ), $(p',\phi_1)\in H^{11}$ for which

$$\|(p-p',\phi-\phi')\|<\delta.$$

Hence L is a homeomorphism and is bijective. Since H^{11} is a hyperplane, H_{11} is a manifold and the theorem is proved.

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4. $H_3 \cup H_4$ as a subset of H

In this section we study some of the properties of $H_3 \cup H_4$. Recall that P^* denotes the equation

$$\ddot{u} - p\dot{u} + ru = 0,$$

where $\mathsf{P} = (p, r)$.

Lemma 4.1. *If* $(\alpha_1(t)\, e^{\lambda_1 t}, \alpha_2(t)\, e^{\lambda_2 t})$ *is a base of* P^* *and* $\lambda_1 \neq \lambda_2$, *then* $\mathsf{P} \notin H_3$.

Proof. It can be checked that the solutions of P are

$$\frac{-c_1(\dot{\alpha}_1 + \lambda_1\alpha_1)\, e^{\lambda_1 t} + c_2(\dot{\alpha}_2 + \lambda_2\alpha_2)\, e^{\lambda_2 t}}{c_1\alpha_1\, e^{\lambda_1 t} + c_2\alpha_2\, e^{\lambda_2 t}} \tag{4.1}$$

where c_1 and c_2 are complex constants. Since $\alpha_1\, e^{\lambda_1 t}$ and $\alpha_2\, e^{\lambda_2 t}$ are linearly independent, the solution (4.1) is ω-periodic if and only if

$$c_1 c_2 (e^{\lambda_1 \omega} - e^{\lambda_2 \omega}) = 0. \tag{4.2}$$

Since $\lambda_1 \neq \lambda_2$ and $|\mathrm{Im}\, \lambda_i| \leqq \pi/\omega$ $(i = 1, 2)$, we see from (4.1) and (4.2) that P has at most two ω-periodic solutions, namely

$$\frac{-\dot{\alpha}_2 + \lambda_2\alpha_2}{\alpha_2} \quad \text{and} \quad \frac{-\dot{\alpha}_1 + \lambda_1\alpha_1}{\alpha_1}.$$

Therefore $\mathsf{P} \notin H_3$ and the lemma is proved.

Now suppose that $(\alpha_1(t)\, e^{\lambda_1 t}, \alpha_2(t)\, e^{\lambda_2 t})$ is a base of P^* and $\lambda_1 \neq \lambda_2$. Then by Lemma 4.1 $\mathsf{P} \notin H_3$. It can be checked that

$$\frac{-\dot{\alpha}_1 + \lambda_1\alpha_1}{\alpha_1}, \quad \text{and} \quad \frac{-\dot{\alpha}_2 + \lambda_2\alpha_2}{\alpha_2}$$

are ω-periodic solutions of P, provided $\alpha_1(t)$ and $\alpha_2(t)$ do not vanish. Hence we have proved the following lemma.

Lemma 4.2. *Let* α_i, λ_i $(i = 1, 2)$ *and* P *be as in Lemma* 4.1. *If* $\alpha_1(t) \neq 0$ *or* $\alpha_2(t) \neq 0$ *for all t, then* $\mathsf{P} \in H_1 \cup H_2$, *otherwise* $\mathsf{P} \in H_4$.

Remark. By the Sturm separation theorem, α_1, α_2 either both have zeroes or both have none.

Theorem 4.3. *If* $\mathsf{P} \in H_3$, *then* P^* *has a base of the form*

$$(\alpha_1(t)\, e^{\lambda t}, \alpha_2(t)\, e^{\lambda t}),$$

where $\lambda \in \mathbb{R}$ *and* α_1, α_2 *are real-valued* ω*-periodic functions.*

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Proof. If $\mathsf{P}\in H_3$, then a base of P^* has one of the following two forms:

$$(\alpha_1(t)\,e^{\lambda t},\alpha_2(t)\,e^{\lambda t}) \tag{4.3}$$

$$(\alpha_1(t)\,e^{\lambda t},(t\alpha_1(t)+\alpha_2(t)\,e^{\lambda t}), \tag{4.4}$$

where $\lambda\in\mathbb{R}$ and α_1,α_2 are real-valued ω-periodic functions. It can be checked that if a base of P^* has the form (4.4), then

$$-\frac{c_1(\dot\alpha_1+\lambda\alpha_1)+c_2((\dot\alpha_1+\lambda\alpha_1)t+\dot\alpha_2+\lambda\alpha_2+\alpha_1)}{c_1\alpha_1+c_2(\alpha_1 t+\alpha_2)}$$

is a solution of P where $c_1,c_2\in\mathbb{C}$. Hence P has at most one period solution, namely

$$-\frac{\dot\alpha_1+\lambda\alpha_1}{\alpha_1},$$

provided $\alpha_1(t)\neq 0$ for all t, and the lemma is proved.

Remark. We saw in Lemma 3.5 that if $\mathsf{P}\in H_{11}$, then P^* has Floquet base of the form (4.4). If P^* has this form of base, either $\mathsf{P}\in H_{11}$ or $\mathsf{P}\in H_4$. We saw that H_{11} is the boundary "between" H_1 and H_2 in the sense that $H_{11}=\bar H_1\cap\bar H_2$. We now show that H_3 is a part of the boundary of H_4.

Theorem 4.4. *If* $\mathsf{P}\in H_3$, *then*

(i) P *is an accumulation point of* H_3,
(ii) P *is an accumulation point of* H_4.

Proof. Since $\mathsf{P}\in H_3$, by Theorem 4.3, P^* has a base of the form $(\alpha_1(t)\,e^{\lambda t},\ \alpha_2(t)\,e^{\lambda(t)})$. Formulae (1.3) and (1.4) give us

$$p=\frac{\alpha_2(\ddot\alpha_1+2\lambda\dot\alpha_1+\lambda^2\alpha_1)-\alpha_1(\ddot\alpha_2+2\lambda\dot\alpha_2+\lambda^2\alpha_2)}{\alpha_2(\dot\alpha_1+\lambda\alpha_1)-\alpha_1(\dot\alpha_2+\lambda\alpha_2)}$$

and

$$r=\frac{(\ddot\alpha_2+2\lambda\dot\alpha_1+\lambda^2\alpha_1)(\dot\alpha_2+\lambda\alpha_2)-(\ddot\alpha_2+2\lambda\dot\alpha_2+\lambda^2\alpha_2)(\dot\alpha_1+\lambda\alpha_1)}{\alpha_2(\dot\alpha_1+\lambda\alpha_1)-\alpha_1(\dot\alpha_2+\lambda\alpha_2)}$$

where $P=(p,r)$ and $\alpha_2(t)\ (\dot\alpha_1(t)+\lambda\alpha_1(t))-\alpha_1(t)(\dot\alpha_2(t)+\lambda\alpha_2(t))\neq 0$ for all t (because $\alpha_1\,e^{\lambda t}$, $\alpha_2\,e^{\lambda t}$ are linearly independent).

To prove (i) consider

$$u_n=(\alpha_1(t)\,e^{(\lambda+1/n)t},\alpha_2(t)\,e^{(\lambda+1/n)t}),\quad n\in\mathbb{Z}^+.$$

It can be checked that u_n is a base of $\mathsf{P}_n=(p_n, r_n)$ where

$$p_n=\frac{\alpha_2(\ddot{\alpha}_1+2(\lambda+1/n)\dot{\alpha}_1+(\lambda+1/n)^2\alpha_1)-\alpha_1(\ddot{\alpha}_2+2(\lambda+1/n)\dot{\alpha}_2+(\lambda+1/n)^2\alpha_2}{\alpha_2(\dot{\alpha}_1+(\lambda+1/n)\alpha_1)-\alpha_1(\dot{\alpha}_2+(\lambda+1/n)\alpha_2)}$$

$$r_n=$$

$$\frac{(\ddot{\alpha}_2+2(\lambda+1/n)\dot{\alpha}_1+(\lambda+1/n)^2\alpha_1)(\dot{\alpha}_2+(\lambda+1/n)\alpha_2)-(\ddot{\alpha}_2+2(\lambda+1/n)\dot{\alpha}_2+(\lambda+1/n)^2\alpha_2)(\dot{\alpha}_1+(\lambda+1/n)\alpha_1)}{\alpha_2(\dot{\alpha}_1+(\lambda+1/n)\alpha_1)-\alpha_1(\dot{\alpha}_2+(\lambda+1/n)\alpha_2)}.$$

Also $\lim_{n\to\infty}\mathsf{P}_n=\mathsf{P}$ and $\mathsf{P}_n\in H_3$. Hence (i) is proved.

To prove (ii) consider

$$u_n=(\alpha_1(t)\,e^{(\lambda+1/n)t},\alpha_2(t)\,e^{\lambda t}).$$

It can be checked that u_n is a base of $\mathsf{P}^n=(p_n, r_n)$ where

$$P_n=\frac{\alpha_2(\ddot{\alpha}_1+2(\lambda+1/n)\dot{\alpha}_1+(\lambda+1/n)^2\alpha_1)-\alpha_1(\ddot{\alpha}_2+2\dot{\alpha}_2+\lambda^2\alpha_2)}{\alpha_2(\dot{\alpha}_1+(\lambda+1/n)\alpha_1)-\alpha_1(\dot{\alpha}_2+\lambda\alpha_2)}$$

$$r_n=\frac{(\ddot{\alpha}_1+2(\lambda+1/n)\dot{\alpha}_1+(\lambda+1/n)^2\alpha_1)(\dot{\alpha}_2+\lambda\alpha_2)-(\ddot{\alpha}_2+2\lambda\dot{\alpha}_2+\lambda^2\alpha_2)(\dot{\alpha}_1+(\lambda+1/n)\alpha_1)}{\alpha_2(\alpha_1^2+(\lambda+1/n)\alpha_1)-\alpha_1(\dot{\alpha}_2+\lambda_2)}$$

Now $\lim_{n\to\infty}\mathsf{P}_n=\mathsf{P}$. Since $\mathsf{P}\in H_3$, we know that α_1 must have a zero, otherwise P has a real ω-periodic solution, namely $-\dot{\alpha}_1+\lambda\alpha_1/\alpha_1$. Hence by Lemma 4.2 $\mathsf{P}_n\in H_4$ and the theorem is proved.

Remark. Theorem 4.4(i) shows that H_3 has no isolated member.

Theorem 4.5. $H_3=\bar{H}_4\backslash H_4$.

Proof. Suppose that $\mathsf{P}\in H_3$. Then by Theorem 4.4(ii) $\mathsf{P}\in\bar{H}_4\backslash H_4$. Hence $H_3\subset H_4\backslash H_4$.

Now if $\mathsf{P}\in\bar{H}_4\backslash H_4$, then either $\mathsf{P}\in H_1\cup H_2$ or $\mathsf{P}\in H_3$. Since $H_1\cup H_2$ is open, $\mathsf{P}\in H_3$ and the theorem is proved.

Remark. Suppose that P^* has a base of the form $(\alpha_1 e^{\lambda t}, (t\alpha_1+\alpha_2)\,e^{\lambda t})$ and α_1 has a zero. Then $\mathsf{P}\,\varepsilon\, H_4$. By using the same method used in the proof of Theorem 4.4 we can show that $\mathsf{P}\in\bar{H}_2$. Therefore $H_3\neq\partial H_4$.

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