GROUPS WHOSE LATTICES OF NORMAL SUBGROUPS ARE DISTRIBUTIVE

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Various authors deal with distributive sublattices of the lattice $\mathscr{L}(G)$ of subgroups of a group G. Perhaps the most basic result in this direction is due to O. Ore **[9]**: $\mathscr{L}(G)$ is distributive if and only if G is locally cyclic.

In **[11]** and **[12]** finite groups with distributive lattices of subnormal subgroups were considered, while **[3]**, **[4]**, **[7]**, **[8]**, **[10]** and **[13]** deal with the case of groups G whose lattice $\mathcal{N}(G)$ of normal subgroups is distributive. Such groups were called DLN-groups in **[10]**.

In the first part of the present note, we develop a somewhat systematic theory of DLN-groups and consider various constructions for such groups. Our main result is

THEOREM. *Let A and B be nontrivial finite groups and assume that A is nilpotent. Then the following are equivalent*:

(i) *the wreath product A wr B is a* DLN-*group*,

(ii) *A and B are cyclic and the orders of A and B are coprime.*

We use standard notation throughout. Moreover, if a group B acts on a group A, then $[A]B$ denotes the corresponding split extension and the sign $\cong_G$ denotes G-isomorphism.

1. How to construct DLN-groups. In this section we develop some methods to construct finite DLN-groups. It will turn out that the DLN-property of a group heavily depends on the distribution of its chief factors. We shall use the following convention. If H and K are normal subgroups of a group G, then we call L/M a G-chief factor of H/K if $K \leq M < L \leq H$ and L/M is a minimal normal subgroup of G/M. When it is clear which group G is meant, we simply call L/M a chief factor of H/K. No confusion should arise.

Although a few results in this section hold for groups satisfying the maximal or the minimal condition for normal subgroups, for reason of simplicity we restrict ourselves to finite groups.

LEMMA 1. *Let N be a normal subgroup in G and let H and K be normal subgroups in G such that $K \leq H$. Assume that no chief factor of the form H/L (resp. L/K) of H/K is G-isomorphic to a chief factor of G/N (resp. N). Then we have $H/K \cong_G H \cap N/K \cap N$ (resp. $H/K \cong_G HN/KN$).*

Proof. In the first case, we have $H \cap N/K \cap N \cong_G K(H \cap N)/K = H \cap KN/K$. We show $H \cap KN = H$. Otherwise, let H/L be a chief factor of $H/H \cap KN$. We have $H/H \cap KN \cong_G HN/KN$, and hence H/L is G-isomorphic to a chief factor of G/N. But this contradicts our hypothesis, and we get $H = H \cap KN$ as claimed. The second case follows from an analogous argument.

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The following simple observation is the key to our results.

LEMMA 2 [**1**]. *For the group G the following are equivalent*:

(i) *G is a* DLN-*group*,

(ii) *for every normal subgroup N of G, the socle* $\mathrm{Soc}(G/N)$ *of G/N does not contain any two distinct G-isomorphic minimal normal subgroups.*

By Lemma 2, a group G is a DLN-group if no two factors in some fixed chief series are G-isomorphic. As the cyclic groups show, the converse of this is not true in general. However, we have the following modification that gives a sufficient condition for an extension to be a DLN-group.

PROPOSITION 3. *Let N be a normal subgroup of G and set $Q = G/N$. Assume that the following conditions hold*:

(i) *Q is a* DLN-*group*,

(ii) *the lattice of all Q-invariant normal subgroups of N is distributive*,

(iii) *no chief factor of N is G-isomorphic to a chief factor of Q.*

Then G is a DLN-*group.*

Proof. If G is not a DLN-group, then by Lemma 2, there exist two distinct G-isomorphic chief factors of the form H_1/K and H_2/K. Then $H/K = \langle H_1/K, H_2/K\rangle = H_1/K \times H_2/K$ is a direct product of two G-isomorphic chief factors. By Lemma 1, either Q or N contains a normal section that is G-isomorphic to H/K. But this contradicts Lemma 2 and our assumptions (i) or (ii).

COROLLARY 4 [**10**]. *Let $G = G_1 \times G_2$ where G_1 and G_2 are* DLN-*groups. Then the following are equivalent*:

(i) *G is a* DLN-*group*,

(ii) *G_1 and G_2 do not both contain a central factor of order p.*

Proof. If G is a DLN-group then Lemma 2 shows that G_1 and G_2 cannot both have a central factor of order p.

Suppose condition (ii) is satisfied. Then G_1 and G_2 cannot have G-isomorphic chief factors H_1/K_1 and H_2/K_2 since $C(H_1/K_1) \geq G_2$ and $C(H_2/K_2) \geq G_1$ and so H_1/K_1 and H_2/K_2 are central. Proposition 3 now shows that G is a DLN-group.

COROLLARY 5. *Let N be an abelian normal subgroup of G and let $Q = G/N$. Assume that the following conditions hold*:

(i) *Q is a* DLN-*group*,

(ii) *the lattice of all Q-invariant subgroups of N is distributive*,

(iii) *Q acts faithfully on every Q-chief factor of N.*

Then G is a DLN-*group.*

Proof. It suffices to show that condition (iii) of Proposition 3 is satisfied. If H/K is a chief factor of N, then (iii) implies $C_G(H/K) = N$. Now let L/M be a chief factor of G/N. If L/M is nonsoluble, then clearly $L/M \not\cong H/K$. Otherwise, we have $L \leq C_G(L/M)$ and we get $C_G(H/K) \neq C_G(L/M)$. As isomorphic chief factors have the same centralizer, we see $L/M \not\cong_G H/K$ in all cases. The result follows from Proposition 3.

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REMARK. In the situation of Corollary 5, it is easy to see that for every normal subgroup R of G, one has $R \le N$ or $N \le R$. In particular, $\mathcal{N}(G)$ can easily be constructed from the corresponding lattices of N and Q.

Conditions (ii) and (iii) of Corollary 5 are trivially satisfied if N is a faithful and irreducible module for Q. Now Lemma 2 together with [**6**] shows that such N in fact exists for every (finite) DLN-group. This proves the first part of the following result while the second follows from Corollary 5.

PROPOSITION 6. *Let Q be a* DLN-*group.*

(a) [**7**, **10**] *Let p be a prime not dividing the order of Q. Then there exists a faithful and irreducible $\mathbb{F}_pQ$-module N.*

(b) *The split extension $G = [N]Q$ is a* DLN-*group.*

Call a group (a module) uniserial, if the lattice of all normal subgroups (all submodules) forms a chain.

COROLLARY 7. (a) *Let Q be a* DLN-*group. Then exists a* DLN-*group G having a unique minimal normal subgroup $N = F(G)$ such that $G/N \cong Q$.*

(b) *For every d there exists a soluble uniserial group of derived length d.*

2. Wreath products. In this section we determine when the wreath product $G = N \operatorname{wr} Q$ of two groups N and Q is a DLN-group. As Q is a quotient of G, it must clearly be a DLN-group. If the order of N is a prime p, then the base group of G is Q-isomorphic to the regular $\mathbb{F}_pQ$-module, and so representation theory will play a rôle here.

LEMMA 8. *Let p be a prime and let Q be a finite group. If $G = \mathbb{Z}_p \operatorname{wr} Q$ is a* DLN-*group, then Q is a cyclic p'-group.*

Proof. The base group M of G is isomorphic to $\mathbb{F}_pQ$. Let $M = P_1 \oplus \ldots \oplus P_r$, where the P_i are directly indecomposable $\mathbb{F}_pQ$-modules. Each irreducible $\mathbb{F}_pQ$-module occurs as a factor module of M and since G is a DLN-group, it occurs only once in this way. Let $V_i = P_i/\operatorname{Jac}(P_i)$; then $V_1, \ldots, V_r$ are non-isomorphic in pairs and, again since G is a DLN-group, no V_i occurs as a composition factor of P_j for $j \ne i$. Thus each block contains just one irreducible $\mathbb{F}_pQ$-module and hence Q is p-nilpotent [**5**, VII 14.9].

Now M contains an $\mathbb{F}_pQ$-submodule R such that $G/R \cong \mathbb{Z}_p \times Q$. Hence if p divides the order of Q, then G would have a quotient isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ and hence G cannot be a DLN-group. So Q is a p'-group.

If Q were nonabelian, then some summand P_i would occur with multiplicity >1 contradicting Lemma 2. So Q is an abelian DLN-group and hence Q is cyclic.

The following is a necessary and sufficient criterion for a wreath product of a nilpotent group with another group to be a DLN-group.

THEOREM 9. *Let N and Q be finite groups and assume that N is nilpotent and* *nontrivial. Let $G = N \operatorname{wr} Q$. Then the following are equivalent*:

(i) *G is a* DLN-*group,*

(ii) *both N and Q are cyclic and of coprime order.*

Proof. (i)$\Rightarrow$(ii). Let R be a normal subgroup of prime index p in N. Then the quotient group $\mathbb{Z}_p \operatorname{wr} Q$ of G is a DLN-group and Lemma 8 implies that Q is a cyclic p'-group. So the orders of N and Q are coprime. If N were noncyclic, then N would have a quotient isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. But the base group of $(\mathbb{Z}_p \times \mathbb{Z}_p) \operatorname{wr} Q$ is isomorphic to $\mathbb{F}_pQ \oplus \mathbb{F}_pQ$ and Lemma 2 gives a contradiction.

(ii)$\Rightarrow$(i). Suppose that G is not a DLN-group. Then Lemma 2 yields the existence of two distinct p-chief factors of G of the form H_1/K and H_2/K. The join $\langle H_1/K, H_2/K\rangle \cong H_1/K \times H_2/K$ is noncyclic and hence p divides the order of N.

Let $P = O_p(G)$. Then we have $P = P_1 \oplus \ldots \oplus P_r$ where the P_i are Q-uniserial. Furthermore, all composition factors of P_i are Q-isomorphic to the $\mathbb{F}_pQ$-module V_i, say. As Q is abelian, we also have $V_i \not\cong V_j$ for $i \neq j$.

Finally, Lemma 1 implies $H_i/K \cong_G H_i \cap P/K \cap P =: W_i$ $(i = 1, 2)$. Furthermore, W_1 and W_2 are distinct and G-isomorphic. But this contradicts the previous paragraph.

Corollary 10. *Let Q be a finite group and assume that $N \operatorname{wr} Q$ is a* DLN-*group. If N has a quotient isomorphic to $\mathbb{Z}$ then $Q = 1$.*

We now develop a sufficient criterion for the wreath product G of a nonnilpotent group N with a group Q to be a DLN-group. If M is a subgroup of N, then the base group of G contains a canonical subgroup isomorphic to the direct product of $|Q|$ copies of M, which we will denote by $M^Q = M_1 \times \ldots \times M_{|Q|}$, where $M_i \cong M$ for all i.

Lemma 11. *Let M be a minimal normal subgroup of N and assume $M \nleq Z(N)$. Then M^Q is a minimal normal subgroup of G.*

Proof. Let R be a minimal normal subgroup of G contained in M^Q and let $1 \neq r = (r_1, \ldots) \in R$ where $r_i \in M_i$. Without loss, assume $r_1 \neq 1$. As M is noncentral, we have $M = [M, N]$, whence $M_1 \leq R$. Now Q acts transitively on the set $\{M_1, \ldots, M_{|Q|}\}$ and we get $M^Q \leq R$, so M^Q is a minimal normal subgroup of G as claimed.

With the above notation, if G is a DLN-group, then so is $(N/N') \operatorname{wr} Q$. Hence, if $N \neq N'$ then Theorem 9 implies that Q is cyclic. We now show that under certain circumstances, a converse of this holds.

Theorem 12. *Let N and Q be finite* DLN-*groups. Assume no chief factor of N in N' is central. In the case when $N \neq N'$, assume in addition that Q is cyclic and that the orders of Q and N/N' are coprime. Then $G = N \operatorname{wr} Q$ is a* DLN-*group.*

Proof. We want to apply Proposition 3 to $A = (N')^Q$ and $B = (N/N') \operatorname{wr} Q$. We have $G/A \cong B$. By Theorem 9 condition (i) of Proposition 3 is satisfied. By Lemma 11, the lattice of all G-invariant subgroups of A is isomorphic to the lattice of all normal subgroups of N contained in N', which by our hypothesis on N is distributive, so (ii) holds. For (iii), let M be a G-chief factor of B. Then clearly $N^Q \leq C_G(M)$. Now let M be a G-chief factor of A. By our hypothesis on N, we see that M is not central in N^Q, so $N^Q \nleq C_G(M)$. As any two G-isomorphic chief factors have the same centraliser, we see that condition (iii) holds and the result follows from Proposition 3.

At least in the case when N is perfect, the hypothesis on the central factors below N' cannot be dispensed with.

Example. Let $N = SL(2, 5)$ and let Q be a noncyclic DLN-group of odd order. Then the lattice of all normal subgroups of $G = N \operatorname{wr} Q$ contained in $Z(N)^Q$ is isomorphic to the lattice of all submodules of $\mathbb{F}_2 Q$, and the latter is not distributive. So G is not a DLN-group.

In some cases, however, some central factors below N' may occur. The following is based on an example of H. Heineken.

Example. Let p and $q \geq 5$ be primes such that $q \equiv 1 \bmod p$, and let $N = SL(p, q)$. Then $Z(N) \leq \Phi(N)$ and $|Z(N)| = p$. Let r be a positive integer and let $G = N \operatorname{wr} \mathbb{Z}_{p^r}$.

Then $Z = O_p(G) = Z_\infty(G)$ is an elementary abelian p-group of rank r and $Q = G/Z \cong PSL(p, q) \operatorname{wr} \mathbb{Z}_{p^r}$ contains a unique minimal normal subgroup B/Z, where B is the base group of G. Hence Q is a DLN-group. Furthermore, the Q-module Z is uniserial.

To prove that G is a DLN-group, it suffices to show that for every normal subgroup R of G either $R \leq Z$ or $Z \leq R$. Suppose that R is not contained in Z. By the above, we have $B \leq RZ$ and we obtain $B = (B \cap R)Z$. But $Z \leq \Phi(B)$ implies $B = B \cap R$ and we arrive at $Z \leq B \leq R$ as claimed.

This leaves us with the following.

Problem. Determine when the wreath product of (not necessarily finite) groups is a DLN-group.

In [**10**] G. Pazderski proves that the hypercentre of a finite DLN-group is abelian and he asks whether it is cyclic. The above is a counterexample to this. Another example that contains only one noncentral composition factor is the following one. The information concerning $PSL(3, 4)$ is taken from [**2**, p. 23].

Example. Let $E = PSL(3, 4)$ and let N be the representation group of E. Then there exists $R \leq N' \cap Z(N)$ such that $N/R \cong E$ and $R = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3$. Furthermore, there exists an automorphism x of N such that $x^2 = 1$, $N = [N, x]$ and $a^x = b$. Let $G = [N]\langle x \rangle$ and set $\bar{G} = G/\langle a^2, b^2, c \rangle$. Then $Z(\bar{G}) = \langle \bar{a}\bar{b} \rangle < Z_\infty(\bar{G}) = \langle \bar{a}, \bar{b} \rangle$ and the lattice of normal subgroups of $\bar{G}$ is a chain.

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