

## ***b*-GENERALIZED DERIVATIONS OF SEMIPRIME RINGS HAVING NILPOTENT VALUES**

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### **Abstract**

Let  $R$  be a semiprime ring with extended centroid  $C$  and with maximal right ring of quotients  $Q_{mr}(R)$ . Let  $d: R \rightarrow Q_{mr}(R)$  be an additive map and  $b \in Q_{mr}(R)$ . An additive map  $\delta: R \rightarrow Q_{mr}(R)$  is called a (left)  $b$ -generalized derivation with associated map  $d$  if  $\delta(xy) = \delta(x)y + bxd(y)$  for all  $x, y \in R$ . This gives a unified viewpoint of derivations, generalized derivations and generalized  $\sigma$ -derivations with an  $X$ -inner automorphism  $\sigma$ . We give a complete characterization of  $b$ -generalized derivations of  $R$  having nilpotent values of bounded index. This extends several known results in the literature.

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### **1. Results**

Throughout the paper, unless specially stated,  $R$  is always a semiprime ring with Martindale symmetric ring of quotients  $Q_s(R)$ . We let  $Q_{mr}(R)$  (respectively  $Q_{ml}(R)$ ) denote the maximal right (respectively left) ring of quotients of  $R$ . It is known that  $R \subseteq Q_s(R) \subseteq Q_{mr}(R)$ . The overrings  $Q_s(R)$  and  $Q_{mr}(R)$  of  $R$  are semiprime rings with the same center  $C$ , which is a regular self-injective ring. The ring  $C$  is called the extended centroid of  $R$ . Also,  $R$  is a prime ring if and only if  $C$  is a field. We refer the reader to the book [1] for details.

An additive map  $d: R \rightarrow R$  is called a *derivation* if  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ . For  $b \in R$ , we let  $\text{ad}(b)$  denote the map  $x \mapsto [b, x] := bx - xb$  for  $x \in R$ . Clearly,  $\text{ad}(b)$  is a derivation of  $R$ , which is called the *inner derivation* of  $R$  induced

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by the element  $b$ . It is known that any derivation  $d$  of  $R$  can be uniquely extended to a derivation of  $Q_{mr}(R)$ . A derivation  $d: R \rightarrow R$  is called *X-inner* if its extension to  $Q_{mr}(R)$  is inner. In this case, it is easy to check that  $d = \text{ad}(q)$  for some  $q \in Q_s(R)$ . An additive map  $\delta: R \rightarrow R$  is called a *generalized derivation* if there exists a derivation  $d$  of  $R$  such that  $\delta(xy) = \delta(x)y + xd(y)$  for all  $x, y \in R$  (see [2, 14, 18]). The derivation  $d$  is uniquely determined by  $\delta$ , and is called the *associated derivation* of  $\delta$ .

Let  $\sigma$  be an automorphism of  $R$ . An additive map  $\delta: R \rightarrow R$  is called a (right)  $\sigma$ -*derivation* if  $\delta(xy) = x\delta(y) + \delta(x)\sigma(y)$  for  $x, y \in R$ . Basic examples of  $\sigma$ -derivations are derivations and  $\sigma - 1$ . Given  $b \in R$ , the map  $x \mapsto xb - b\sigma(x)$  for  $x \in R$  obviously defines a  $\sigma$ -derivation, which is called the *inner  $\sigma$ -derivation* induced by  $b$ . It is clear that any  $\sigma$ -derivation of  $R$  can be uniquely extended to a  $\sigma$ -derivation of  $Q_{mr}(R)$ . In [21], Lee and Liu gave a common generalization of both generalized derivations and  $\sigma$ -derivations. An additive map  $g: R \rightarrow R$  is called a right *generalized  $\sigma$ -derivation* if there exists an additive map  $\delta: R \rightarrow R$  such that  $g(xy) = xg(y) + \delta(x)\sigma(y)$  for all  $x, y \in R$ . It is clear that  $\delta$  is uniquely determined by the map  $g$ . The additive map  $\delta$  is called the *associated map* of  $g$ . Our present study is motivated by the following results.

Let  $d: R \rightarrow R$  be a derivation,  $\delta: R \rightarrow R$  a generalized derivation,  $g: R \rightarrow R$  a right generalized  $\sigma$ -derivation, and  $n$  a fixed positive integer. Also, the rings  $R$  in (4)–(6) are prime.

- (1) Suppose that  $d(x)^n = 0$  for all  $x \in R$ . Then  $d = 0$  (see [10, 12, 13]).
- (2) Let  $\lambda$  be a left ideal of  $R$ . Suppose that  $d(x)^n = 0$  for all  $x \in \lambda$ . Then  $\lambda d(\lambda) = 0$  (see [16, Theorem 6]).
- (3) Suppose that  $\delta(x)^n = 0$  for all  $x \in R$ . Then  $\delta = 0$  (see [18, Theorem 5]).
- (4) Suppose that  $\delta(x)^n = 0$  for all  $x \in \rho$ , a right ideal of  $R$ . Then there exist  $b, c \in Q_{mr}(R)$  and  $\beta \in C$  such that  $\delta(x) = bx - xc$  for all  $x \in R$  and  $(b - \beta)\rho = 0 = (c - \beta)\rho$  (see [18, Theorem 6]).
- (5) Suppose that  $g(x)^n = 0$  for all  $x \in R$ . Then  $g = 0$  (see [21, Theorem 2.7]).
- (6) Let  $a, b, q \in Q_{mr}(R)$ . Suppose that  $(a\delta(qx) - bx)^n = 0$  for all  $x \in R$ . Then either  $a\delta(q) - b = 0 = aq$  or there exist  $a_0, b_0 \in Q_{mr}(R)$  and  $\mu \in C$  such that  $\delta(x) = a_0x + xb_0$  for  $x \in R$  and  $aa_0q - b = -b_0aq = \mu aq$ .

Let us consider a special case of (5). Suppose that the extension of  $\sigma$  to  $Q_{ml}(R)$  is inner; that is, there exists a unit  $u \in Q_{ml}(R)$  such that  $\sigma(x) = x u u^{-1}$  for  $x \in R$ . Let  $\delta$  be the associated map of  $g$ . Then  $g(xy) = xg(y) + d(x)yu^{-1}$  for all  $x, y \in R$ , where  $d(x) := \delta(x)u$  for  $x \in R$ . Notice that  $d: R \rightarrow Q_{ml}(R)$ . See [3, 4] for the Lie ideal case.

In (6), let  $d: R \rightarrow R$  be the associated derivation of  $\delta$ ; that is,  $\delta(xy) = \delta(x)y + xd(y)$  for  $x, y \in R$ . We let  $\tilde{\delta}(x) := a\delta(qx) - bx$  for  $x \in R$ . Then  $\tilde{\delta}(x) = aqd(x) + (a\delta(q) - b)x$  for  $x \in R$ . A direct computation shows that  $\tilde{\delta}(xy) = \tilde{\delta}(x)y + (aq)xd(y)$  for  $x, y \in R$ . Since  $d$  can be uniquely extended to  $Q_{mr}(R)$ , so can  $\tilde{\delta}$ . In view of [17, Theorem 3] (or see Fact 1.5 below),  $R$  and  $Q_{mr}(R)$  satisfy the same differential identities. Thus,  $\tilde{\delta}(x)^n = 0$  for all  $x \in Q_{mr}(R)$ .

Motivated by the results (1)–(6) above, we give the following definition.

**DEFINITION 1.1.** (1) Let  $d: R \rightarrow Q_{mr}(R)$  be an additive map and  $b \in Q_{mr}(R)$ . An additive map  $\delta: R \rightarrow Q_{mr}(R)$  is called a (left)  $b$ -generalized derivation with associated map  $d$  if  $\delta(xy) = \delta(x)y + bxd(y)$  for all  $x, y \in R$ .

(2) Let  $d: R \rightarrow Q_{ml}(R)$  be an additive map and  $b \in Q_{ml}(R)$ . An additive map  $\delta: R \rightarrow Q_{ml}(R)$  is called a right  $b$ -generalized derivation with associated map  $d$  if  $\delta(xy) = x\delta(y) + d(x)yb$  for all  $x, y \in R$ .

Clearly, a generalized derivation is a 1-generalized derivation and a right generalized  $\sigma$ -derivation is a right  $u^{-1}$ -generalized derivation if  $\sigma(x) = uxu^{-1}$  for  $x \in R$ , where  $u$  is a unit in  $Q_{ml}(R)$ . For  $a, b, c \in Q_{mr}(R)$ , the map  $x \mapsto ax + bxc$  for  $x \in R$  is a left  $b$ -generalized derivation. Analogously, for  $a, b, c \in Q_{ml}(R)$ , the map  $x \mapsto xa + bxc$  for  $x \in R$  is a right  $c$ -generalized derivation. We note that left or right  $b$ -generalized derivations appear canonically in [7, Theorems 1.1 and 1.3]. The goal of the paper is to give a complete characterization of  $b$ -generalized derivations having nilpotent values of bounded index. By symmetry, it suffices to deal with one of left and right  $b$ -generalized derivations. For simplicity of notation, a  $b$ -generalized derivation always means a left  $b$ -generalized derivation.

To state the main theorem of the paper, we have to recall some basic properties of idempotents of  $C$ . We write  $\mathbf{B}$  for the set of all idempotents of  $C$ . The set  $\mathbf{B}$  forms a Boolean algebra with respect to the operations  $e + h := e + h - 2eh$  and  $e \cdot h := eh$  for all  $e, h \in \mathbf{B}$ . It is complete with respect to the partial order  $e \leq h$  (defined by  $eh = e$ ) in the sense that any subset  $S$  of  $\mathbf{B}$  has a supremum  $\bigvee S$  and an infimum  $\bigwedge S$ . Given a subset  $S$  of  $Q_{mr}(R)$ , we define  $E[S]$  to be the infimum of  $e \in \mathbf{B}$  such that  $ex = x$  for all  $x \in S$ . If  $S = \{b\}$ , we write  $E[b]$  instead of  $E[S]$  for simplicity. Note that, for  $a, b \in Q_{mr}(R)$ ,  $aRb = 0$  if and only if  $E[a]E[b] = 0$ . By the characterization, it is easy to see that if a  $b$ -generalized derivation  $\delta$  has associated maps  $d$  and  $d'$ , then  $E[b]d(x) = E[b]d'(x)$  for all  $x \in R$ . We refer the reader to the book [1] for details.

We are now in a position to state the main theorems of the paper.

**THEOREM 1.2.** Let  $R$  be a semiprime ring,  $b \in Q_{mr}(R)$ , and let  $\delta: R \rightarrow Q_{mr}(R)$  be a  $b$ -generalized derivation with associated map  $d$ . Suppose that  $\delta(x)^n = 0$  for all  $x \in R$ , where  $n$  is a positive integer. Then there exists  $q \in Q_{mr}(R)$  such that  $E[b]d(x) = [q, x]$  for  $x \in R$ ,  $\delta(x) = -bxq$  for  $x \in R$ , and  $qb = 0$ .

By symmetry, we also have the following result whose proof parallels that of Theorem 1.2.

**THEOREM 1.3.** Let  $R$  be a semiprime ring,  $b \in Q_{ml}(R)$ , and let  $\delta: R \rightarrow Q_{ml}(R)$  be a right  $b$ -generalized derivation with associated map  $d$ . Suppose that  $\delta(x)^n = 0$  for all  $x \in R$ , where  $n$  is a positive integer. Then there exists  $q \in Q_{ml}(R)$  such that  $E[b]d(x) = [q, x]$  for  $x \in R$ ,  $\delta(x) = qxb$  for  $x \in R$ , and  $bq = 0$ .

Let  $I$  be an ideal of  $R$ . By the semiprimeness of  $R$ , the left annihilator of  $I$  in  $R$  coincides with the right annihilator of  $I$  in  $R$ . The ideal  $I$  is called *dense* if

its left annihilator in  $R$  is zero. We write  $C\{X_1, X_2, \dots\}$  for the free algebra over  $C$  in noncommutative indeterminates  $X_1, X_2, \dots$  and  $Q_{mr}(R) *_C C\{X_1, X_2, \dots\}$  for the free product of the  $C$ -algebras  $Q_{mr}(R)$  and  $C\{X_1, X_2, \dots\}$ . Let  $f(X_i) \in Q_{mr}(R) *_C C\{X_1, X_2, \dots\}$  and  $T$  be a subring of  $Q_{mr}(R)$ . We say that  $f$  is a GPI (that is, a *generalized polynomial identity*) of  $T$  if  $f(x_i) = 0$  for all  $x_i \in T$ . By a *derivation word*  $\Delta$ , we mean that  $\Delta$  is of the form  $d_1 d_2 \cdots d_s$ , where each  $d_i$  is either a derivation of  $Q_{mr}(R)$  or the identity map of  $Q_{mr}(R)$ . By a *differential polynomial*  $f(X_i^{\Delta_j})$ , we mean that all  $\Delta_j$  are derivation words and  $f(Z_{ij})$  is a generalized polynomial in noncommutative indeterminates  $Z_{ij}$ . The differential polynomial  $f(X_i^{\Delta_j})$  is called a *differential identity* of  $T$  if  $f(x_i^{\Delta_j}) = 0$  for all  $x_i \in T$ . We will use the following facts in the proofs below.

**FACT 1.4.** Let  $I$  be a dense ideal of  $R$ . Then  $I$  and  $Q_{mr}(R)$  satisfy the same GPIs with coefficients in  $Q_{mr}(R)$  (see [1, Theorem 6.4.1] for a semiprime ring  $R$  and [6, Theorem 2] for a prime ring  $R$ ).

**FACT 1.5.** Let  $I$  be a dense ideal of  $R$ . Then  $I$  and  $Q_{mr}(R)$  satisfy the same differential identities with coefficients in  $Q_{mr}(R)$  (see [17, Theorem 3]).

**FACT 1.6.** Let  $\rho$  be a right ideal of  $R$  and  $a \in Q_{mr}(R)$ . Suppose that  $(ax)^n = 0$  for all  $x \in \rho$ . Then  $a\rho = 0$  (see Fact 1.4 and [11, Lemma 1.1]).

**FACT 1.7.** Let  $\phi: I \rightarrow Q_{mr}(R)$  be a right  $R$ -module map, where  $I$  is a dense ideal of  $R$ . Then there exists  $a \in Q_{mr}(R)$  such that  $\phi(x) = ax$  for all  $x \in I$  (see [19, Lemma 2.1] with the same proof by replacing ‘a nonzero ideal in a prime ring’ with ‘a dense ideal in a semiprime ring’).

**FACT 1.8.** Let  $d: R \rightarrow Q_{mr}(R)$  be a derivation. Then  $d$  can be uniquely extended to a derivation from  $Q_{mr}(R)$  to itself (see, for instance, [17, Lemma 2]).

## 2. The prime case

We begin with the following key result.

**PROPOSITION 2.1.** *Let  $R$  be a prime ring,  $a, b, c \in R$ , and  $n$  a positive integer. Suppose that  $(ax + bxc)^n = 0$  for all  $x \in R$ . Then there exists  $\beta \in C$  such that  $a = \beta b$  and  $(c + \beta)b = 0$ .*

A prime ring  $R$  is called a *GPI-ring* if it satisfies a nontrivial (that is, nonzero) generalized polynomial with coefficients in  $Q_{mr}(R)$ . The prime ring  $R$  is called *centrally closed* if  $R = RC$ . In particular, the prime ring  $Q_{mr}(R)$  is centrally closed. The following lemma is a special case of [24, Theorem 1]. Since the proof below is neat and self-contained, we give its proof here for the convenience of the reader. We also remark that Chang proved the following lemma with the extra assumption that  $b$  is invertible in  $R$  (see [5, Lemma 2.1]).

**LEMMA 2.2.** *Let  $R$  be a prime ring,  $a, b, c \in R$ , and  $n$  a positive integer. Suppose that  $(b(ax + xc))^n = 0$  for all  $x \in R$ . Then there exists  $\beta \in C$  such that  $b(a - \beta) = 0$  and  $(c + \beta)b = 0$ .*

**PROOF.** Suppose first that  $R$  is not a GPI-ring. This implies that  $(b(aX + Xc))^n$  is a trivial generalized polynomial. In particular,  $ba$  and  $b$  are dependent over  $C$ . That is,  $b(a - \beta) = 0$  for some  $\beta \in C$ . Thus,

$$\begin{aligned} 0 &= (c + \beta)(b(ax + xc))^n bx \\ &= (c + \beta)(b((a - \beta)x + x(c + \beta)))^n bx = ((c + \beta)bx)^{n+1} \end{aligned} \tag{2.1}$$

for all  $x \in R$ . In view of Fact 1.6,  $(c + \beta)b = 0$ .

Suppose next that  $R$  is a GPI-ring. It follows from Fact 1.4 that

$$(b(ax + xc))^n = 0 \tag{2.2}$$

for all  $x \in RC$ . Let  $F$  denote the algebraic closure of  $C$  if  $C$  is an infinite field and let  $F = C$  if  $C$  is a finite field. Then (2.2) holds for all  $x \in \widetilde{R}$  (see [22, Lemma 2.3]), where  $\widetilde{R} := RC \otimes_C F$ . In view of [8, Theorem 3.5],  $\widetilde{R}$  is a centrally closed prime  $F$ -algebra. By [23, Theorem 3],  $\widetilde{R}$  is a primitive ring with a minimal idempotent  $e$  such that  $e\widetilde{R}e = Fe$ . Hence, there exists a left vector space  $V$  over  $F$  such that  $\widetilde{R}$  acts densely on  ${}_FV$ .

Given  $v \in V$ , we claim that  $v(ba)$  and  $vb$  are dependent over  $F$ . Suppose not; then there exists  $x \in \widetilde{R}$  such that  $v(ba)x = v$  and  $vb x = 0$ . Then  $0 = v(b(ax + xc))^n = v$ , which is a contradiction. This proves the claim.

If  $\dim_F Vb \geq 2$ , it is routine to prove that there exists  $\widetilde{\beta} \in C$  such that  $ba = \widetilde{\beta}b$ ; that is,  $b(a - \widetilde{\beta}) = 0$ . Thus, by (2.1) we have  $(c + \widetilde{\beta})b = 0$ . Suppose next that  $\dim_F Vb = 1$ . Choose  $v_0 \in V$  such that  $Vb = Fv_0b$ . Write  $v_0ba = \widetilde{\gamma}v_0b$  for some  $\widetilde{\gamma} \in F$ . Let  $v \in V$ . Then  $vb = \widetilde{\alpha}v_0b$  for some  $\widetilde{\alpha} \in F$ . Then  $vba = \widetilde{\alpha}v_0ba = \widetilde{\alpha}\widetilde{\gamma}v_0b = \widetilde{\gamma}vb$ .

In either case, there exists  $\widetilde{\beta} \in F$  such that  $ba = \widetilde{\beta}b$ . Choose a basis  $\mu_0, \mu_1, \dots$  for  $F$  over  $C$ , where  $\mu_0 = 1$ , and write  $\widetilde{\beta} = \beta\mu_0 + \beta_1\mu_1 + \dots$ , where  $\beta, \beta_1, \dots \in C$ . Then  $ba = \beta a$ . That is,  $b(a - \beta) = 0$ . It follows from (2.1) that  $(c + \beta)b = 0$ .  $\square$

**PROOF OF PROPOSITION 2.1.** It follows from Fact 1.4 that

$$(ax + bxc)^n = 0 \tag{2.3}$$

for all  $x \in Q_{mr}(R)$ . We claim that  $a \in bQ_{mr}(R)$ . Clearly, we may assume  $a \neq 0$ .

Suppose that  $R$  is not a GPI-ring. Then  $a$  and  $b$  are dependent over  $C$ . In particular,  $a \in bQ_{mr}(R)$ , as asserted. Suppose next that  $R$  is a GPI-ring. In this case,  $Q_{mr}(R)$  is also a prime GPI-ring (see Fact 1.4). Since  $Q_{mr}(R)$  is a centrally closed prime ring, it follows from [23, Theorem 3] that  $Q_{mr}(R)$  is a primitive ring with nonzero socle. Write  $H := \text{soc}(Q_{mr}(R))$ , the socle of  $Q_{mr}(R)$ . Note that  $H$  is a regular ring (see [9]); that is, for any  $w \in H$ ,  $wz w = w$  for some  $z \in H$ . For  $z \in H$ , we write  $\ell_H(z)$  for the left annihilator of  $z$  in  $H$ ; that is,  $\ell_H(z) = \{x \in H \mid xz = 0\}$ .

We first consider the case that  $a, b \in H$ . Let  $w \in \ell_H(b)$ . By (2.3),

$$0 = w(a(xw) + b(xw)c)^n ax = (wax)^{n+1}$$

for all  $x \in Q_{mr}(R)$ . In view of Fact 1.6,  $wa = 0$ . That is,  $w \in \ell_H(a)$ . Up to now, we have proved that  $\ell_H(b) \subseteq \ell_H(a)$

Since  $a, b \in H$ , there exist  $u, v \in H$  such that  $aua = a$  and  $bvb = b$ . Set  $f := au$  and  $g := bv$ . Then  $f, g$  are idempotents. Then  $\ell_H(g) \subseteq \ell_H(f)$ ; that is,  $H(1 - g) \subseteq H(1 - f)$ . So  $(1 - g)f = 0$ . Then  $a = fa = gfa = bvfa \in bH$ , as asserted.

For the general case, let  $w \in H$ . We see that  $(awx + bwxc)^n = 0$  for all  $x \in Q_{mr}(R)$ . Since  $aw, bw \in H$ , the first case implies that  $aw \in bwH$ . Write  $aw = bwt$  for some  $t \in H$ , depending on  $w$ . Replacing  $x$  by  $wx$  in (2.3),

$$(bw(tx + xc))^n = (a(wx) + b(wx)c)^n = 0$$

for all  $x \in Q_{mr}(R)$ . By Lemma 2.2, there exists  $\beta_w \in C$ , depending on  $w$ , such that  $bw(t - \beta_w) = 0$ . That is,  $aw = \beta_w bw$  for  $w \in H$ .

Fix an idempotent  $e_0 \in H$  such that  $ae_0 \neq 0$ . Then  $ae_0 = \beta e_0$  for some  $\beta \in C$ . Let  $f$  be an idempotent of  $H$ . Then  $af = \beta_f bf$  for some  $\beta_f \in C$ . We claim that  $\beta_f = \beta$  if  $af \neq 0$ . Indeed, there exists  $h = h^2 \in H$  such that  $e_0H + fH = hH$  and  $ah = \beta_h bh$  for some  $\beta_h \in C$ . Note that  $he_0 = e_0$  and  $hf = f$ . Thus,

$$ae_0 = ahe_0 = \beta_h bhe_0 = \beta_h be_0,$$

implying that  $\beta_h = \beta$ . Similarly,  $\beta_h = \beta_f$  and so  $\beta = \beta_f$ . Thus,  $(a - \beta b)f = 0$  if  $af \neq 0$ .

Let  $f = f^2 \in H$  with  $af = 0$ . We claim that  $bf = 0$ . By Litoff's theorem [9], there exists an idempotent  $h \in H$  such that  $e_0, f \in hHh$ . If  $ah = 0$  then  $ae_0 = ahe_0 = 0$ , which is a contradiction. Thus, neither  $ah$  nor  $a(h - f)$  is zero. Note that  $h - f$  is an idempotent. Then

$$ah = \beta bh \quad \text{and} \quad a(h - f) = \beta b(h - f).$$

This implies that  $\beta bf = 0$ , so  $bf = 0$  follows. Up to now, we have proved that  $(a - \beta b)f = 0$  for any idempotent  $f \in H$  with  $af = 0$ .

In either case,  $(a - \beta b)f = 0$  for any idempotent  $f \in H$ . Since  $H$  is a regular ring,  $(a - \beta b)H = 0$  and so  $a = \beta b$ . Rewrite (2.3) as  $(bx(c + \beta))^n = 0$  for all  $x \in Q_{mr}(R)$ . So  $(x(c + \beta)b)^{n+1} = 0$  for all  $x \in Q_{mr}(R)$ . By Fact 1.6,  $(c + \beta)b = 0$  follows.  $\square$

The following characterizes *b*-generalized derivations of semiprime rings.

**THEOREM 2.3.** *Let  $R$  be a semiprime ring,  $b \in Q_{mr}(R)$ , and let  $\delta: R \rightarrow Q_{mr}(R)$  be a *b*-generalized derivation with associated map  $d$ . Then  $E[\underline{b}]d: R \rightarrow Q_{mr}(R)$  is a derivation and there exists  $\widetilde{b} \in Q_{mr}(R)$  such that  $\delta(x) = bd(x) + \widetilde{b}x$  for all  $x \in R$ .*

**PROOF.** Expanding  $\delta((xy)z)$  and  $\delta(x(yz))$  respectively, we see that

$$bx(d(yz) - yd(z) - d(y)z) = 0$$

for all  $x, y, z \in R$ . The semiprimeness of  $R$  implies that  $E[b]d(yz) = yE[b]d(z) + E[b]d(y)z$  for all  $y, z \in R$ ; that is,  $E[b]d: R \rightarrow Q_{mr}(R)$  is a derivation. Let  $\mu: R \rightarrow Q_{mr}(R)$  be the map defined by  $\mu(x) = bd(x)$  for  $x \in R$ . Then

$$\begin{aligned} \mu(xy) &= bE[b]d(xy) = bE[b]d(x)y + bxE[b]d(y) \\ &= bd(x)y + bxd(y) = \mu(x)y + bxd(y) \end{aligned}$$

for all  $x, y \in R$ . Thus, we have  $(\delta - \mu)(xy) = (\delta - \mu)(x)y$  for all  $x, y \in R$ . In view of Fact 1.7, there exists  $\tilde{b} \in Q_{mr}(R)$  such that  $\delta(x) = bd(x) + \tilde{b}x$  for all  $x \in R$ .  $\square$

**THEOREM 2.4.** *Let  $R$  be a prime ring,  $b \in Q_{mr}(R)$ , and let  $\delta: R \rightarrow Q_{mr}(R)$  be a nonzero  $b$ -generalized derivation with associated map  $d$ . Suppose that  $\delta(x)^n = 0$  for all  $x \in R$ , where  $n$  is a positive integer. Then there exists  $q \in Q_{mr}(R)$  such that  $d = \text{ad}(q)$ ,  $\delta(x) = -bxq$  for  $x \in R$ , and  $qb = 0$ .*

**PROOF.** In view of Theorem 2.3, there exists  $\tilde{b} \in Q_{mr}(R)$  such that  $\delta(x) = bd(x) + \tilde{b}x$  for all  $x \in R$ . By assumption,

$$(bd(x) + \tilde{b}x)^n = 0 \tag{2.4}$$

for all  $x \in R$ . By Fact 1.8,  $d$  can be uniquely extended to a derivation from  $Q_{mr}(R)$  to itself, also denoted by  $d$ . In view of Fact 1.5, (2.4) holds for all  $x \in Q_{mr}(R)$ . Note that  $Q_{mr}(Q_{mr}(R)) = Q_{mr}(R)$ .

Suppose first that  $d$  is  $X$ -outer. In view of [15, Theorem 2],  $(by + \tilde{b}x)^n = 0$  for all  $x, y \in Q_{mr}(R)$ . Then  $b = 0 = \tilde{b}$  (see Fact 1.6). This implies that  $\delta = 0$ , which is a contradiction. Thus,  $d$  is  $X$ -inner. Then there exists  $q' \in Q_{mr}(R)$  such that  $d(x) = [q', x]$  for  $x \in R$ . Since  $R$  and  $Q_{mr}(R)$  satisfy the same GPIs (see Fact 1.4), we rewrite (2.4) as

$$((bq' + \tilde{b})x - bxq')^n = 0$$

for all  $x \in Q_{mr}(R)$ . In view of Proposition 2.1, there exists  $\mu \in C$  such that  $bq' + \tilde{b} = \mu b$  and  $(q' - \mu)b = 0$ . Let  $q := q' - \mu$ . Then  $d = \text{ad}(q)$ ,  $bq = -\tilde{b}$  and  $qb = 0$ . Therefore,

$$\delta(x) = bd(x) + \tilde{b}x = b(qx - xq) - bqx = -bxq \quad \text{for } x \in R,$$

as asserted.  $\square$

### 3. Proof of Theorem 1.2

Let  $R$  be a semiprime ring with extended centroid  $C$ . We call  $\{e_\nu \mid \nu \in \Lambda\} \subseteq \mathbf{B}$  an *orthogonal subset* if  $e_\nu e_\mu = 0$  for  $\nu \neq \mu$  and a *dense subset* of  $\mathbf{B}$  if  $\sum_{\nu \in \Lambda} e_\nu C$  is an essential ideal of  $C$ . The ring  $Q_{mr}(R)$  is *orthogonally complete* in the following sense: Given any dense orthogonal subset  $\{e_\nu \mid \nu \in \Lambda\}$  of  $\mathbf{B}$ ,  $Q_{mr}(R)$  is ring-isomorphic to the direct product  $\prod_{\nu \in \Lambda} Q_{mr}(R)e_\nu$  via the map

$$x \mapsto \langle xe_\nu \rangle \in \prod_{\nu \in \Lambda} Q_{mr}(R)e_\nu \quad \text{for } x \in Q_{mr}(R).$$



Therefore, given any subset  $\{a_\nu \in Q_{mr}(R) \mid \nu \in \Lambda\}$ , there exists a unique  $a \in Q_{mr}(R)$  such that  $a \mapsto \langle a_\nu e_\nu \rangle$ . The element  $a$  is written as  $\sum_{\nu \in \Lambda}^\perp a_\nu e_\nu$  and is characterized by the property that  $ae_\nu = a_\nu e_\nu$  for all  $\nu \in \Lambda$ . A subset  $T$  of  $Q_{mr}(R)$  is called orthogonally complete if  $0 \in T$  and  $\sum_{\nu \in \Lambda}^\perp a_\nu e_\nu \in T$  for any dense orthogonal subset  $\{e_\nu \mid \nu \in \Lambda\}$  of  $\mathbf{B}$  and any subset  $\{a_\nu \mid \nu \in \Lambda\} \subseteq T$ . Denote by  $\text{Spec}(\mathbf{B})$  the set of all maximal ideals of the complete Boolean algebra  $\mathbf{B}$ . Let  $T$  be a subset of  $Q_{mr}(R)$ . The intersection of all orthogonally complete subsets of  $Q_{mr}(R)$  containing  $T$  is called the *orthogonal completion* of  $T$  and is denoted by  $\widehat{T}$ . In view of [1, Proposition 3.1.14 and Corollary 3.1.15],  $\widehat{R}$  is a subring of  $Q_{mr}(R)$  and

$$\widehat{R} = \left\{ \sum_{\alpha \in \Lambda}^\perp x_\alpha e_\alpha \mid \{e_\alpha \mid \alpha \in \Lambda\} \text{ is a dense orthogonal subset of } \mathbf{B} \text{ and } x_\alpha \in R \text{ for all } \alpha \in \Lambda \right\}.$$

Moreover,  $\widehat{R} \cap \mathfrak{m}Q_{mr}(R)$  is a prime ideal of  $\widehat{R}$  for all  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$  (see [1, Theorem 3.2.15]).

**PROPOSITION 3.1.** *A derivation  $d: Q_{mr}(R) \rightarrow Q_{mr}(R)$  is  $X$ -inner if and only if  $\bar{d}: Q_{mr}(R)/\mathfrak{m}Q_{mr}(R) \rightarrow Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$  is  $X$ -inner for any  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$ .*

The proof of Proposition 3.1 is the same as that of [20, Proposition 2.2]. Let  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$ . It is known that  $\mathfrak{m}Q_{mr}(R)$  is a prime ideal of  $Q_{mr}(R)$ . We use the notations:  $\overline{Q_{mr}(R)} = Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ ,  $\overline{C} = C + \mathfrak{m}Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ , and  $\overline{\widehat{R}} = \widehat{R} + \mathfrak{m}Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ . Then both  $\overline{Q_{mr}(R)}$  and  $\overline{\widehat{R}}$  are prime rings having the same extended centroid  $\overline{C}$  (see [1]). Keeping these notations we have the following.

**LEMMA 3.2.** *Let  $v, x \in Q_{mr}(R)$ . Suppose that  $\bar{x} \in \overline{Cv}$  for any  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$ , where  $\bar{z} := z + \mathfrak{m}Q_{mr}(R)$  for  $z \in Q_{mr}(R)$ . Then  $x \in Cv$ .*

**PROOF.** Consider the set  $\Sigma = \{e \in \mathbf{B} \mid ex \in Cv\}$ . We see that if  $e \leq f \in \Sigma$  then  $e \in \Sigma$ . Also, if  $e, f \in \Sigma$  are orthogonal then clearly  $e+f \in \Sigma$ . This means that  $\Sigma$  is an ideal of the complete Boolean algebra  $\mathbf{B}$ . If  $1 \in \Sigma$  then  $x \in Cv$ , as asserted. Suppose on the contrary that  $1 \notin \Sigma$ . By Zorn’s lemma, there exists  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$  such that  $\Sigma \subseteq \mathfrak{m}$ . We work in  $Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ . Since  $\bar{x} \in \overline{Cv}$ , there exists  $a \in Cv$  such that  $\bar{x} = \bar{a}$ . Therefore,  $ex = ea$  for some  $e \in \mathbf{B} \setminus \mathfrak{m}$ . Note that  $ea \in Cv$ , implying  $e \in \Sigma$ . This is a contradiction.  $\square$

The next theorem extends Proposition 2.1 to the semiprime case.

**THEOREM 3.3.** *Let  $R$  be a semiprime ring,  $a, b, c \in R$ , and  $n$  a positive integer. Suppose that  $(ax + bxc)^n = 0$  for all  $x \in R$ . Then there exists  $\beta \in C$  such that  $a = \beta b$  and  $(c + \beta)b = 0$ .*

**PROOF.** By Fact 1.4,  $(ax + bxc)^n = 0$  for all  $x \in Q_{mr}(R)$ . Let  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$ . Working in  $Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ , we see that  $(\bar{a}\bar{x} + \bar{b}\bar{x}\bar{c})^n = 0$  for all  $\bar{x} \in Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ . In view of Proposition 2.1,  $\bar{a} \in \overline{C}\bar{b}$ . Since  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$  is arbitrary, it follows from Lemma 3.2



that  $a \in Cb$ . Write  $a = \beta b$  for some  $\beta \in C$ . Then  $(bx(c + \beta))^n = 0$  for all  $x \in R$ . By Fact 1.6,  $(c + \beta)b = 0$  follows, as asserted.  $\square$

**LEMMA 3.4.** *Theorem 1.2 holds if  $E[b] = 1$ .*

**PROOF.** Since  $E[b] = 1$ , it follows from Theorem 2.3 that  $d: R \rightarrow Q_{mr}(R)$  is a derivation. By Fact 1.8,  $d$  can be uniquely extended to a derivation  $\widetilde{d}: Q_{mr}(R) \rightarrow Q_{mr}(R)$ . Clearly,

$$\widetilde{d}\left(\sum_{v \in \Lambda}^{\perp} x_v e_v\right) = \sum_{v \in \Lambda}^{\perp} d(x_v) e_v,$$

where  $x_v \in R$ . We claim that  $\delta$  can be also uniquely extended to a  $b$ -generalized derivation of  $\widehat{R}$ , say  $\widetilde{\delta}$ , with associated map  $\widetilde{d}: \widehat{R} \rightarrow Q_{mr}(R)$ , by defining

$$\widetilde{\delta}\left(\sum_{v \in \Lambda}^{\perp} x_v e_v\right) = \sum_{v \in \Lambda}^{\perp} \delta(x_v) e_v,$$

where  $x_v \in R$ . Indeed, let  $\sum_{v \in \Lambda}^{\perp} x_v e_v = 0$ , where  $x_v \in R$ . Then  $x_v e_v = 0$  for any  $v$ . Fix an  $x_v$ . Choose a dense ideal  $I$  of  $R$  such that  $x_v I \cup e_v I \subseteq R$ . Note that  $d(ye_v) = \widetilde{d}(ye_v) = \widetilde{d}(y)e_v = d(y)e_v$  for  $y \in I$  since  $\widetilde{d}$  is a derivation. Thus,

$$0 = \delta(x_v(ye_v))^n = (\delta(x_v)ye_v + bx_v d(ye_v))^n = (\delta(x_v)ye_v)^n,$$

implying that  $(\delta(x_v)e_v y)^n = 0$  for all  $y \in I$ . Fact 1.4 implies that  $(\delta(x_v)e_v y)^n = 0$  for all  $y \in Q_{mr}(R)$ . By Fact 1.6,  $\delta(x_v)e_v = 0$ . So  $\sum_{v \in \Lambda}^{\perp} \delta(x_v)e_v = 0$ . This proves that  $\widetilde{\delta}$  is well defined. It is routine to check that  $\widetilde{\delta}$  is an additive map.

We claim that  $\widetilde{\delta}: \widehat{R} \rightarrow Q_{mr}(R)$  is a  $b$ -generalized derivation with associated map  $\widetilde{d}$ . Indeed, let  $\widetilde{x}, \widetilde{y} \in \widehat{R}$ . Write

$$\widetilde{x} = \sum_{v \in \Lambda}^{\perp} x_v e_v \quad \text{and} \quad \widetilde{y} = \sum_{v \in \Lambda}^{\perp} y_v e_v,$$

where  $x_v, y_v \in R$ . Then  $\widetilde{x}\widetilde{y} = \sum_{v \in \Lambda}^{\perp} (x_v y_v) e_v$  and

$$\begin{aligned} \widetilde{\delta}(\widetilde{x}\widetilde{y}) &= \sum_{v \in \Lambda}^{\perp} \delta(x_v y_v) e_v \\ &= \sum_{v \in \Lambda}^{\perp} (\delta(x_v) y_v + b x_v d(y_v)) e_v \\ &= \left(\sum_{v \in \Lambda}^{\perp} \delta(x_v) e_v\right) \left(\sum_{v \in \Lambda}^{\perp} y_v e_v\right) + b \left(\sum_{v \in \Lambda}^{\perp} x_v e_v\right) \left(\sum_{v \in \Lambda}^{\perp} d(y_v) e_v\right) \\ &= \widetilde{\delta}(\widetilde{x})\widetilde{y} + b\widetilde{x}\widetilde{d}(\widetilde{y}), \end{aligned}$$

as asserted.

Let  $\mathfrak{m} \in \text{Spec}(\mathbf{B})$ . Clearly,  $\widetilde{d}(\mathfrak{m}\widehat{R}) \subseteq \mathfrak{m}Q_{mr}(R)$  since  $\widetilde{d}$  is a derivation. We claim that  $\widetilde{\delta}(\mathfrak{m}\widehat{R}) \subseteq \mathfrak{m}Q_{mr}(R)$ . Let  $x \in \mathfrak{m}\widehat{R}$ . Then  $xe = 0$  for some  $e \in \mathbf{B} \setminus \mathfrak{m}$ . Applying the same argument as in the first paragraph, we see that  $\widetilde{\delta}(x)e = 0$ . Thus  $\delta(x) \in \mathfrak{m}Q_{mr}(R)$ . This proves our claim.

Thus,  $\widetilde{\delta}$  and  $\widetilde{d}$  canonically induce the maps  $\widetilde{\delta}_{\mathfrak{m}}: \widehat{R}/\mathfrak{m}\widehat{R} \rightarrow Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$  and  $\widetilde{d}_{\mathfrak{m}}: \widehat{R}/\mathfrak{m}\widehat{R} \rightarrow Q_{mr}(R)/\mathfrak{m}Q_{mr}(R)$ , where

$$\widetilde{\delta}_{\mathfrak{m}}(\overline{\widetilde{x}}) := \overline{\widetilde{\delta}(\widetilde{x})} \quad \text{and} \quad \widetilde{d}_{\mathfrak{m}}(\overline{\widetilde{x}}) := \overline{\widetilde{d}(\widetilde{x})}$$

for  $\widetilde{x} = \widetilde{x} + \mathbf{m}\widehat{R}$ , where  $\widetilde{x} \in \widehat{R}$ . Note that  $Q_{mr}(R)/\mathbf{m}Q_{mr}(R) \subseteq Q_{mr}(\widehat{R}/\mathbf{m}\widehat{R})$ . It is clear that  $\widetilde{\delta}_{\mathbf{m}}$  is a  $\widetilde{b}$ -generalized derivation with associated map  $\widetilde{d}_{\mathbf{m}}$ . Note that  $\widetilde{b} \neq \widetilde{0}$  since  $E[b] = 1$ .

We work in the prime ring  $\widehat{R}/\mathbf{m}\widehat{R}$  with extended centroid  $\overline{C} (= C + \mathbf{m}\widehat{R}/\mathbf{m}\widehat{R})$ . Let  $\widetilde{x} = \widetilde{x} + \mathbf{m}\widehat{R} \in \widehat{R}/\mathbf{m}\widehat{R}$ , where  $\widetilde{x} \in \widehat{R}$ . Write  $\widetilde{x} = \sum_{v \in \Lambda}^{\perp} x_v e_v$ , where  $x_v \in R$ . Then  $\widetilde{\delta}(\widetilde{x}) = \sum_{v \in \Lambda}^{\perp} \delta(x_v) e_v$  and

$$\widetilde{\delta}_{\mathbf{m}}(\widetilde{x})^n = \overline{\widetilde{\delta}(\widetilde{x})}^n = \overline{\left(\sum_{v \in \Lambda}^{\perp} \delta(x_v) e_v\right)^n} = \sum_{v \in \Lambda}^{\perp} \overline{\delta(x_v)^n e_v} = \overline{0}.$$

In view of Theorem 2.4, the derivation  $\widetilde{d}_{\mathbf{m}}$  is X-inner. It follows from Proposition 3.1 that  $\widetilde{d}$  is X-inner. Thus,  $\widetilde{d} = \text{ad}(q')$  for some  $q' \in Q_{mr}(R)$ . Moreover, in view of Theorem 2.4, for any  $\mathbf{m} \in \text{Spec}(\mathbf{B})$  we have  $\widetilde{q}' \widetilde{b} = q' b \in \overline{C} \widetilde{b}$ . By Lemma 3.2,  $q' b = \beta b$  for some  $\beta \in C$ . Set  $q := q' - \beta$ . Then  $d = \text{ad}(q)$  and  $qb = 0$ .

Let  $x, y \in R$ . Then

$$\delta(xy) = \delta(x)y + bxd(y) = \delta(x)y + bx(qy - yq),$$

implying that

$$\delta(xy) + bxyq = (\delta(x) + bxq)y.$$

By Fact 1.7, there exists  $w \in Q_{mr}(R)$  such that  $\delta(x) = -bxq + wx$  for all  $x \in R$ . Thus,  $(wx - bxq)^n = 0$  for all  $x \in R$  and hence for all  $x \in Q_{mr}(R)$  (see Fact 1.4). In view of Theorem 3.3, there exists  $\mu \in C$  such that  $w = \mu b$  and  $(q - \mu)b = 0$ . Thus, by the fact that  $qb = 0$ , we see that  $\mu = 0$  and  $w = 0$ . That is,  $\delta(x) = -bxq$  for all  $x \in R$ , as asserted. □

**PROOF OF THEOREM 1.2.** Let  $e := E[b]$ ,  $\delta_1(x) := e\delta(x)$  and  $d_1(x) := ed(x)$  for  $x \in R$ . Then  $(1 - e)\delta(xy) = (1 - e)\delta(x)y$  for all  $x, y \in R$ . By Fact 1.7, there exists  $w \in Q_{mr}(R)$  such that  $(1 - e)\delta(x) = wx$  for all  $x \in R$ . But  $(wx)^n = 0$  for all  $x \in R$ . This implies that  $w = 0$ ; that is,  $(1 - e)\delta(x) = 0$  for all  $x \in R$ .

Note that  $\delta_1 : R \rightarrow Q_{mr}(R)$ ,  $d_1 : R \rightarrow Q_{mr}(R)$ , and  $\delta_1(xy) = \delta_1(x)y + bxd_1(y)$  for all  $x, y \in R$ . Applying the same argument given in the proof of Lemma 3.4,  $d_1$  is a derivation and can be uniquely extended to a derivation  $\widetilde{d}_1 : \widehat{R} \rightarrow Q_{mr}(R)$  by defining

$$\widetilde{d}_1\left(\sum_{v \in \Lambda}^{\perp} x_v e_v\right) = \sum_{v \in \Lambda}^{\perp} (ed(x_v))e_v, \quad \text{where } x_v \in R.$$

On the other hand,  $\delta_1$  can be extended to a map  $\widetilde{\delta}_1 : \widehat{R} \rightarrow Q_{mr}(R)$  by defining

$$\widetilde{\delta}_1\left(\sum_{v \in \Lambda}^{\perp} x_v e_v\right) = \sum_{v \in \Lambda}^{\perp} (e\delta(x_v))e_v, \quad \text{where } x_v \in R.$$

Note that  $\widetilde{d}_1(e\widehat{R}) \subseteq eQ_{mr}(R)$  and  $\widetilde{\delta}_1(e\widehat{R}) \subseteq eQ_{mr}(R)$ . Working on  $eQ_{mr}(R)$ ,

$$\widetilde{\delta}_1(xy) = \widetilde{\delta}_1(x)y + bx\widetilde{d}_1(y)$$

for all  $x, y \in e\widehat{R}$ . Note that  $Q_{mr}(e\widehat{R}) = eQ_{mr}(R)$  and that  $(\widetilde{\delta}_1(x))^n = 0$  for all  $x \in e\widehat{R}$ . Since  $E[b] = e$  and the extended centroid of  $e\widehat{R}$  is equal to  $eC$ , it follows from Lemma 3.4 that

there exists  $q \in eQ_{mr}(R)$  such that  $ed(x) = [q, x]$  for  $x \in e\widehat{R}$ ,  $e\delta(x) = -bxq$  for  $x \in e\widehat{R}$ , and  $qb = 0$ .

Choose a dense ideal  $I$  of  $R$  such that  $(1 - e)I \subseteq R$ . Let  $x, y, z \in I$ . Then

$$\begin{aligned}\delta(x(1 - e)y) &= \delta(x)(1 - e)y + bxd((1 - e)y) \\ &= bxd((1 - e)y) = bx(ed(y) - ed(e)y - ed(y)) = 0,\end{aligned}$$

since  $\delta(x)(1 - e) = 0$  and  $ed$  is a derivation on  $Q_{mr}(R)$ . So  $\delta((1 - e)I^2) = 0$ . Let  $x \in I^2$ . Then

$$\delta(x) = e\delta(x) = e\delta(ex + (1 - e)x) = e\delta(ex) = -b(ex)q = -bxq.$$

Up to now, we have proved that  $\delta(x) = -bxq$  for  $x \in I^2$ . Let  $y \in R$  and  $x \in I^2$ . We notice that  $ed(x) = ed(ex) = e[q, ex] = [q, x]$ . Then  $yx \in I^2$  and

$$-byxq = \delta(yx) = \delta(y)x + byd(x) = \delta(y)x + byed(x) = \delta(y)x + by[q, x],$$

implying that  $(\delta(y) + byq)x = 0$ . That is,  $(\delta(y) + byq)I^2 = 0$  and so  $\delta(y) = -byq$ , as asserted.  $\square$

## References

- [1] K. I. Beidar, W. S. Martindale III and A. V. Mikhaev, *Rings with Generalized Identities*, Monographs and Textbooks in Pure and Applied Mathematics, 196 (Marcel Dekker, Inc, New York, 1996).
- [2] M. Brešar, 'On the distance of the composition of two derivations to the generalized derivations', *Glasg. Math. J.* **33**(1) (1991), 89–93.
- [3] J. C. Chang, 'Generalized skew derivations with nilpotent values on Lie ideals', *Monatsh. Math.* **161**(2) (2010), 155–160.
- [4] J. C. Chang, 'Generalized skew derivations with power central values on Lie ideals', *Comm. Algebra* **39**(6) (2011), 2241–2248.
- [5] J. C. Chang and J. S. Lin, ' $(\alpha, \beta)$ -derivations with nilpotent values', *Chinese J. Math.* **22**(4) (1994), 349–355.
- [6] C.-L. Chuang, 'GPIs having coefficients in Utumi quotient rings', *Proc. Amer. Math. Soc.* **103** (1988), 723–728.
- [7] C.-L. Chuang and T.-K. Lee, 'Derivations modulo elementary operators', *J. Algebra* **338** (2011), 56–70.
- [8] T. S. Erickson, W. S. Martindale III and J. M. Osborn, 'Prime nonassociative algebras', *Pacific J. Math.* **60**(1) (1975), 49–63.
- [9] C. Faith and Y. Utumi, 'On a new proof of Litoff's theorem', *Acta Math. Acad. Sci. Hung.* **14** (1967), 369–371.
- [10] A. Giambruno and I. N. Herstein, 'Derivations with nilpotent values', *Rend. Circ. Mat. Palermo* (2) **30**(2) (1981), 199–206.
- [11] I. N. Herstein, *Topics in Ring Theory* (The University of Chicago Press, Chicago, Ill.-London, 1969).
- [12] I. N. Herstein, 'Center-like elements in prime rings', *J. Algebra* **60** (1979), 567–574.
- [13] I. N. Herstein, 'Derivations of prime rings having power central values', in: *Algebraists' Homage: Papers in Ring Theory and Related Topics (New Haven, Conn., 1981)*, Contemporary Mathematics, 13 (American Mathematical Society, Providence, RI, 1982), 163–171.
- [14] B. Hvala, 'Generalized derivations in rings', *Comm. Algebra* **26**(4) (1998), 1147–1166.
- [15] V. K. Kharchenko, 'Differential identities of semiprime rings', *Algebra Logika* **18** (1979), 86–119; *Algebra Logic* **18** (1979), 58–80 (English translation).

- [16] C. Lanski, 'Derivations with nilpotent values on left ideals', *Comm. Algebra* **22**(4) (1994), 1305–1320.
- [17] T.-K. Lee, 'Semiprime rings with differential identities', *Bull. Inst. Math. Acad. Sin.* **20** (1992), 27–38.
- [18] T.-K. Lee, 'Generalized derivations of left faithful rings', *Comm. Algebra* **27**(8) (1999), 4057–4073.
- [19] T.-K. Lee, 'Generalized skew derivations characterized by acting on zero products', *Pacific J. Math.* **216**(2) (2004), 293–301.
- [20] T.-K. Lee and K.-S. Liu, 'The Skolem-Noether theorem for semiprime rings satisfying a strict identity', *Comm. Algebra* **35**(6) (2007), 1949–1955.
- [21] T.-K. Lee and K.-S. Liu, 'Generalized skew derivations with algebraic values of bounded degree', *Houston J. Math.* **39**(3) (2013), 733–740.
- [22] T.-K. Lee and Y. Zhou, 'An identity with generalized derivations', *J. Algebra Appl.* **8**(3) (2009), 307–317.
- [23] W. S. Martindale III, 'Prime rings satisfying a generalized polynomial identity', *J. Algebra* **12** (1969), 576–584.
- [24] X. Xu, J. Ma and F. Niu, 'Annihilators of power central values of generalized derivations', *Adv. Math. (China)* **41**(1) (2012), 113–119.

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