Canad. Math. Bull. Vol. **40** (2), 1997 pp. 174–182

NON-UNIQUENESS FOR THE p-HARMONIC FLOW

NORBERT HUNGERBÜHLER

ABSTRACT. If $f_0: \Omega \subset \mathbb{R}^m \longrightarrow S^n$ is a weakly p-harmonic map from a bounded smooth domain Ω in $\mathbb{R}^m$ (with $2 < p < m$) into a sphere and if f_0 is not stationary p-harmonic, then there exist infinitely many weak solutions of the p-harmonic flow with initial and boundary data f_0, *i.e.*, there are infinitely many global weak solutions $f: \Omega \times \mathbb{R}_+ \longrightarrow S^n$ of

$$\partial_t f - \operatorname{div}(|\nabla f|^{p-2}\nabla f) = |\nabla f|^p f \quad \text{weakly on } \Omega \times \mathbb{R}_+$$
$$f = f_0 \quad \text{on the parabolic boundary of } \Omega \times \mathbb{R}_+.$$

We also show that there exist non-stationary weakly $(m-1)$-harmonic maps $f_0: B^m \longrightarrow S^{m-1}$.

1. **Introduction.** Let M and N be compact smooth Riemannian manifolds (M possibly having a boundary) with metrics γ and g respectively. Let m and n denote the dimensions of M and N. For a C^1-map $f: M \longrightarrow N$ the p-energy density is defined by

$$e(f)(x) := \frac{1}{p}|df_x|^p \tag{1}$$

and the p-energy by

$$E(f) := \int_M e(f)\, d\mu. \tag{2}$$

Here, p denotes a real number in $[2, \infty[$, $|df_x|$ is the Hilbert-Schmidt norm with respect to γ and g of the differential $df_x \in T_x^*(M) \otimes T_{f(x)}(N)$ and μ is the measure on M which is induced by the metric. In local coordinates $E(f)$ is given by:

$$E_U(f) = \frac{1}{p}\int_\Omega \Big(\gamma^{\alpha\beta}(g_{ij}\circ f)\partial_\alpha f^i \partial_\beta f^j\Big)^{\frac{p}{2}} \sqrt{\gamma}\, dx.$$

Here, $U \subset M$ and $\Omega \subset \mathbb{R}^m$ denote the domain and the range of the coordinates on M and it is assumed that $f(U)$ is contained in the domain of the coordinates chosen on N. Upper indices denote components, whereas ∂_α denotes the derivative with respect to the coordinate variable x^α. We use the usual summation convention.

First we consider variations of the energy-functional of the form $f_\varepsilon = f + \varepsilon\varphi$ with $\varphi \in C_0^\infty\big(B_\rho(x), \mathbb{R}^n\big)$ with $\bar{B}_\rho(x) \subset U$ such that $f_\varepsilon(U)$ is contained in the domain of the coordinates chosen on N provided $|\varepsilon|$ is small enough. The resulting Euler-Lagrange equations are

$$\Delta_p f = -(\gamma^{\alpha\beta} g_{ij}\partial_\alpha f^i \partial_\beta f^j)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\Gamma^l_{ij}\partial_\alpha f^i \partial_\beta f^j \tag{3}$$

This work is supported by the Swiss National Science Foundation.
Received by the editors October 6, 1995.
AMS subject classification: 35K40, 35K55, 35K65.

https://doi.org/10.4153/CMB-1997-021-9 Published online by Cambridge University Press

in local coordinates. Here, the operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}} \partial_\beta \Big(\sqrt{\gamma} (\gamma^{\alpha\beta} g_{ij} \partial_\alpha f^i \partial_\beta f^j)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \partial_\alpha f^l \Big)$$

is called *p-Laplace operator* (for $p = 2$ this is just the Laplace-Beltrami operator and does not depend on N). On the right hand side of (3) the Γ^l_{ij} denote the Christoffel-symbols related to the manifold N. According to Nash's embedding theorem we can think of N as being isometrically embedded in some Euclidean space $\mathbb{R}^k$ since N is compact. Then, if we regard f as a function into $N \subset \mathbb{R}^k$, equation (3) admits a geometric interpretation, namely

$$\Delta_p f \perp T_f N \tag{4}$$

with Δ_p being the p-Laplace operator with respect to the manifolds M and $\mathbb{R}^k$. If N is a unit sphere, then (4) becomes $\Delta_p f = \lambda f$ for a function $\lambda \colon \Omega \longrightarrow \mathbb{R}$. Multiplying the equation with f one obtains $\lambda = -|\nabla f|^p$.

For $p > 2$ the p-Laplace operator is degenerated elliptic. (Weak) solutions of (3) are called (weakly) p-harmonic maps.

On the other hand, vanishing variations of the form

$$f_\varepsilon(x) = f\big(x + \varepsilon \varphi(x)\big) \tag{5}$$

with $\varphi \in C_0^\infty\big(B_\rho(x), \mathbb{R}^m\big)$ and $|\varepsilon|$ small enough, lead to

$$0 = \int_\Omega \sqrt{\gamma} (\gamma^{\sigma\rho} \partial_\sigma f^i \partial_\rho f^i)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \partial_\alpha f^j \partial_\beta (\partial_\delta f^j \varphi^\delta)\, dx.$$

If M is locally Euclidean, this can be rewritten as

$$\int_\Omega |\nabla f|^{p-2} \partial_\alpha f^i \partial_\beta f^i \partial_\beta \varphi^\alpha \, dx = \int_\Omega \frac{1}{p} |\nabla f|^p \partial_j \varphi^j \, dx. \tag{6}$$

For a non-constant metric γ this reformulation is in general not possible since

$$h_\delta := \sqrt{\gamma} (\gamma^{\sigma\rho} \partial_\sigma f^i \partial_\rho f^i)^{\frac{p}{2}-1} \gamma^{\alpha\beta} \partial_\alpha f^j \partial_{\beta\delta} f^j$$

is not a gradient then.

Weakly p-harmonic maps in $W^{1,p}(M, N)$ which satisfy (6) are called *stationary p-harmonic* and satisfy the system

$$\partial_\alpha |\nabla f|^p = p\, \partial_\beta (|\nabla f|^{p-2} \partial_\alpha f^j \partial_\beta f^j) \tag{7}$$

in distributional sense. Here, $W^{1,p}(M, N)$ denotes the nonlinear Sobolev space of functions $g \in W^{1,p}(M, \mathbb{R}^k)$ with $g(x) \in N$ for almost every $x \in M$. Notice that (7) is formally obtained from (4) by multiplication with ∇f. So, (smooth) p-harmonic maps are always stationary. This is in general not true for weakly p-harmonic maps and we will see examples for this later on.

The heat flow related to the p-energy is described by

$$\partial_t f - \Delta_p f \perp T_f N \tag{8}$$

$$f|_{t=0} = f_0 \tag{9}$$

or explicitly for (8)

$$\partial_t f - \Delta_p f = \big(pe(f)\big)^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f) \tag{10}$$

where $A(f)(\cdot,\cdot)$ is the second fundamental form on N. For $p = 2$ Eells and Sampson showed in their famous work [6] of 1964, that there exist global solutions of (8)–(9) provided N has non-positive sectional curvature and that the flow tends for suitable $t_k \longrightarrow \infty$ to a harmonic map. Existence and uniqueness of partially regular solutions of the harmonic flow on Riemannian surfaces (*i.e.* $p = m = 2$) has been shown by Struwe in [10] and recently Freire [7] proved uniqueness in this case in the class of weak solutions. Existence for $p = 2$ in higher dimensions (*i.e.* $m > 2$) has been obtained by Chen and Struwe in [4]. Coron [5] constructed maps u_0 such that the 2-flow $u\colon B^3 \times \mathbb{R}_+ \longrightarrow S^2$ with initial and boundary data u_0 has infinitely many weak solutions. In fact, Coron showed that for suitable weakly 2-harmonic maps $u_0\colon B^3 \longrightarrow S^2$ the construction of Chen [2] and Chen-Struwe [4] leads to a weak solution $\underline{u}(x,t)$ of the flow which satisfies a certain monotonicity property (see [11] or [4]) in contrast to $\overline{u}(x,t) := u_0(x)$ which is also a weak solution of the flow. Since a monotonicity formula is not available for the p-harmonic flow for $p > 2$, this approach cannot be carried over to the latter situation. Recently, Coron's result has been reproved in [1] by a different technique. In this paper, we prove non-uniqueness of the p-harmonic flow in case $p > 2$ by combining ideas of [1] and [9].

We will establish the following theorem:

THEOREM 1. *If $f_0\colon \Omega \subset \mathbb{R}^m \longrightarrow S^n$ is a weakly p-harmonic map from a bounded smooth domain Ω of $\mathbb{R}^m$ (with $2 < p < m$) into a sphere, and if f_0 is not stationary p-harmonic then there exist infinitely many weak solutions of the p-harmonic flow with initial and boundary data f_0, i.e., there are infinitely many global weak solutions $f\colon \Omega \times \mathbb{R}_+ \longrightarrow S^n$ of*

$$\partial_t f - \operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right) = |\nabla f|^p f \quad \textit{weakly on } \Omega \times \mathbb{R}_+ \tag{11}$$

$$f = f_0 \quad \textit{on the parabolic boundary of } \Omega \times \mathbb{R}_+. \tag{12}$$

REMARKS. (a) Ω can be replaced by any smooth compact Riemannian manifold M which is locally flat.

(b) In the last section we will actually construct examples of non-stationary weakly p-harmonic maps.

2**. Existence of a global weak solution.** In this section we establish a special case of the following theorem, which is proved in [9]. The approximate solutions of the p-harmonic flow constructed in the proof will be used in Section 3 to establish the existence of multiple solutions.

THEOREM 2. *For* $2 < p < \dim(M)$ *there exists a global weak solution of the p-harmonic flow between Riemannian manifolds M and N for arbitrary initial data having finite p-energy in the case when the target N is a homogeneous space with a left invariant metric. The solution* $f\colon M \times [0,\infty) \longrightarrow N$ *satisfies the energy inequality*

$$\frac{1}{2}\int_0^T \int_M |\partial_t f|^2 \, dt \, d\mu + \frac{1}{p}\int_M |df(T)|^p \, d\mu \leq \frac{1}{p}\int_M |df(0)|^p \, d\mu \tag{13}$$

for all $T > 0$.

PROOF. (in the case $M = \Omega$ and $N = S^n$):
For $f_0, g \in W^{1,p}(\Omega, S^n)$ fixed,

$$f \in W^{1,p}_{f_0}(\Omega, S^n) := \{w \in W^{1,p}(\Omega, S^n) : w - f_0 \in W^{1,p}_0(\Omega, S^n)\}$$

and $h > 0$ let

$$E_g(f) := \int_\Omega \Big(\frac{1}{p}|\nabla f|^p + \frac{1}{2h}|f - g|^2\Big) \, dx.$$

By the direct method of the calculus of variations we find a function $w \in W^{1,p}_{f_0}(\Omega, S^n)$ such that

$$E_g(w) = \inf_{f \in W^{1,p}_{f_0}(\Omega, S^n)} E_g(f).$$

The set of arguments for which the infimum of E_g is attained is usually denoted by $\arg\min E_g$. Now we define recursively a family $f_i \in W^{1,p}_{f_0}(\Omega, S^n)$ by

$$f_{i+1} \in \arg\min E_{f_i} \quad \text{for } i = 0, 1, \ldots.$$

Notice that f_i is a weak solution of the Euler-Lagrange equation to energy $E_{f_{i-1}}$, *i.e.*, there holds for every $i = 1, 2, \ldots$

$$\Pi_{T_{f_i}S^n}\Big(\frac{1}{h}(f_i - f_{i-1})\Big) - \operatorname{div}(|\nabla f_i|^{p-2}\nabla f_i) = |\nabla f_i|^p f_i \tag{14}$$

in distributional sense and $f_i = f_0$ on $\partial\Omega$ in the trace sense. In (14) $\Pi_{T_f S^n}$ denotes the orthogonal projection onto the tangent space $T_f S^n$.

Since f_i minimizes $E_{f_{i-1}}$ we have in particular $E_{f_{i-1}}(f_i) \leq E_{f_{i-1}}(f_{i-1})$, *i.e.*,

$$\int_\Omega \Big(\frac{1}{p}|\nabla f_i|^p + \frac{1}{2h}\,|f_i - f_{i-1}|^2\Big) \, dx \leq \int_\Omega \frac{1}{p}|\nabla f_{i-1}|^p \, dx. \tag{15}$$

Now we define the function $f^{(h)}\colon \Omega \times \mathbb{R}_+ \longrightarrow S^n$ by

$$f^{(h)}(t, \cdot) := f_i \quad \text{for } t \in \big[ih, (i+1)h\big).$$

Thus, rewriting (14) by using the notation $\partial^{(h)}$ for the forward difference quotient in time with step length h, *i.e.*, $(\partial^{(h)}f)(t,x) = \frac{1}{h}\big(f(t+h,x) - f(t,x)\big)$, we get

$$\Pi_{T_{f^{(h)}}S^n}\partial^{(-h)}f^{(h)} - \operatorname{div}(|\nabla f^{(h)}|^{p-2}\nabla f^{(h)}) = |\nabla f^{(h)}|^p f^{(h)} \tag{16}$$

in distributional sense on $\Omega \times (h, \infty)$ and $f^{(h)} = f_0$ on the parabolic boundary of $\Omega \times \mathbb{R}_+$. Summing up (15) we obtain

$$\frac{1}{2}\int_0^{kh}\int_\Omega |\partial^{(h)} f^{(h)}|^2\,dx\,dt + \frac{1}{p}\int_\Omega |\nabla f^{(h)}(kh)|^p\,dx \le \frac{1}{p}\int_\Omega |\nabla f_0|^p\,dx. \tag{17}$$

So, in particular we see that $\{f^{(h)}\}_{h>0}$ is a bounded set in $L^\infty\big(0,\infty; W^{1,p}(\Omega, S^n)\big)$ and hence every sequence in $\{f^{(h)}\}_{h>0}$ has a subsequence $f_i := f^{(h_i)}$ such that

$$f_i - f_0 \overset{*}{\rightharpoonup} f - f_0 \quad \text{weakly}^* \text{ in } L^\infty\big(0,\infty; W_0^{1,p}(\Omega)\big) \tag{18}$$

for a map $f \in L^\infty(0,\infty; W_{f_0}^{1,p}\big(\Omega, S^n)\big)$.

It is now easy to see, that the difference quotient for *fixed* step length $H > 0$ of the sequence $\{f^{(h)}\}_{H>h>0}$ is bounded in $L^2\big(0,\infty; L^2(\Omega)\big)$ by a constant which does not depend on H. Since the set $\{f^{(h)}\}_{h>0}$ is precompact in the space $L^r\big(0, T; L^r(\Omega)\big)$ for all $r < p^* = \frac{mp}{m-p}$ (see [9, Lemma 2]) we have that $\{\partial^{(H)} f\}_{H>0}$ is bounded in $L^2\big(0,\infty; L^2(\Omega)\big)$ and hence f has a distributional time derivative in the latter space. In fact, as a consequence of the partial integration rule for the discrete operator $\partial^{(h)}$, we obtain

$$\partial^{(h)} f^{(h)} \rightharpoonup \partial_t f \quad \text{weakly in } L^2\big(0,\infty; L^2(\Omega)\big)$$

and moreover

$$\Pi_{T_{f^{(h)}}S^n}\,\partial^{(-h)} f^{(h)} \rightharpoonup \partial_t f \quad \text{weakly in } L^2\big(\varepsilon,\infty; L^2(\Omega)\big) \tag{19}$$

for arbitrary $\varepsilon > 0$. This allows to pass to the limit in the first term in equation (16).

Now, by the compactness result in [8] there exists a sequence $h \longrightarrow 0$ such that

$$\nabla f^{(h)} \longrightarrow \nabla f \quad \text{strongly in } L^q\big(0, T; L^q(\Omega)\big) \text{ for all } q < p \tag{20}$$

and hence (since $\{\nabla f^{(h)}\}_{h>0}$ is bounded in $L^p\big(0, T; L^p(\Omega)\big)$)

$$|\nabla f^{(h)}|^{p-2}\nabla f^{(h)} \rightharpoonup |\nabla f|^{p-2}\nabla f \quad \text{weakly in } L^{p'}\big(0, T; L^{p'}(\Omega)\big). \tag{21}$$

This allows to pass to the limit in the p-Laplace term of equation (16) (and in the boundary condition). Now, we are left with the problem to do this also on the right-hand side of (16). To overcome this difficulty, we use a similar technique as in [3].

By taking the wedge product of (16) with $f^{(h)}$, we get

$$\Pi_{T_{f^{(h)}}S^n}\,\partial^{(-h)} f^{(h)} \wedge f^{(h)} - \operatorname{div}\big(|\nabla f^{(h)}|^{p-2}\nabla f^{(h)} \wedge f^{(h)}\big) = 0. \tag{22}$$

Using the previously stated results, we can pass to the limit in (22) and obtain

$$\partial_t f \wedge f - \operatorname{div}\big(|\nabla f|^{p-2}\nabla f \wedge f\big) = 0 \tag{23}$$

in distributional sense. A short calculation shows that for every function $\tau \in C_0^\infty(\Omega \times \mathbb{R}_+; \mathbb{R})$ the relation

$$\left(\partial_t f - \operatorname{div}(|\nabla f|^{p-2}\nabla f) - |\nabla f|^p f\right) \cdot f\tau = 0 \tag{24}$$

automatically holds in distributional sense, provided $|f| = 1$ a.e. in Ω. Note, that any function $\varphi \in C_0^\infty(\Omega \times \mathbb{R}_+; \mathbb{R}^{n+1})$ can be decomposed in the following way

$$\varphi = f(f \cdot \varphi) - f \wedge (f \wedge \varphi). \tag{25}$$

Using the facts that $f \in L^\infty\left(0, \infty; W^{1,p}(\Omega, S^n)\right)$ and $\partial_t f \in L^2\left(0, \infty; L^2(\Omega)\right)$ and an approximation argument we obtain that $\psi = f \wedge \varphi$ and $\tau = f \cdot \varphi$ are admissible test-functions in (23) and in (24). Subtracting the resulting equations and using (25), we get

$$\partial_t f - \operatorname{div}(|\nabla f|^{p-2}\nabla f) = |\nabla f|^p f$$

in distributional sense. Thus f is a weak solution of (11) and (12).

By passing to the limit in (17) we get the energy inequality stated in the theorem.

3**. Proof of Theorem 1.** We assume that $f_0 \colon \Omega \longrightarrow S^n$ is weakly p-harmonic but not stationary. That means that for some index α

$$\partial_\alpha |\nabla f_0|^p \neq p\, \partial_\beta(|\nabla f_0|^{p-2} \partial_\beta f_0^j \partial_\alpha f_0^j). \tag{26}$$

In order to prove Theorem 1, it is sufficient to show that the weak solution f constructed in the previous section is not constant in time. To do this we use a similar idea as in [1].

Using variations of the form (5) for the energy E_g we find that for the approximating solutions $f^{(h)}$ introduced in the previous section there holds

$$\partial^{(-h)} f^{(h)} \cdot \partial_\alpha f^{(h)} + \frac{1}{p}\partial_\alpha |\nabla f^{(h)}|^p = \partial_\beta(|\nabla f^{(h)}|^{p-2} \partial_\alpha f^{(h)} \cdot \partial_\beta f^{(h)}). \tag{27}$$

Let us assume by contradiction that the limit function f is constant

$$f(\cdot, t) = f_0 \tag{28}$$

for all times $t \geq 0$ and hence especially $E_p\left(f(\cdot, t)\right) = E_p(f_0)$. By (20) we may assume that $\nabla f^{(h)} \longrightarrow \nabla f = \nabla f_0$ a.e. on $[0, \infty) \times \Omega$. Thus, using Fatou's Lemma and the energy inequality (17), we have that

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla f_0|^p \, dx\, dt &= \int_0^T \int_\Omega \lim_{h \to 0} |\nabla f^{(h)}|^p \, dx\, dt \\
&\leq \liminf_{h \to 0} \int_0^T \int_\Omega |\nabla f^{(h)}|^p \, dx\, dt \\
&\leq \int_0^T \int_\Omega |\nabla f_0|^p \, dx\, dt
\end{aligned}$$

https://doi.org/10.4153/CMB-1997-021-9 Published online by Cambridge University Press

which implies that

$$\nabla f^{(h)} \longrightarrow \nabla f \text{ strongly in } L^p_{\mathrm{loc}}(\Omega \times [0,\infty]).$$

Using the energy inequality once more we get

$$\begin{aligned}
&\frac{1}{p}\int_0^T\int_\Omega |\nabla f_0|^p\,dx\,dt \\
&\quad\le \liminf_{h\to 0}\Big(\frac{1}{2}\int_0^T\int_0^t\int_\Omega |\partial^{(h)} f^{(h)}(\tau)|^2\,dx\,d\tau\,dt + \frac{1}{p}\int_0^T\int_\Omega |\nabla f^{(h)}(t)|^p\,dx\,dt\Big) \\
&\quad\le \frac{1}{p}\int_0^T\int_\Omega |\nabla f_0|^p\,dx\,dt
\end{aligned}$$

and this yields that

$$\partial^{(h)} f^{(h)} \longrightarrow 0 \quad \text{strongly in } L^2(\Omega\times\mathbb{R}_+).$$

This allows to pass to the limit in (27) and we get

$$\partial_\alpha |\nabla f|^p = p\,\partial_\beta(|\nabla f|^{p-2}\partial_\beta f^j \partial_\alpha f^j)$$

in contradiction to (26) which proves that (28) cannot hold true.

Now we have the two distinct solutions $\tilde f(\cdot,t) = f_0$ and the limit f of the approximating functions $f^{(h)}$. Then

$$f_\tau(x,t) = \begin{cases} f_0(x) & \text{for } t \le \tau \\ f(x,t-\tau) & \text{for } t > \tau \end{cases}$$

is an infinite family of solutions of (11) and (12) and the theorem is proved.

REMARKS. (a) Theorem 1 remains true for an arbitrary homogeneous space N with a left invariant metric as target manifold in place of a sphere: The construction of the approximating solutions and the passage to the limit are described in [9], the second part of the proof (*i.e.* Section 3) remains unchanged.

(b) It would be a challenging problem to investigate whether the inverse implication of Theorem 1 holds true: If $f_0 \colon \Omega \longrightarrow N$ is a stationary weakly p-harmonic function, then the solution of (11) and (12) is unique.

4. **Examples of non-stationary p-harmonic maps.** We show now that there exist non-stationary weakly $(m-1)$-harmonic maps $f_0 \colon B^m \longrightarrow S^{m-1}$. The idea to use the conformal invariance of the p-energy in dimension p is similar to the construction of Coron in [5].

Let us consider the map

$$w \colon S^{m-1} \longrightarrow S^{m-1}, \quad x \longmapsto \pi^{-1}\circ v\circ \pi(x)$$

where $\pi \colon S^{m-1} \longrightarrow \mathbb{R}^{m-1}\cup\{\infty\} =: \overline{\mathbb{R}}^{m-1}$ is the stereographic projection and $v \colon \overline{\mathbb{R}}^{m-1} \longrightarrow \overline{\mathbb{R}}^{m-1}$ is a bijective conformal map, *i.e.*, v belongs to the Möbius group of $\overline{\mathbb{R}}^{m-1}$ which is generated by dilatations $x \longmapsto \lambda x$, isometric maps and inversions $x \longmapsto x/|x|^2$ (and in two dimensions by the complex Möbius transformations). Hence w is a conformal map.

https://doi.org/10.4153/CMB-1997-021-9 Published online by Cambridge University Press

The function $B^m \longrightarrow S^{m-1}$, $x \longmapsto \frac{x}{|x|}$ belongs to $W^{1,p}$ if $p < m$ and is weakly p-harmonic. Because of the conformal invariance of the p-energy in dimension p the map

$$f_0 \colon B^m \longrightarrow S^{m-1}, \quad x \longmapsto w\left(\frac{x}{|x|}\right)$$

is also weakly p-harmonic if $p = m - 1$. Now for f_0 there holds

LEMMA 1. *If f_0 is stationary then*

$$\int_{S^{m-1}} |\nabla w|^{m-1} y \, d\sigma(y) = 0. \tag{29}$$

PROOF. First, we observe that for $p = m - 1$

$$\begin{aligned} \int_{B^m} |\nabla f_0|^p \frac{x}{|x|} \, dx &= \int_{S^{m-1}} |\nabla f_0|^p y \, d\sigma(y) \\ &= \int_{S^{m-1}} |\nabla w|^p y \, d\sigma(y). \end{aligned} \tag{30}$$

We now use $|x| - 1$ as a test-function in (7) and obtain that for every index α

$$\begin{aligned} \int_{B^m} |\nabla f_0|^p \frac{x^\alpha}{|x|} \, dx &= \int_{B^m} |\nabla f_0|^p \partial_\alpha |x| \, dx \\ &= p \int_{B^m} |\nabla f_0|^{p-2} \partial_\alpha f_0 \cdot \underbrace{\partial_\beta f_0 \frac{x^\beta}{|x|}}_{=0} \, dx = 0. \end{aligned} \tag{31}$$

The combination of (30) and (31) gives the desired result. ∎

It is easy to see that *e.g.* for a dilatation $v \colon x \longmapsto \lambda x$ with $\lambda \neq 1$ the condition (29) is violated.

ACKNOWLEDGEMENT. I would like to thank Prof. Michael Struwe for his encouragement and Prof. Stefan Müller for his useful comments and valuable suggestions. I am grateful to the Institute for Mathematics and its Applications (IMA) of the University of Minnesota, Minneapolis, for the hospitality and to the Swiss National Science Foundation for supporting this research.

REFERENCES

1. F. Bethuel, J.-M. Coron, J.-M. Ghidaglia and A. Soyeur, *Heat flows and relaxed energies for harmonic maps*, Progr. Nonlinear Differential Equations Appl. **7**(1992), 99–110.

2. Y. Chen, *The weak solutions to the evolution problem of harmonic maps*, Math. Z. **201**(1989), 69–74.

3. Y. Chen, M.-C. Hong and N. Hungerbühler, *Heat flow of p-harmonic maps with values into spheres*, Math. Z. **215**(1994), 25–35.

4. Y. Chen and M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201**(1989), 83–103.

5. J.-M. Coron, *Nonuniqueness for the heat flow of harmonic maps*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **7**(1990), 335–344.

6. J. Eells and H. J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86**(1964), 109–169.

7. A. Freire, *Uniqueness for the harmonic map flow from surfaces to general targets*, Comment. Math. Helv. **70**(1995), 310–338.
8. N. Hungerbühler, *Compactness properties of the p-harmonic flow into homogeneous spaces*, Nonlinear Anal., to appear.
9. ———, *Global weak solutions of the p-harmonic flow into homogeneous spaces*, Indiana Univ. Math. J. (1)**45**(1996), 275–288.
10. M. Struwe, *On the evolution of harmonic maps of Riemannian surfaces*, Comment. Math. Helv. **60**(1985), 558–581.
11. ———, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom. **28**(1988), 485–502.

ETH-Zentrum
Departement Mathematik
CH-8092 Zürich
Switzerland
e-mail: buhler@math.ethz.ch