Appendix C Some practice with the path integral in field theory

The path integral is extremely useful, both in field theory and in string theory. This appendix provides a brief review of path integration, and some applications. Many of the examples are drawn from finite-temperature field theory. These are instructive since one can easily write explicit expressions. They are also useful for understanding the high-temperature universe and are closely connected to the computations which arise in compactified theories.

C.1 Path integral review

Feynman gave an alternative formulation of quantum mechanics in which one calculates amplitudes by summing over the possible trajectories of a system, weighting by $e^{iS/\hbar}$, where S is th classical action of the trajectory. For a particle, the path integral is

$$Z = \int [dx]\, e^{iS/\hbar}. \tag{C1}$$

Here $\int [dx]$ implies an instruction to sum over all possible paths of the particle.

This generalizes immediately to field theory, where surprisingly it is often more useful than in the case of quantum systems with a small number of degrees of freedom:

$$Z = \int [d\phi]\, e^{iS}. \tag{C2}$$

For a single field ϕ it is useful to introduce sources $J(x)$ and to define

$$Z[J] = \int [d\phi] \exp\left\{ i \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - V(\phi) + J\phi \right] \right\}. \tag{C3}$$

Green's functions for ϕ can then be obtained by the functional differentiation of Z with respect to J:

$$T\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} Z[J]. \tag{C4}$$

For free fields the integral can be performed by completing the squares. Writing the action as

$$S_{\text{free}} = \int d^4x \left[\frac{1}{2}\phi(x) D^{-1} \phi(x) + \phi(x) J(x) \right], \tag{C5}$$

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with

$$D^{-1} = \partial^2 - m^2 = p^2 - m^2, \tag{C6}$$

we can complete the squares in the action:

$$S_{\text{free}} = \int d^4x \left[\frac{1}{2}\phi(x) + \int d^4y J(y)D(y,x)\right] D^{-1}\left[\phi(x) + \int d^4z J(z)D(z,x)\right] - \int d^4x d^4y J(x)D(x,y)J(y). \tag{C7}$$

Now, in the free field functional integral one can shift the ϕ integral, obtaining

$$Z_0[J] = \Delta \exp\left[\frac{-i}{2}\int d^4x d^4y J(x)D(x,y)J(y)\right]. \tag{C8}$$

Here Δ is the free field functional integral at $J = 0$. It is the square root of the functional determinant of the operator D; D itself is the propagator of the scalar. This expression can then be used to develop perturbation theory. For example, with a $(\lambda/4!)\phi^4$ interaction we can write

$$Z[J] = \exp\left[i\int d^4x \frac{\lambda}{4!}\left(\frac{\delta}{i\delta J(x)}\right)^4\right] Z_0[J]. \tag{C9}$$

Working out the terms in the power series reproduces precisely the Feynman diagram expansion.

This has generalizations to non-Abelian gauge theories, both those with unbroken and those with broken symmetries, which we discuss in Section 2.3. We will also find it useful for addressing other questions.

C.2 Finite-temperature field theory

As an application of path integral methods and because of its importance in cosmology, we consider at some length the problem of field theory at finite temperatures.

In statistical mechanics one is interested in the partition function,

$$Z[\beta] = \operatorname{Tr} e^{-\beta H}. \tag{C10}$$

For a quantum mechanical system in contact with a heat bath, we have

$$Z[\beta] = \sum_n \langle n|e^{-\beta E_n}|n\rangle, \tag{C11}$$

where n labels the energy eigenstates.

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For a harmonic oscillator of unit mass, $H = [(p^2/2) + (\omega^2/2)]x^2$ and the partition function is:

$$e^{-\beta F} = \sum_n e^{-\beta\omega(n+1/2)}$$
$$= e^{-\omega\beta/2}\frac{1}{1-e^{-\beta\omega}}. \tag{C12}$$

Now, we can think of

$$\langle x|e^{-\beta H}|x\rangle \tag{C13}$$

as the amplitude for starting at x and ending up at x after propagating through an imaginary time $-i\beta$. This can be represented as a path integral:

$$\langle x|e^{-\beta H}|x\rangle = \int_{x(0)=x(\beta)=x} [dx]\exp\left(-\int_0^\beta dt\, L_{\rm E}\right), \tag{C14}$$

where $L_{\rm E}$ is the Euclidean Lagrangian,

$$L_{\rm E} = \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\omega^2 x^2 \tag{C15}$$

(note the signs here!). The partition function is now

$$Z[\beta] = \int_{x(0)=x(\beta)=x_0}^{dx_0} [dx]\exp\left(-\int_0^\beta dt\, L_{\rm E}\right), \tag{C16}$$

i.e. we integrate over the possible values of x at $t = 0$ in order to take the trace. This is the problem of a box periodic in the time direction. For this simple system with one degree of freedom, we can write:

$$x(t) = \sum_n \frac{1}{\sqrt{T}} a_n e^{-2\pi int/\beta}. \tag{C17}$$

We will simplify the problem slightly by taking $x(t)$ to be complex (you can think of this simply as corresponding to an isotropic harmonic oscillator in two dimensions). The action of this configuration is

$$S = \sum_{n=-\infty}^{\infty} \frac{1}{2}(\omega_n^2 + \omega^2)|a_n|^2. \tag{C18}$$

The path integral is now

$$Z[\beta] = \prod \int da_n da_n^* \, e^{-S_{\rm E}}. \tag{C19}$$

The integrals are just Gaussian integrals. For a complex variable z we have

$$\int d^2z\, e^{-a|z|^2} = \frac{\pi}{a}, \tag{C20}$$

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so we have the following result for Z:

$$Z[\beta] = \prod \frac{1}{\omega^2 + \omega_n^2}, \tag{C21}$$

where $\omega_n = 2\pi n/T$.

Now, before trying to evaluate this product, it is useful to pause and note that it can be expressed in terms of the determinant of a matrix. Quite generally, Gaussian path integrals take the form of (inverse) determinants. In this case, if we write $\mathcal{M}$ as the differential operator

$$\mathcal{M} = \frac{1}{2}\left(-\frac{d^2}{dt^2} + \omega^2\right), \tag{C22}$$

its eigenfunctions are just $e^{i\omega_n t}$, with eigenvalues $\omega_n^2 + \omega^2$. So Z is just the inverse determinant of $\mathcal{M}$. Had we worked with only one real coordinate, we would have obtained the square root of the inverse determinant.

The determinant of an infinite matrix may seem a daunting object, but there are some tricks that permit evaluation in many cases. The first thing is to write the determinant as a sum, by taking logarithms. In general,

$$\det M = \exp(\mathrm{Tr} \ln M) \tag{C23}$$

(to see this, diagonalize M). It is easier to evaluate derivatives of the determinant rather than the determinant itself. We can obtain a very useful formula for the derivative of a determinant by writing

$$\begin{aligned} \det(M + \delta M) &= \exp\left[\mathrm{Tr} \ln(M + \delta M)\right] = \exp\left[\mathrm{Tr} \ln M + \ln(1 + M^{-1}\delta M)\right] \\ &= \exp\left(\mathrm{Tr} \ln M\right) \exp\left(\mathrm{Tr}\, M^{-1}\delta M\right) \approx \det M(1 + \mathrm{Tr}\, M^{-1}\, \delta M). \end{aligned} \tag{C24}$$

Dividing by δM gives the derivative.

In our case, it is convenient to study

$$\frac{1}{Z}\frac{d}{d\omega^2} Z = \sum_n \frac{1}{\omega^2 + \omega_n^2}. \tag{C25}$$

This is progress. Our infinite product is now an infinite sum. The question is: how do we do the sum? The trick is to look for a periodic function which is well-behaved at infinity but has poles at the integers. A suitable choice is

$$\frac{1}{e^{iz\beta} - 1}. \tag{C26}$$

We can then replace any sum of the form $\sum f(n)$ by a contour integral,

$$\frac{1}{2\pi} \int dz\, f(z) \frac{1}{e^{iz\beta} - 1}. \tag{C27}$$

Here the contour is a line running just above the real z axis and back again just below it. The residues of the (infinite number of) poles give back the original sum.

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Now one can deform the contour, taking one line into the upper half plane and the other into the lower, picking up the poles at $z = \pm i\omega$. This leaves us with

$$\frac{dF}{d\omega^2} = \left(\frac{1}{e^{-\omega\beta} - 1} - \frac{1}{e^{\omega\beta} - 1}\right)\frac{1}{2\omega}. \tag{C28}$$

We could analyze this problem further, but let us jump instead to free-field theory. Then

$$Z[\beta] = \int_{\phi(\beta)=\phi(0)} [d\phi] \exp\left\{-\int d^4x[(\partial_\mu\phi)^2 + m^2\phi^2]\right\}. \tag{C29}$$

In a finite box, with periodic boundary conditions, we can make the following expression:

$$\phi(\vec{x}, t) = \sum_{\vec{k},m} \exp(i\vec{k}_n \cdot \vec{x} + i\omega_m t)\phi_{\vec{k},m}, \tag{C30}$$

where $\omega_m = 2\pi mT$.

In this form we have that

$$Z[\beta] = \det(-\partial^2 + m^2)^{-1/2}. \tag{C31}$$

Again, this is somewhat awkward to work with. It is easier to differentiate it:

$$\frac{1}{Z}\frac{\partial Z}{\partial m^2} = \frac{1}{Z}\int [d\phi] \exp\left(-\int d^4x\, \mathcal{L}_{\mathcal{E}}\right)\int d^4z\, \frac{1}{2}\phi^2(z). \tag{C32}$$

This is just the propagator, with periodic boundary conditions in the time direction:

$$\int d^4z\, \langle\phi(z)\phi(z)\rangle = \beta V\langle\phi(0)\phi(0)\rangle. \tag{C33}$$

The propagator is given by

$$\langle\phi(0)\phi(0)\rangle = \sum_m \sum_k \frac{1}{\omega_m^2 + \vec{k}^2 + m^2}. \tag{C34}$$

We can convert this into a more recognizable form by means of the same trick as above. The propagator is given by the expression below:

$$\langle\phi(0)\phi(0)\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\pi}\int \frac{dz}{e^{iz\beta} - 1}\frac{1}{(2\pi nT)^2 + \vec{k}^2 + m^2}. \tag{C35}$$

Now deform the contour as before, picking up the poles at $\pm i\sqrt{\vec{k}^2 + m^2}$. Both poles make the same contribution, yielding

$$\begin{aligned} &\frac{1}{2\sqrt{\vec{k}^2 + m^2}}\left(\frac{1}{\exp\left(-\beta\sqrt{\vec{k}^2 + m^2}\right) - 1} - \frac{1}{\exp\left(\beta\sqrt{\vec{k}^2 + m^2}\right) - 1}\right) \\ &= \frac{1}{2\sqrt{\vec{k}^2 + m^2}}\left(1 + \frac{2}{\exp\left(\beta\sqrt{\vec{k}^2 + m^2}\right) - 1}\right). \end{aligned} \tag{C36}$$

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Note the appearance of the Bose–Einstein factors here. Note also that the first term has the structure of the zero-temperature expression for the energy; the second is the finite-temperature expression. This is what we find on differentiating Eq. (C36):

$$\beta F = V \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} E_k + \beta^{-1} \ln(1 - e^{-\beta E_k}) \right]. \tag{C37}$$

Note the connection with the result for the single oscillator. So far our discussion has been for free-field theory but we can extend it immediately to interacting theories by developing a perturbation order-by-order in the couplings, just as at zero temperature.

C.3 QCD at high temperatures

Two particularly important cases are QCD and the weak-interaction theory. At low energies QCD is a complicated theory but, at high temperatures, things simplify drastically. In perturbation theory, if we are studying the free energy, for example, over above path integral analysis instructs us to study a Euclidean problem with discrete energies which are multiples of T. So, provided that we do not encounter infrared problems, the free energy should be a power series in $g^2(T)$, calculable in perturbation theory.

One can argue that there is actually a phase transition between a confined phase and a deconfined phase. To find an order parameter for this transition, we start by considering a Wilson line, running between imaginary times $t = 0$ and $t = \beta$,

$$U_T(\vec{x}) = P \exp \left[i \int_0^\beta A^0(\vec{x}, t) dt \right]. \tag{C38}$$

Because of the periodic boundary conditions, this expression is gauge invariant. The correlation of two such operators is related to the potential of two static quarks:

$$P(R) = \langle U_T(\vec{R}) U_T(0) \rangle = C \exp[-\beta V(R)]. \tag{C39}$$

In a confining phase, with a linear potential between the quarks, $P(R)$ vanishes exponentially with R. In a Coulomb phase (nearly free quarks), it will tend to a constant. At very high temperatures we would expect that we could compute P in a power series in $g^2(T)$ and that we will find free-quark behavior. Numerical studies show that there is indeed a phase transition at a particular temperature between confined and unconfined phases. The order of the transition depends on the group.

Finite-temperature perturbation theory suffers from infrared divergences, even at very high temperatures. The problem is the zero-frequency modes in the sum over frequencies. If we simply set all the frequencies to zero, we have the Feynman diagrams of a three-dimensional field theory. At four loops the divergence is logarithmic. At higher loops it is power law.

We can understand this directly in the path integral. Consider a massless scalar field. The exponent in the path integral is

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$$\int_0^{\beta} dtd^3x\,(\partial_\mu\phi)^2. \tag{C40}$$

For small β, assuming it makes sense to treat fields as constant in β, the path integral thus becomes

$$\int [d\phi(\vec{x})]\, e^{-\beta H}, \tag{C41}$$

which is the classical partition function for the three-dimensional system.

Thought of in this way, there is a natural guess for how the infrared divergences are cut off. A three-dimensional gauge theory has a dimensionful coupling λ^2. One might expect that such a theory has a mass gap proportional to λ^2 (in three dimensions, the gauge coupling has the dimensions of $\sqrt{M}$). In the present case the coupling is $\lambda = g^2 T$. This scale then would cut off the infrared divergence. This suggests that the theory at finite temperature makes sense but does not help a great deal with computations. The problem is that in four loops we obtain a contribution $g^8 \ln g^2$ but, at higher orders, we obtain a power series in g^2/g^2, i.e. we can at best compute the leading logarithmic term at four loops. It is possible to study some of these issues numerically in lattice gauge theory, which provides some support for this picture.

Instanton effects at high temperatures

In QCD at zero temperature we saw that instanton calculations were plagued by infrared divergences. At high temperatures this is not the case. The scale invariance of the zero-energy theory is lost and the instanton solution has a definite scale, of order the temperature. As a result, instanton effects behave as $\exp[-8\pi^2/g^2(T)]$ and are calculable. Thus it is possible to compute the θ-dependence systematically. This is particularly relevant to the understanding of the axion in the early universe.

C.4 Weak interactions at high temperatures

The weak interactions exhibit different phenomena at high temperatures. Most strikingly, there is a transition between a phase in which the gauge bosons are massive and one in which they are massless. This transition can be uncovered in perturbation theory. By analogy with the phase transition in the Landau–Ginzburg model of superconductivity, one might expect that the value of $\langle\Phi\rangle$ will change as the temperature increases. To determine the value of Φ one must compute the free energy as a function of Φ. The leading temperature-dependent corrections are obtained by simply noting that the masses of the various fields in the theory (the W and Z bosons and the Higgs field, in particular) depend on Φ. So the contributions of each species to the free energy are Φ-dependent:

$$\mathcal{F}(\Phi)V_T(\Phi) = \pm\sum_i \int \frac{d^3p}{2\pi^3} \ln\left\{1 \mp \exp\left[-\beta\sqrt{p^2 + m_i^2(\Phi)}\right]\right\}, \tag{C42}$$

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where $\beta = 1/T$, T is the temperature, the sum is over all particle species (physical helicity states) and the plus sign is for bosons, the minus for fermions. In the Standard Model, for temperature $T \sim 10^2$ GeV, one can treat all the quarks as massless except for the top quark. The effective potential (C42) then depends on the top quark mass m_t, the vector boson masses M_Z and m_W and the Higgs mass m_H. Performing the integral in the equation yields

$$V(\Phi, T) = D\big(T^2 - T_0^2\big)\Phi^2 - ET\Phi^3 + \frac{\lambda}{4}\Phi^4 + \cdots . \tag{C43}$$

The parameters T_0, D and E are given in terms of the gauge boson masses and the gauge couplings. For the moment, though, it is useful to note certain features of this expression. The quantity E turns out to be a rather small dimensionless number, of order 10^{-2}. If we ignore the ϕ^3 term then we have a second-order transition, at temperature T_0, between a phase with $\phi \neq 0$ and a phase with $\phi = 0$. Because the W and Z masses are proportional to ϕ, this is a transition between states with massive and massless gauge bosons.

Because of the ϕ^3 term in the potential, the phase transition is potentially at least weakly first order. A second, distinct, minimum appears at a critical temperature. A first-order transition is not, in general, an adiabatic process. As we lower the temperature to the transition temperature, the transition proceeds by the formation of bubbles; inside the bubble the system is in the true equilibrium state (the state which minimizes the free energy) while outside it tends to the original state. These bubbles form through thermal fluctuations at different points in the system and grow until they collide, completing the phase transition. The moving bubble walls are regions where the Higgs fields are changing and all Sakharov's conditions are satisfied.

C.5 Electroweak baryon number violation

We have seen that, at low temperatures, violations of baryon and lepton number are extremely small. This is not the case at high temperatures, where baryon number violation is a rapid process which can come to thermal equilibrium. This has at least two possible implications. First, it is conceivable that these sphaleron (see below) processes can themselves be responsible for generating a baryon asymmetry. This is called electroweak baryogenesis. Second, sphaleron processes can change an existing lepton number, producing a net lepton and baryon number. This is the process called leptogenesis. In this section, we summarize the main arguments showing that the electroweak interactions violate baryon number at high temperature.

Recall that, classically, the ground states are field configurations for which the energy vanishes. The trivial solution of this condition is $\vec{A} = 0$, where $\vec{A}$ is the vector potential. More generally, one can consider an $\vec{A}$ which is a "pure gauge",

$$\vec{A} = \frac{1}{i} g^{-1} \vec{\nabla} g, \tag{C44}$$

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where g is a gauge transformation matrix. In an Abelian ($U(1)$) gauge theory, fixing the gauge eliminates all but the trivial solution, $\vec{A} = 0$.[1] This is not the case for non-Abelian gauge theories. There is a class of gauge transformations, labeled by a discrete index n, which do not tend to unity as $|\vec{x}| \to \infty$ and which therefore must be considered to be distinct states. These have the form:

$$g_n(\vec{x}) = e^{inf(\vec{x})\hat{x}\cdot\tau/2}, \tag{C45}$$

where $f(x) \to 2\pi$ as $\vec{x} \to \infty$ and $f(\vec{x}) \to 0$ as $\vec{x} \to 0$.

So, the ground states of the gauge theory are labeled by an integer n. Now if we evaluate the integral of the current K^0, we obtain a quantity known as the *Chern–Simons number*:

$$n_{\rm CS} = \frac{1}{16\pi^2}\int d^3x\, K^0 = \frac{2/3}{16\pi^2}\int d^3x\, \epsilon_{ijk}{\rm Tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g). \tag{C46}$$

For $g = g_n$, $n_{\rm CS} = n$. The reader can also check that for $g' = g_n(x)h(x)$, where h is a gauge transformation which tends to unity at infinity (a so-called "small gauge transformation"), this quantity is unchanged. The Chern–Simons number $n_{\rm CS}$, is topological in this sense (for $\vec{A}$s which are not pure gauge, $n_{\rm CS}$ is in no sense quantized).

Schematically, we can thus think of the vacuum structure of a Yang–Mills theory as indicated in Fig. C.1. We have, at weak coupling, an infinite set of states, labeled by integers, and separated by barriers from one another. In tunneling processes which change the Chern–Simons number, because of the anomaly the baryon and lepton numbers will change. The exponential suppression found in the instanton calculation is typical of tunneling processes, and in fact the instanton calculation which leads to the result for the amplitude is nothing other than a field-theoretic WKB calculation.

One can determine the height of the barrier separating configurations having different $n_{\rm CS}$ by looking for the field configuration which corresponds to a particle top of the barrier. This is a solution of the static equations of motion with finite energy. It is known as a *sphaleron*. When one studies the small fluctuations about this solution, one finds that there is a single negative mode, corresponding to the possibility that the system will roll downhill into one or the other well. The sphaleron energy is of order

$$E_{\rm sp} = \frac{c}{g^2}M_W. \tag{C47}$$

This can be seen by using scaling arguments on the classical equations; determining the coefficient c requires a more detailed analysis. The rate for thermal fluctuations to cross the barrier per unit time per unit volume should be of order the Boltzmann factor for this configuration, multiplied by a suitable prefactor:

$$\Gamma_{\rm sp} = T^4 e^{-E_{\rm sp}/T}. \tag{C48}$$

Note that the rate becomes large as the temperature approaches the W boson mass. The W boson mass itself goes to zero as one approaches the electroweak phase transition.

[1] More precisely, this is true in axial gauge. In the gauge $A_0 = 0$, it is necessary to sum over all time-independent transformations in order to construct a state which obeys Gauss's law.

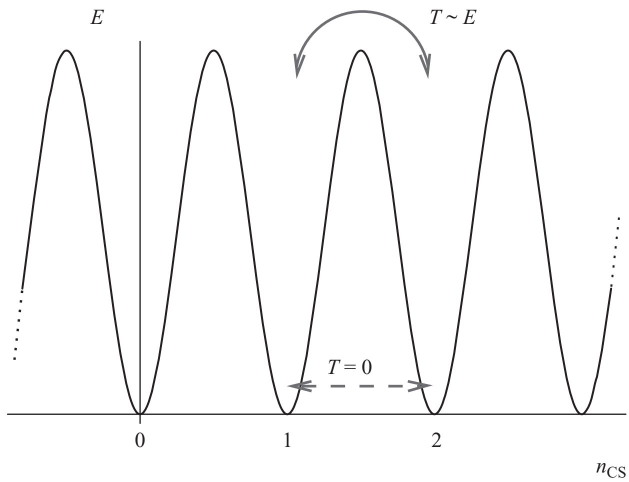

Fig. C.1 Schematic Yang–Mills vacuum structure. At zero temperature instanton transitions between vacua with different Chern–Simons numbers are suppressed. At finite temperature these transitions can proceed via sphalerons.

At this point the computation of the transition rate is a difficult problem – there is no small parameter – but general scaling arguments show that the transition rate is of the form:[2]

$$\Gamma_{bv} = \alpha_W^4 T^4. \tag{C49}$$

Suggested reading

The path integral is well treated in most modern field theory textbooks. Peskin and Schroder (1995) provide a concise introduction. High-temperature field theory is developed in a number of textbooks, such as that of Kapusta (1989).

Exercises

(1) Go through the calculation of the free energy of a free scalar field, being careful about factors of 2 and π.
(2) Compute the constants appearing in Eq. (C43). Plot the free energy, and show that the transition is weakly first order.
(3) Show, by power counting, that infrared divergences first appear in the free energy of a gauge theory at three loops. To do this you can look at the zero-frequency terms in the sums over frequency. Show that the divergences become more severe at higher orders.

[2] More detailed considerations alter slightly the parametric form of the rate.

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