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Computing character tables of groups of type $M.G.A$

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Abstract

We describe a method for constructing the character table of a group of type $M.G.A$ from the character tables of the subgroup $M.G$ and the factor group $G.A$, provided that A acts suitably on $M.G$. This simplifies and generalizes a recently published method.

1. *Introduction*

In [**2**], a method based on the technique of Clifford matrices (see [**6**]) for computing the ordinary character tables of groups with the structure $(2 \times 2.G){:}2$ is presented.

The aim of this paper is to show that a more appropriate approach to compute such character tables is to use the underlying 'Clifford theory' directly. This yields a technically simpler and more general method. It avoids the overhead of Clifford matrices and it covers more situations, in particular it allows one to compute also Brauer character tables (see Remark 4 and the examples in the Sections 3.3 and 3.4).

An implementation of our method is available in the computer algebra system GAP at least since version 4.4.2 (see the functions `PossibleCharacterTablesOfTypeMGA` and `PossibleActionsForTypeMGA` in [**8**]) and has been used, for example, to encode many character tables from the Atlas of finite groups [**4**] in a compact way (see Section 3.1).

More examples and the description of similar methods for computing character tables can be found at http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib/doc/, in the files poster_valencia_2009.pdf and ctblcons.pdf.

2. *The method*

For a finite group G, let $\mathrm{Irr}(G)$ denote the set of its complex irreducible characters. For a normal subgroup N of G, $\chi \in \mathrm{Irr}(N)$ and $g \in G$, we define χ^g by $\chi^g(n) = \chi(gng^{-1})$ for $n \in N$. This defines an action of G (which can be regarded as an action of G/N) on $\mathrm{Irr}(N)$, in which the stabilizer of χ is $I_G(\chi) = \{g \in G; \chi^g = \chi\}$. We will consider the following situation.

Hypothesis 1. Let H be a finite group, and let N and M be normal subgroups of H such that $M \leqslant N$ holds. Let $G = N/M$, $F = H/M$, and $A = H/N$. In Atlas notation (see [**4**, Chapter 6]), H has the structures $M.F$, $N.A$, and $M.G.A$. Set

$$X(N, M) = \{\chi \in \mathrm{Irr}(N); M \subseteq \ker(\chi)\} \quad \text{and} \quad Y(N, M) = \mathrm{Irr}(N) \setminus X(N, M),$$

where $\ker(\chi) = \{n \in N; \chi(n) = \chi(1)\}$. Assume that $I_H(\chi) = N$ holds for any $\chi \in Y(N, M)$. Equivalently, A acts semiregularly on $Y(N, M)$, that is, any A-orbit on $Y(N, M)$ has length $|A|$.

Further assume that we know the character tables of F and N, together with the class fusions that are induced by the embedding of G into F and the epimorphism from N to G, and that we know also the orbits of the conjugation action of H on the classes of N.

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Our aim is to compute the character table of H. The main ingredients are the following.

THEOREM 2. *Let N be a normal subgroup of H.*

(i) *If $I_H(\varphi) = N$ for $\varphi \in \mathrm{Irr}(N)$ then the induced character φ^H is an irreducible character of H (see* [**5**, Lemma III 2.11] *or* [**12**, Corollary 3.6.4]*).*

(ii) *The restriction of any irreducible character of H to N has the form $e \sum_g \varphi^g$ for a positive integer e and $\varphi \in \mathrm{Irr}(N)$, where the summation runs over a set of coset representatives g of $I_H(\varphi)$ in H (see* [**5**, Theorem III 2.12] *or* [**12**, Theorem 3.6.2]*).*

First we claim that $Y(H, M) = \{\varphi^H; \varphi \in Y(N, M)\}$ holds. For that, take $\chi \in Y(H, M)$ and let φ be any irreducible constituent of the restriction of χ to N. Then $M \not\subseteq \ker(\varphi)$, $\varphi^H \in \mathrm{Irr}(H)$, and χ is a constituent of φ^H by Frobenius–Nakayama reciprocity (see [**5**, Theorem III 2.5] or [**12**, Theorem 3.2.12]), thus $\chi = \varphi^H$. The reverse inclusion follows from the general fact that $\ker(\varphi^H) \subseteq \ker(\varphi)$ holds.

Hence all irreducible characters of H that are not inflated from F vanish outside N. Thus the preimage of any conjugacy class in $F \backslash G$ under the natural epimorphism π from H to F cannot split into several classes of H. Hence each conjugacy class of H is either the union of an H-orbit of conjugacy classes of N or the full preimage of a conjugacy class of $F \backslash G$ under π.

This means that we know the class fusions from H to F and from N to H, from which we can compute $X(H, M)$ by inflating $\mathrm{Irr}(F)$ to H and $Y(H, M)$ by inducing $Y(N, M)$ to H.

It remains to determine the power maps of H. For any prime p not dividing $|A|$, the p-power map of H is determined by the p-power maps of N and F together with the assumed class fusions. For a prime p that divides $|A|$, ambiguities in the p-power map of H can arise in the case that g^p lies inside N, for an element $g \in H \backslash N$, and that the coset gM contains elements from several conjugacy classes of H. These ambiguities can in general not be avoided because the input data do not determine the group H up to isomorphism. For example, let each of M, G, A have order two, let N be cyclic of order four, and F be a Klein four group, such that A swaps the two faithful irreducible characters of N. The group H can then be any of the two non-abelian groups of order eight. These groups have different 2-power maps.

See Section 3.3 for another example where different 2-power maps arise. Under stronger assumptions than those from Hypothesis 1, all power maps of H are determined by the input data, see Section 3.1.

REMARK 3. We have assumed that we know the orbits of the action of A on the classes of N. When dealing only with character tables (and avoiding computations with the underlying groups), a usual way to get this information is the following: the permutation of the classes of N induced by the action of A leaves $\mathrm{Irr}(N)$ invariant and commutes with the power maps of N. The character table of N determines the full group of these permutations (which are called *table automorphisms*). Often it contains only few elements that act on the classes inside M in the right way and that are compatible with the permutation of A on the classes of G, via the epimorphism from N to G and the embedding of G into F (which we know by assumption).

REMARK 4. Let p be a prime integer, and let $\mathrm{IBr}_p(H)$ denote the set of irreducible p-modular Brauer characters of H. The statements of Theorem 2 hold also for *Brauer* characters, see [**5**] or [**12**]; for explicit computations with induced Brauer characters, see [**12**, Theorem 4.3.7].

We assume Hypothesis 1 and additionally that p does not divide $|M|$, and set $Y(N, M, p) = \{\varphi \in \mathrm{IBr}_p(N); M \not\subseteq \ker(\varphi)\}$.

For the case that p does not divide the order of A, one can show that A acts semiregularly on $Y(N, M, p)$, as follows. Without loss of generality A has prime order different from p. In this case, the fact that the preimage of each conjugacy class in $F \backslash G$ is a single class of H implies that no character in $Y(N, M, p)$ can extend to H; thus any character in $Y(N, M, p)$ induces to an irreducible character of H.

Alternatively, we can simply *assume* that A acts semiregularly on $Y(N, M, p)$, without using the semiregularity of the action on $Y(N, M)$, cf. Section 3.4.

Then the same arguments as above yield $Y(H, M, p) = \{\varphi^H; \varphi \in Y(N, M, p)\}$. Hence we can use our method for computing $\mathrm{IBr}_p(H)$ from $\mathrm{IBr}_p(N)$ and $\mathrm{IBr}_p(F)$.

3. *Examples*

3.1. *Fixed point free action of A on M*

In this section, let us assume that M is central in N and that A acts semiregularly on the non-identity elements of M.

By Brauer's permutation lemma [**9**, Theorem 6.32], A acts semiregularly on the non-trivial irreducible characters of M. The restriction of any $\chi \in Y(N, M)$ to M is a multiple of an irreducible character of M, so already the restrictions of the characters in the A-orbit of χ to M are $|A|$ different characters. This implies that A acts semiregularly on $Y(N, M)$, hence Hypothesis 1 holds.

The easiest examples for the situation of this section are Frobenius groups H (cf. [**9**, Chapter 7]) with abelian kernel $N = M$ and complement isomorphic with $A \cong F$, so G is trivial in these cases. If H is a dihedral group of order not divisible by four then N is the unique index two subgroup. Examples with non-cyclic complement A are the primitive groups of the structure $p^2 : \mathrm{Q}_8$ and degree p^2, where p is an odd prime integer, $N = M$ is an elementary abelian group of order p^2, and $F \cong A$ is the quaternion group of order eight.

Now let us assume that A is a cyclic group of prime order p, say. We claim that the p-power map and hence the element orders of H are uniquely determined by the input data. For that, take $g \in H \backslash N$ such that $g^p \in N$ holds. The action of A on the classes of N is given by the action of $\langle g \rangle$, so A acts on the coset $g^p M$ in the same way as it acts on M. That is, g^p is the only fixed point. In other words, the class containing the p-power of g is the unique class of H that is a preimage of the F-class containing $g^p M$ and that consists of a single class of N.

The Atlas of finite groups [**4**] contains many examples which have the structure $3.G.2$ or $2^2.G.3$, for a simple group G. In the first case, G has an involutory outer automorphism that lifts to an automorphism of the triple cover $3.G$ of G and inverts the central elements in this cover. In the second case, G possesses a central extension $N = 2^2.G$ by a Klein four group and an outer automorphism of order three that lifts to N such that the three involutions in the Klein four group are permuted cyclicly.

3.2. *The examples from Barraclough's paper*

In [**2**], the construction of the ordinary character tables of groups of the shape $(2 \times 2.G)$:2 from the character tables of the groups $2.G$ and $2.G.2$ is described. Also our method can be used for this purpose, by choosing N to be the subgroup of type $2 \times 2.G$ called G^0 in [**2**], and M the subgroup called $\langle z \rangle$ in [**2**]. Hence F corresponds to the factor group called $\langle x \rangle \times G{:}\langle \sigma \rangle$ in [**2**]. The set $Y(N, M)$ consists of those irreducible characters of N whose kernels contain x or xz but not z, and by the general assumption of [**2**] that σ conjugates x and xz, σ acts semiregularly on $Y(N, M)$. Thus Hypothesis 1 holds if we know the orbits of the action of σ on the classes of N, and these orbits can be derived from the class fusion of $2.G$ into $2.G.2$, which is part of the input data of the construction in [**2**].

Concerning the main example in [**2**], let Fi_{22} and $2.\mathrm{Fi}_{22}$ denote the smallest sporadic simple Fischer group and its double cover, respectively. The group H whose character table is computed in [**2**] has the structure $(2 \times 2.\mathrm{Fi}_{22})$:2. We construct the character table of H from the character tables of $N \cong 2 \times 2.\mathrm{Fi}_{22}$ and $F \cong 2 \times \mathrm{Fi}_{22}.2$, where $\mathrm{Fi}_{22}.2$ denotes the automorphism group of Fi_{22}. The character tables of (the direct factors of) N and F are known (see [**4**, pp. 156–161]), and the table of N admits a unique involutory table automorphism that is compatible with the action of F on the classes of $G \cong 2 \times \mathrm{Fi}_{22}$ and does not centralize the normal Klein four group of N. Also the 2-power map of H is determined by the input data in this example.

Remark 5. Contrary to the approach used in [**2**], our construction does not require the knowledge of the character tables of the $2.\mathrm{Fi}_{22}.2$ type subgroups in H. Note that the character table of N turned out to provide enough information for determining the action of H on the classes of N.

One consequence of the smaller input data of our method compared to the one in [**2**] is that also character tables such as that of $(2 \times 2.A_6).2_3$ mentioned in [**2**] can be computed, in this case from the character tables of $N \cong 2 \times 2.A_6$ and the subdirect product F of $M_{10} = A_6.2_3$ with a cyclic group of order four. (The character table of N admits three table automorphisms that could belong to the action of F but only one of them leads to a consistent character table.)

Remark 6. The group H occurs as a maximal subgroup of the automorphism group Fi_{24} of the sporadic simple Fischer group Fi_{24}', and the intersection of H with Fi_{24}' has the structure $2.\mathrm{Fi}_{22}.2$ (see [**4**, p. 207]). The action of Fi_{24} on Fi_{24}' lifts to the triple cover $3.\mathrm{Fi}_{24}'$ and inverts its center. Hence there is a group $3.\mathrm{Fi}_{24}$ that satisfies the conditions of Section 3.1. If we consider H as a subgroup of Fi_{24} then its preimage $3.H$ in $3.\mathrm{Fi}_{24}$ has the structure $(\mathrm{S}_3 \times 2.\mathrm{Fi}_{22}).2$ and occurs as a subgroup of the sporadic simple Monster group M. Namely, $3.H$ is the full normalizer of an element x in the class `6A` of M and thus is contained in the centralizer of x^3 in M, which has the structure 2.B, the double cover of the sporadic simple Baby Monster group. In fact, $3.H$ is a maximal subgroup of 2.B (see [**4**, p. 217]).

Also the character table of $3.H$ can be computed using our method, from the character tables of its normal subgroup of the type $\mathrm{S}_3 \times 2.\mathrm{Fi}_{22}$ and its factor group of the type $\mathrm{S}_3 \times \mathrm{Fi}_{22}.2$. This construction can be found in the file `ctblcons.pdf` mentioned in Section 1. (This table is available in GAP's character table library since version 1.1, see [**3**].)

3.3. *The character table of the subgroup* 4.HS.2 *of* HN.2

The sporadic simple Harada–Norton group HN contains a maximal subgroup of the type 2.HS.2 that extends to a group H of the structure 4.HS.2 in the automorphism group HN.2 of HN (see [**4**, p. 166]). The subgroup H is the normalizer of a `4D` element $g \in \mathrm{HN}.2 \backslash \mathrm{HN}$. The center of H is $M = \langle g^2 \rangle$, and the centralizer N of g has the structure 4.HS, a central product of 2.HS and the cyclic group $\langle g \rangle$ of order four, with respect to the common subgroup M of order two. We have $F = H/M \cong \mathrm{HS}.2 \times 2$ and $G = N/M \cong 2 \times \mathrm{HS}$. Conjugation with any element in $H \backslash N$ inverts g, so H acts semiregularly on the faithful irreducible characters of N.

Thus we are in the situation of Hypothesis 1, and we construct the character table of H using our method. The character tables of N and F can be derived from the known character tables of 2.HS and HS.2 (see [**4**, p. 81]). The table of N has two table automorphisms that are compatible with the induced action of F on G. One of these permutations leads to a result table that does not admit a 2-power map. The other permutation yields two different possible character tables for H, which belong to non-isomorphic groups that are isoclinic in the sense of [**4**, Chapter 6.7]. Only one of the two tables admits a class fusion into HN.2.

Remark 7. The group 4.HS.2 occurs as the normalizer of a radical 2-chain in HN.2, so it is interesting for the computations in [**1**, Section 6]. In fact only the irreducible degrees are needed for that, and the authors of that paper compute them by first determining the permutation induced by the action of $\langle g \rangle$ on the classes of 2.HS.2, and then deriving the corresponding permutation of irreducible characters of 2.HS.2, using Brauer's permutation lemma. In our construction of the character table of 4.HS.2, we did not use the table of 2.HS.2.

REMARK 8. All Brauer character tables of 2.HS and HS.2 are known, see [**10**, pp. 210–214]. So also the Brauer character tables of $N = 4.\mathrm{HS}$ and $F = 2 \times \mathrm{HS}.2$ can be written down easily. Hence we can use our method to construct all Brauer character tables of $H = 4.\mathrm{HS}.2$. The ordinary character tables admit 32 possibilities for the class fusion from H to HN.2, and the 5- and 7-modular tables of H and HN.2 impose additional conditions, which exclude 16 possibilities in each case. It turns out that the choice for the 5-modular table of HN.2 made in [**11**, last paragraph] is not compatible with the 7-modular table of HN.2 that is available in [**3**].

3.4. *Constructions only for Brauer character tables*

If a group H has normal subgroups M and N as in Hypothesis 1 such that A does *not* act semiregularly on $Y(N, M)$ then the ordinary character table of H cannot be constructed with our method. However, it may happen that the p-modular variant of Hypothesis 1 as sketched in Remark 4 is satisfied for some p-modular Brauer table, so we can construct this Brauer table with our method. In such cases, all classes of $F\backslash G$ that split into several classes of $H\backslash N$ with respect to the natural epimorphism from H to F are p-singular.

Examples are the group $2.\mathrm{Fi}_{22}.2$ (cf. Section 3.2), for $p = 3$, the double cover $2.\mathrm{A}_6.2_1 \cong \Sigma\mathrm{L}_2(9)$ of the symmetric group on six points, for $p = 3$ (see [**10**, p. 4]), and the group $2.\mathrm{L}_2(25).2_2 = \Sigma\mathrm{L}_2(25)$, for $p = 5$ (see [**10**, p. 31]). It is straightforward to show that for any odd prime p, the semilinear group $\Sigma\mathrm{L}_2(p^2)$ is an example in characteristic p — exactly $p + 4$ conjugacy classes of the group $\Sigma\mathrm{L}_2(p^2)$ are not contained in $\mathrm{SL}_2(p^2) = 2.\mathrm{L}_2(p^2)$, the number of conjugacy classes in $\mathrm{P}\Sigma\mathrm{L}_2(p^2) = \mathrm{L}_2(p^2).2$ not contained in $\mathrm{L}_2(p^2)$ is $p + 2$, and the two outer classes of $\mathrm{P}\Sigma\mathrm{L}_2(p^2)$ whose preimages in $\Sigma\mathrm{L}_2(p^2)$ split into two classes are p-singular.

3.5. *A pseudo character table*

We can apply our method to the character tables of groups $M.G$ and $G.A$ where no common group of the structure $M.G.A$ exists such that A acts on M as required in Hypothesis 1. This may yield a table which has many properties of a character table but which is not the character table of a group, cf. [**7**].

For example, let G be the alternating group on six points, let $M.G$ be its triple cover, and $G.A$ be the non-split extension of G by an outer automorphism σ of order two. The group $G.A$ is known as M_{10}, it occurs as a subgroup of index 11 in the sporadic simple Mathieu group M_{11}. The automorphism σ of G lifts to $M.G$ but any such lift centralizes M.

However, the character table of $M.G$ admits a table automorphism that lifts the permutation of classes of G induced by σ and swaps the classes of non-identity elements in M. Furthermore, the output table of our method satisfies the orthogonality relations, any tensor product of two 'irreducible characters' decomposes into 'irreducibles' such that the coefficients are non-negative integers, the 'class multiplication coefficients' are non-negative integers, and also the nth symmetrizations of 'irreducible characters' decompose into 'irreducibles', for $n \leqslant 5$.

In general, tables obtained this way do not have all these properties.

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