A new numerical scheme in the solution of the geodynamo Z-model

Pavel Hejda
Geophysical Institute, 141 31 Prague 4,Czechoslovakia

Alexander P. Anufrijev
Geophysical Institute, Sofia, Bulgaria

ABSTRACT. The hydromagnetic dynamo Z-model represents a non-linear dynamic system. Its steady-state solution is derived by step-by-step integration of parabolic partial differential equations of the second order with the use of the finite difference method. Until now, two methods have been used: the semi-implicit method in which the θ-diffusion was carried out implicitly while the *r*-diffusion explicitly, and the implicit method in which the complete diffusion was carried out implicitly. In the present contribution, a combined semi-implicit method is suggested which reflects not only the singularity at the coordinate system origin but also the decreasing mesh size near the core-mantle boundary. This procedure preserves the advantage of semi-implicit methods and, simultaneously, increases the stability in the most critical boundary layer.

1. Problem formulation

In model Z, the magnetic field, **B**, and flux velocity, **v**, are axisymmetric, and are split into meridional and zonal parts:

$$\mathbf{B} = \mathbf{B}_p + B\mathbf{1}_\varphi, \quad \mathbf{v} = \mathbf{v}_p + v\mathbf{1}_\varphi,$$

$$\mathbf{B}_p = s^{-1}\nabla\psi\times\mathbf{1}_\varphi, \quad \mathbf{v}_p = s^{-1}\nabla\chi\times\mathbf{1}_\varphi,$$

where B, ψ, v and χ are functions of the time t and space coordinates s, z in the cylindrical coordinates (s,φ,z), or r, θ in the spherical coordinates (r,θ,φ). The stream function of the meridional flow is given by

$$\chi = -s\int_0^z v_s \mathrm{d}z, \quad v_s = s^{-1}\mathbf{B}_p.\nabla(sB),$$

while the zonal shear consists of magnetic, geostrophic and thermal parts

$$v/s = B^2/s^2 + \omega + f,$$

where ω is given by

$$\mathrm{d}(s^2\tau)/\mathrm{d}s = \varepsilon s^3\omega(s)/\left(2\sqrt{z_1}\right), \quad \tau = \int_0^{z_1} BB_s\mathrm{d}z, \quad z_1 = \sqrt{\left(1-s^2\right)},$$

and buoyancy f is prescribed. The induction equations are non-linear and read:

F. Krause et al. (eds.), The Cosmic Dynamo, 441–445.

$$\partial\psi / \partial t = -\nabla.\left(\psi \mathbf{v}_p\right) + \Delta_- \psi + s\alpha B, \tag{1.1}$$

$$\begin{aligned} \partial B / \partial t = &-\nabla.\left(B\mathbf{v}_p\right) + \Delta_s B + s\mathbf{B}_p.\nabla f + \\ &+ s^{-1}B\left(3\mathbf{B}_p.\nabla B - s^{-1}B_s B\right) + sB_s \mathrm{d}\omega / \mathrm{d}s, \end{aligned} \tag{1.2}$$

where

$$\Delta_- \psi = \nabla.\left(\nabla\psi - \mathbf{1}_s \psi / s\right), \quad \Delta_s B = \nabla^2 B - B / s^2. \tag{1.3a,b}$$

In the viscous case and under consideration of the usual symmetry with respect to the equatorial plane, the boundary conditions are

$$B = 0,\ \psi = 0 \text{ at } s = 0,\ B = 0,\ \partial\psi/\partial n = 0 \text{ at } z = 0, \tag{1.4}$$

$$B = 0,\ \partial\psi/\partial r = \partial\psi^{\mathrm{m}}/\partial \mathrm{r} - \chi \mathrm{B}_r \text{ at } r=1, \tag{1.5}$$

where ψ^{m} is a stream function of the source free outer potential field.

2. Numerical methods

The main aim of the numerical process is to find steady state solutions of the model equations. In contrast to the kinematic case, the hydromagnetic models do not lead to the eigenvalue problem. The steady state solution is usually sought by integrating the equations step by step in time by means of the finite difference approximation of the differential operators. Although some terms of the equations are expressed in the cylindrical coordinates, the application of a spherical grid is justified by the benefit which follows from a simple formulation of boundary conditions. Moreover, the spherical network enables us to increase the density of grid points in r near the core-mantle boundary and thus reflect the fact that the solutions change more rapidly near the boundary. In our numerical method a network $G = \{(r_i,\theta_j),\ \theta_j=(j-1/2)h_\theta,\ i=1, \ldots, I,\ j=1, \ldots, J\}$ with increased density of points r_i as $r \to 1$ is used.

It is well-known that explicit numerical schemes are stable only under a severe limitation on the time step length, τ. Usually, $\tau \le Ch^2$, where h is the smallest mesh size and constant C depends on the form of operators. As the mesh size contracts in the origin of the spherical grid, a pure explicit scheme is out of question. At the same time, a pure implicit scheme is impossible because of the intricacy of the problem. It means that some intermediate treatment is necessary.

As higher derivatives are more sensitive with respect to the stability, the diffusion terms $\Delta_-\psi$ and $\Delta_s B$ deserve more attention. Braginsky (1978) made use of the fact that the distance between mesh points is shrinking only in the θ direction. He proposed an algorithm in which the θ-diffusion was carried out implicitly but the r-diffusion explicitly. The merit of this method is similar to that of ADI (Alternating Direction

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Implicit) methods: on each r-layer a linear tridiagonal system of algebraic equations is obtained, which can be solved by a direct method with negligible time demands (the number of operations is linearly proportional to the number of equations). In (Braginsky and Roberts, 1987) the total diffusion was carried out implicitly. It has improved the stability, but on each time step a large sparse system of algebraic equations must be solved by an iterative (over-relaxation) method and the time demands are higher.

We have realized that the stability is deteriorated not only by the θ-diffusion near the origin of coordinates but also by the r-diffusion near the core-mantle boundary. Therefore, a new algorithm was suggested in which the original semiimplicit method was applied only in the inner core, while in the outer core the r-diffusion was carried out implicitly and the θ-diffusion explicitly. The algorithm, which will be referred to as a combined method, is obvious. The only part, which needs an explanation is the way how the smooth fit of the poloidal field onto a source-free potential field (1.5) can be included directly into the implicit scheme.

Let us express the diffusion term (1.3a) in the form

$$\Delta_- \psi = \Delta_r \psi + \Delta_\theta \psi, \quad \Delta_r \psi = \frac{\partial^2 \psi}{\partial r^2}, \quad \Delta_\theta = \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

and discretize it by

$$\langle \Delta_r \psi \rangle_{i,j} \approx \frac{1}{\hbar_i}\left(\frac{\psi_{i+1,j}-\psi_{i,j}}{h_{i+1}} - \frac{\psi_{i,j}-\psi_{i-1,j}}{h_i}\right) = a_i \psi_{i-1,j} - (a_i + b_i)\psi_{i,j} + b_i \psi_{i+1,j},$$

$$\langle \Delta_\theta \psi \rangle_{i,j} \approx \frac{1}{r_i^2 h_\theta^2}\left(\frac{\sin\theta_j}{\sin\theta_{j+1/2}}(\psi_{i,j+1}-\psi_{i,j}) - \frac{\sin\theta_j}{\sin\theta_{j-1/2}}(\psi_{i,j}-\psi_{i,j-1})\right) = c_{i,j}\psi_{i,j-1} - (c_{i,j}+d_{i,j})\psi_{i,j} + d_{i,j}\psi_{i,j+1}$$

If we denote the time step by superscript k, we get equations

$$\frac{\psi_{i,j}^{k+1}-\psi_{i,j}^{k}}{\tau} = \langle \Delta_r \psi \rangle_{i,j}^{k} + \langle \Delta_\theta \psi \rangle_{i,j}^{k+1} - \langle \nabla.(\psi \mathbf{v}_p) \rangle_{i,j}^{k} + s\alpha B_{i,j}^{k}, \qquad (2.1)$$

for $1 \le i \le i_0$, $1 \le j \le J$ and

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$$\frac{\psi_{i,j}^{k+1}-\psi_{i,j}^{k}}{\tau}=\langle\Delta_r\psi\rangle_{i,j}^{k+1}+\langle\Delta_\theta\psi\rangle_{i,j}^{k}-\langle\nabla.\left(\psi\mathbf{v}_p\right)\rangle_{i,j}^{k}+s\alpha B_{i,j}^{k}, \qquad (2.2)$$

for $i_0<i\le I$-1, $1\le j\le J$.

Both the parts lead to systems of linear equations with three diagonal matrices. The outer part (2.2) can be expressed in the form

$$-\tau a_i\psi_{i-1,j}^{k+1}+\left(1+\tau a_i+\tau b_i\right)\psi_{i,j}^{k+1}-\tau b_i\psi_{i+1,j}^{k+1}=f_{i,j}\left(B^k,\psi^k\right).$$

Applying Gauss elimination, we get J systems of equations with upper bidiagonal matrices

$$C_i\,\psi_{i,j}-B_i\,\psi_{i+1,j}=F_{i,j},\quad i_0<i\le I-1. \qquad (2.3)$$

Particularly, for $i=I$-1 we get

$$C_{I-1}\,\psi_{I-1,j}-B_{I-1}\,\psi_{I,j}=F_{I-1,j},\ j=1,\ldots,J. \qquad (2.4)$$

As was shown in (Hejda, 1983) and (Cupal and Hejda, 1989), the boundary condition (1.5) can be expressed in the form

$$L\psi_{I-1,j}+\Sigma\,M_{k,j}\,\psi_{I,k}=G_j,\ j=1,\ldots,J. \qquad (2.5)$$

Combining (2.4) and (2.5), we get a linear system of J equations

$$\Sigma(C_{I-1}M_{k,j}+L\,B_{I-1}\delta_{k,j})\,\psi_{I,k}=C_{I-1}G_j-L\,F_{I-1,j}.$$

After the solution on the boundary, $\psi_{I,k}$, has been obtained, the back substitutions (2.3) for $i=I$-1, I-2,...,i_0+1 can be carried out.

3. Results of tests

In order to test the efficiency of the combined method, we have compared it with the "standard" semiimplicit method. The instability was manifest by an unbounded growth of the solution (overflow) which usually occurred during a dozen time steps. The integration of the equations was then carried out with a reduced length of the time step and the procedure was repeated until a stable solution was found. These values are shown in Table 1. Two cases were investigated: Free decay solutions, i.e. solutions of the equations

$$\partial\psi/\partial t=\Delta_{-}\psi,\ \partial B/\partial t=\Delta_S B,$$

and solution of the full problem (started from a partly stabilized steady state).

The smallest mesh size involved in explicit calculations is in the combined method about four times larger than that in the standard method. If we suppose that the time step changes with the square of mesh size, we get to the ratio $\tau_{combined}/\tau_{standard}\approx$10-20. This corresponds well with the free decay solutions. As the stability of the full problem

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solutions by the standard method is the same as that of the free decay solutions, we can deduce that the implicit treatment of the *r*-diffusion is the main cause of instabilities. After this source of instabilities has been removed, the instabilities due to the explicit processing of all the remaining terms become transparent and that is why the stability of the combined method is much worse in the full problem solutions.
We can conclude that the new method increases the stability of the numerical process without increasing the number of operations per one time step. Further improvement of the stability would require a more detailed analysis of the role of the other terms.

Table 1. The longest time steps which ensure stable solutions of "standard" and combined semiimplicit methods.

Grid size	Free decay solutions		Full problem	
	Standard	Combined	Standard	Combined
16	0.0002	0.002	0.0001	0.0002
32	0.00002	0.0005	0.00002	0.0001
64	0.000005	0.0001	0.000005	0.00002

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