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LIMIT SETS AND COMMENSURABILITY OF KLEINIAN GROUPS

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Abstract

In this paper, we obtain several results on the commensurability of two Kleinian groups and their limit sets. We prove that two finitely generated subgroups G_1 and G_2 of an infinite co-volume Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$ having $\Lambda(G_1) = \Lambda(G_2)$ are commensurable. In particular, we prove that any finitely generated subgroup H of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$ with $\Lambda(H) = \Lambda(G)$ is of finite index if and only if H is not a virtually fibered subgroup.

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1. Introduction

Two groups G_1 and G_2 are commensurable if their intersection $G_1 \cap G_2$ is of finite index in both G_1 and G_2. In this paper, we investigate the following question posed by Anderson [5]: if $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^n)$ are finitely generated and discrete, does $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ imply that G_1 and G_2 are commensurable? Here we use $\mathrm{Ax}(G)$ to denote the set of axes of the hyperbolic elements of $G \subset \mathrm{Isom}(\mathbf{H}^n)$.

The question has been discussed by several authors. In 1990, Mess [11] showed that if G_1 and G_2 are nonelementary finitely generated Fuchsian groups having the same nonempty set of simple axes, then G_1 and G_2 are commensurable. Using some technical results on arithmetic Kleinian groups, Long and Reid [9] gave an affirmative answer to this question in the case where G_1 and G_2 are arithmetic Kleinian groups. Note that all the confirmed cases for the question are geometrically finite groups. So it is natural to ask if the question is true with the assumption that G_1 and G_2 are geometrically finite. Recently, Susskind [13] constructed two geometrically finite Kleinian groups in $\mathrm{Isom}(\mathbf{H}^n)$ (for $n \geq 4$) having the same action on some invariant 2-hyperbolic plane but whose intersection is infinitely generated. So this implies that

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these two geometrically finite groups are not commensurable although they have the same set of axes. But it is worth pointing that these two geometrically finite Kleinian groups generate a nondiscrete group. This example suggests that some additional conditions need to be imposed to eliminate such 'bad' groups.

In higher dimensions, we have the following consequence of Susskind and Swarup's results [14] on the limit set of the intersection of two geometrically finite Kleinian groups.

PROPOSITION 1.1. *Let G_1 and G_2 be two nonelementary geometrically finite subgroups of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^n)$. Then G_1 and G_2 are commensurable if and only if the limit sets $\Lambda(G_1)$ and $\Lambda(G_2)$ are equal. In particular, $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$.*

In several cases, the condition that both subgroups lie in a larger discrete group can be dropped.

COROLLARY 1.2. *Let $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^n)$ be two nonelementary geometrically finite Kleinian groups of the second kind leaving no m hyperbolic planes invariant for $m < n - 1$. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$.*

In dimension three, we can refine the analysis of the limit sets using Anderson's results [3] to get the following result modulo the recent solution of the tameness conjecture (see [1, 6]) which states that all finitely generated Kleinian groups in $\mathrm{Isom}(\mathbf{H}^3)$ are topologically tame.

THEOREM 1.3. *Let G_1, G_2 be two nonelementary finitely generated subgroups of an infinite co-volume Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$. In particular, $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$.*

Similarly, in the following case we are able to remove the ambient discrete group.

COROLLARY 1.4. *Let $G_1, G_2 \subset \mathrm{Isom}(\mathbf{H}^3)$ be two nonelementary finitely generated Kleinian groups of the second kind whose limit sets are not circles. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$.*

In fact, under the hypotheses of the above results, the condition of having the same limit sets of two Kleinian subgroups exactly implies having the same sets of axes. But Anderson's original formulation of the question is only to suppose that two Kleinian groups have the same set of axes. So it is interesting to explore whether there is some essential difference between the limit set and set of axes. The following theorem is a result in this direction, suggesting that the 'same set of axes' condition is necessary in general for Anderson's question.

THEOREM 1.5. *Let H be a nonelementary finitely generated subgroup of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Suppose that $\Lambda(H) = \Lambda(G)$. Then $[G : H]$ is finite if and* *only if G is not virtually fibered over H. In particular, $[G : H]$ is finite if and only if* $\mathrm{Ax}(H) = \mathrm{Ax}(G)$.

REMARK 1.6. We remark that the case of H being geometrically finite is proved by Susskind and Swarup [14, Theorem 1]. Theorem 1.5 actually proves the special case of Anderson's question where G_1 is a subgroup of G_2.

The paper is organized as follows. In Section 2, we gather together some results on the limit set of the intersection of two Kleinian groups and prove some useful lemmas for later use. In Section 3, we prove Theorems 1.3 and 1.5.

2. Preliminaries

Let $\mathbf{B}^n$ denote the closed ball $\mathbf{H}^n \cup \mathbf{S}^{n-1}$, whose boundary $\mathbf{S}^{n-1}$ is identified via stereographic projection with $\overline{\mathbf{R}}^{n-1} = \mathbf{R}^{n-1} \cup \infty$. Let $\mathrm{Isom}(\mathbf{H}^n)$ be the full group of isometries of $\mathbf{H}^n$ and let $G \subset \mathrm{Isom}(\mathbf{H}^n)$ be a *Kleinian* group; that is, a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^n)$. Then G acts discontinuously on $\mathbf{H}^n$ if and only if G is discrete. Furthermore, G acts on $\mathbf{S}^{n-1}$ as a group of conformal homeomorphisms. The *set of discontinuity* $\Omega(G)$ of G is the subset of $\mathbf{S}^{n-1}$ on which G acts discontinuously; the *limit set* $\Lambda(G)$ is the complement of $\Omega(G)$ in $\mathbf{S}^{n-1}$. A Kleinian group is said to be of the *second kind* if $\Omega(G)$ is nonempty, otherwise it is said to be of the *first kind*.

The elements of $\mathrm{Isom}(\mathbf{H}^n)$ are classified in terms of their fixed point sets. An element $g \neq \mathrm{id}$ in $\mathrm{Isom}(\mathbf{H}^n)$ is *elliptic* if it has a fixed point in $\mathbf{H}^n$, *parabolic* if it has exactly one fixed point lying in $\mathbf{S}^{n-1}$, or *hyperbolic* if it has exactly two fixed points lying in $\mathbf{S}^{n-1}$. The unique geodesic joining the two fixed points of the hyperbolic element g, which is invariant under g, is called the *axis* of the hyperbolic element and is denoted by $\mathrm{ax}(g)$. The limit set $\Lambda(G)$ is the closure of the set of fixed points of hyperbolic and parabolic elements of G. A Kleinian group whose limit set contains fewer than three points is called *elementary*, and is otherwise called *nonelementary*.

For a nonelementary Kleinian group G, define $\widetilde{C}(G)$ to be the smallest nonempty convex set in $\mathbf{H}^n$ which is invariant under the action of G; this is the *convex hull* of $\Lambda(G)$. The quotient $C(G) = \widetilde{C}(G)/G$ is the *convex core* of $M = \mathbf{H}^n/G$. The group G is *geometrically finite* if the convex core $C(G)$ has finite volume.

By Margulis's lemma, it is known that there is a positive constant ϵ_0 such that for any Kleinian group $G \in \mathrm{Isom}(\mathbf{H}^n)$ and $\epsilon < \epsilon_0$, the part of $\mathbf{H}^n/G$ where the injectivity radius is less than ϵ is a disjoint union of tubular neighborhoods of closed geodesics, whose lengths are less than 2ϵ, and cusp neighborhoods. In dimensions two and three, these cusp neighborhoods can be taken to be disjoint quotients of horoballs by the corresponding parabolic subgroup. This set of disjoint horoballs is called a *precisely invariant system of horoballs* for G. In dimension three, it is often helpful to identify the infinity boundary $\mathbf{S}^2$ of $\mathbf{H}^3$ with the extended complex plane $\overline{\mathbf{C}}$. In particular, the fixed point of a rank-one parabolic subgroup J of G is called *doubly cusped* if there are two disjoint circular discs $B_1, B_2 \subset \overline{\mathbf{C}}$ such that $B_1 \cup B_2$ is precisely invariant under J in G. In this case, the parabolic elements in J are also called doubly cusped.

In dimension three, we call a Kleinian group G *topologically tame* if the manifold $M = \mathbf{H}^3/G$ is homeomorphic to the interior of a compact 3-manifold. Denote by M^c the complement of these cusp neighborhoods. Using the relative core theorem of McCullough [10], there exists a compact submanifold N of M^c such that the inclusion of N in M^c is a homotopy equivalence, every torus component of $\partial(M^c)$ lies in N, and N meets each annular component of $\partial(M^c)$ in an annulus. Call such an N a *relative compact core* for M. The components of $\partial(N) - \partial(M^c)$ are the *relative boundary components* of N. The *ends* of M^c are in one-to-one correspondence with the components of $M^c - N$. An end E of M^c is *geometrically finite* if it has a neighborhood disjoint from $C(G)$, otherwise E is *geometrically infinite*.

A Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$ is *virtually fibered* over a subgroup H if there are finite-index subgroups G^0 of G and H^0 of H such that $\mathbf{H}^3/G^0$ has finite volume and fibers over the circle with the fiber subgroup H^0. Note that H^0 is then a normal subgroup of G^0, and so $\Lambda(H^0) = \Lambda(G^0) = \mathbf{S}^2$.

In order to analyze the geometry of a geometrically infinite Kleinian group, we will use Canary's covering theorem, which generalizes a theorem of Thurston [15]. Note that the tameness theorem [1, 6] states that all finitely generated Kleinian groups in $\mathrm{Isom}(\mathbf{H}^3)$ are topologically tame.

THEOREM 2.1 [7, The covering theorem]. *Let G be a torsion-free Kleinian group in $\mathrm{Isom}(\mathbf{H}^3)$ and let H be a nonelementary finitely generated subgroup of G. Let $N = \mathbf{H}^3/G$, let $M = \mathbf{H}^3/H$, and let $p : M \to N$ be the covering map. If M has a geometrically infinite end E, then either G is virtually fibered over H or E has a neighborhood U such that p is finite-to-one on U.*

We now list several results on the limit set of the intersection of two Kleinian groups, which describe $\Lambda(G_1 \cap G_2)$ in terms of $\Lambda(G_1)$ and $\Lambda(G_2)$, where G_1 and G_2 are subgroups of a Kleinian group G. Here we only give the results used in this paper, stating them in a form appropriate for our purposes. See [4] for a useful survey and the bibliography therein for the results in full detail.

THEOREM 2.2 [14, Theorem 3]. *Let G_1, G_2 be two geometrically finite subgroups of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^n)$. Then $\Lambda(G_1) \cap \Lambda(G_2) = \Lambda(G_1 \cap G_2) \cup P$ where P consists of some parabolic fixed points of G_1 and G_2.*

PROPOSITION 2.3 [14, Corollary 1]. *Let H be geometrically finite and j be a hyperbolic element with a fixed point in $\Lambda(H)$. If $\langle H, j\rangle$ is discrete, then $j^n \in H$ for some $n > 0$.*

Based on the above results, we get the following lemma characterizing the relationship between limit sets and sets of axes.

LEMMA 2.4. *Let G_1, G_2 be two geometrically finite subgroups of a Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^n)$. Then $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$.*

PROOF. If G_1 and G_2 are geometrically finite subgroups of a Kleinian group G, then $G_1 \cap G_2$ is again geometrically finite [14, Theorem 4]. It is well known that a hyperbolic element cannot share one fixed point with a parabolic element in a discrete group. Thus, by applying Theorem 2.2 to $\Lambda(G_1) \cap \Lambda(G_2)$, we can conclude that any hyperbolic element $h \in G_i$ has at least one fixed point in $\Lambda(G_1 \cap G_2)$ for $i = 1, 2$. Now by Proposition 2.3, we have $h^j \in G_1 \cap G_2$ for some large integer $j > 0$. This implies the axis $\mathrm{ax}(h)$ of h belongs to $\mathrm{Ax}(G_1 \cap G_2)$. Therefore we have $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$ and it also follows that P is the empty set.

The other direction is easy to see using the fact that the set of fixed points of hyperbolic elements of G is dense in $\Lambda(G)$. □

In dimension three, Anderson [3] carried out a more careful analysis on the limit set of the intersection of two topologically tame Kleinian groups. Combined with the recent solution of the tameness conjecture in [1] and [6], we have the following theorem.

THEOREM 2.5 [3, Theorem C]. *Let $G \subset \mathrm{Isom}(\mathbf{H}^3)$ be a Kleinian group, and let G_1 and G_2 be nonelementary finitely generated subgroups of G. Then $\Lambda(G_1) \cap \Lambda(G_2) = \Lambda(G_1 \cap G_2) \cup P$ where P is empty or consists of some parabolic fixed points of G_1 and G_2.*

PROPOSITION 2.6 [3, Theorem A]. *Let H be a finitely generated Kleinian group and j be a hyperbolic element with a fixed point in $\Lambda(H)$. If $\langle H, j\rangle$ is discrete, then either $\langle H, j\rangle$ is virtually fibered over H or $j^n \in H$ for some $n > 0$.*

Similarly, we obtain the following lemma.

LEMMA 2.7. *Let G_1 and G_2 be two nonelementary finitely generated subgroups of an infinite co-volume Kleinian group $G \subset \mathrm{Isom}(\mathbf{H}^3)$. Then $\Lambda(G_1) = \Lambda(G_2)$ if and only if $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$.*

PROOF. Observe that our hypothesis 'G is an infinite co-volume Kleinian group' implies that G is not virtually fibered over G_1. Since the intersection of any pair of finitely generated subgroups of a Kleinian group is finitely generated (see [2]), we see that $G_1 \cap G_2$ is finitely generated. Using Theorem 2.5 and Proposition 2.6, we can argue exactly as in Lemma 2.4 to obtain $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$, if we suppose that $\Lambda(G_1) = \Lambda(G_2)$. □

REMARK 2.8. The condition of G being infinite co-volume cannot be dropped, as will be seen in the proof of Theorem 1.5. The Kleinian group G fibering over H has a different set of axes from that of its fiber subgroup H.

In the following two lemmas, we give some useful properties about the same sets of axes of two Kleinian groups.

LEMMA 2.9. *Let G be a nonelementary finitely generated Kleinian group and H a subgroup of finite index in G. Then $\mathrm{Ax}(G) = \mathrm{Ax}(H)$.*

PROOF. It is obvious that $\mathrm{Ax}(H) \subset \mathrm{Ax}(G)$. Conversely, since $[G : H]$ is finite, for any hyperbolic element g with axis $\mathrm{ax}(g) \in \mathrm{Ax}(G)$, there are two integers i and j such that $g^i H = g^j H$ and thus $g^{i-j} \in H$. It follows that $\mathrm{ax}(g) \in \mathrm{Ax}(H)$. The proof is complete. □

REMARK 2.10. In fact, our Theorem 1.5 proves that the converse of the above lemma is also true when H is a finitely generated subgroup of $G \subset \mathrm{Isom}(\mathbf{H}^3)$.

LEMMA 2.11. *Let G be a nonelementary finitely generated, torsion-free Kleinian group and H be a subgroup of G. Suppose that $\mathrm{Ax}(G) = \mathrm{Ax}(H)$. Then for every hyperbolic element $g \in G$, $g^n \in H$ for some $n > 0$.*

PROOF. For any hyperbolic element $g \in G$, we can choose a hyperbolic element h from H such that $\mathrm{ax}(g) = \mathrm{ax}(h)$ by the hypothesis $\mathrm{Ax}(G) = \mathrm{Ax}(H)$. It follows that the subgroup $\langle g, h\rangle$ is elementary and torsion-free. By the characterization of elementary Kleinian groups it follows that $\langle g, h\rangle$ is actually a cyclic subgroup $\langle f\rangle$ of G. Thus we can write $g = f^m$ and $h = f^n$ for two appropriate integers m, n. Thus we have found the integer n such that $g^n = h^m \in H$, which proves the lemma. □

3. Proofs

PROOF OF PROPOSITION 1.1. Recall that G_1 and G_2 are commensurable if the intersection $G_1 \cap G_2$ is of finite index in both G_1 and G_2. By Lemma 2.9, we have $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ and thus $\Lambda(G_1) = \Lambda(G_2)$, if G_1 and G_2 are commensurable. So it remains to prove the converse.

If $\Lambda(G_1) = \Lambda(G_2)$, we have $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$ by Lemma 2.4. Therefore it follows that $\Lambda(G_1) = \Lambda(G_2) = \Lambda(G_1 \cap G_2)$, since the set of fixed points of hyperbolic elements of G is dense in $\Lambda(G)$. Now we can conclude that $G_1 \cap G_2$ is of finite index in both G_1 and G_2, by using [14, Theorem 1] which states that any geometrically finite subgroup H of a Kleinian group G is of finite index in G if $\Lambda(H) = \Lambda(G)$.

The second assertion is just Lemma 2.4. This completes the proof. □

PROOF OF COROLLARY 1.2. It is well known that the stabilizer in $\mathrm{Isom}(\mathbf{H}^n)$ of the limit set of a nonelementary Kleinian group G of the second kind, leaving no m hyperbolic planes invariant for $m < n - 1$, is itself a Kleinian group. See, for example, Greenberg [8], where the discreteness of the stabilizer of that limit set is proved.

Thus $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ implies that G_1 and G_2 together lie in a common Kleinian group, which is the stabilizer of the common limit set of G_1 and G_2. Thus Proposition 1.1 completes the proof. □

PROOF OF THEOREM 1.3. Observe that the fundamental domain of the subgroup H is the union of translates of the fundamental domain of G by left H-coset representatives in G. So the subgroup $\langle G_1, G_2\rangle$ also has infinite co-volume, and we can assume that G is finitely generated by replacing G by $\langle G_1, G_2\rangle$. As the conclusion is easily seen to be unaffected by passage to a finite-index subgroup, we may use

Selberg's lemma to pass to a finite-index, torsion-free subgroup of G. Hence, without loss of generality, we may assume that G is finitely generated and torsion-free.

By Proposition 1.1, the conclusion is trivial if G is geometrically finite. So we suppose that G is geometrically infinite. Then $M = \mathbf{H}^3/G$ has infinite volume. Let C be a compact core for M. Since M has infinite volume, ∂C contains a surface of genus at least two. Then using Thurston's geometrization theorem for Haken 3-manifolds (see [12]), there exists a geometrically finite Kleinian group with nonempty discontinuity domain, which is isomorphic to G.

Now our task is to give an algebraic characterization of the limit set of $G_1 \cap G_2$ in G_i such that the relationships between $\Lambda(G_1 \cap G_2)$ and $\Lambda(G_i)$ can be passed to those between target isomorphic groups under the above isomorphism of G. Then the conclusion of Theorem 1.3 follows from Proposition 1.1. We claim that for every element $g \in G_1$, there exists an integer k such that $g^k \in G_1 \cap G_2$.

Firstly, by Lemma 2.7, we obtain that $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$. So for any hyperbolic element $g \in G_1$, the integer k obtained in Lemma 2.11 is such that $g^k \in G_1 \cap G_2$. Now we consider the remaining parabolic elements. Theorem B of [3] says that if no nontrivial power of a parabolic element $h \in G_1$ lies in $G_1 \cap G_2$, then there exists a doubly cusped parabolic element $f \in G_1 \cap G_2$ with the same fixed point ξ as h. Normalizing their fixed point ξ to ∞, we can suppose that $f(z) = z + 1$ and $h(z) = z + \tau$, where $\mathrm{Im}(\tau) \neq 0$. Since f is doubly cusped in $G_1 \cap G_2$, then $\Lambda(G_1 \cap G_2) \subset \{z : |\mathrm{Im}(z)| < c\}$, for some constant c. But on the other hand, $\Lambda(G_1 \cap G_2)$ is also kept invariant under h, which contradicts the fact that $\Lambda(G_1 \cap G_2)$ is invariant under f. Therefore the claim is proved for all elements including parabolic elements. A similar claim holds for $G_1 \cap G_2$ and G_2.

Under the isomorphism, using the above claims, we can conclude that the limit set of the (isomorphic) image of $G_1 \cap G_2$ is equal to those of the (isomorphic) images of G_1 and G_2. The proof is complete as a consequence of Proposition 1.1. □

REMARK 3.1. Theorem 1.3 can be thought of as a geometric version of [3, Lemma 5.4], which uses an algebraic assumption on the limit sets of the groups involved.

PROOF OF COROLLARY 1.4. This is proved similarly to Corollary 1.2. □

PROOF OF THEOREM 1.5. In view of Lemma 2.9, we may assume, without loss of generality, that H is finitely generated and torsion-free by using Selberg's lemma to pass to a finite-index, torsion-free subgroup of H.

If H is geometrically finite, then the conclusion follows from a result of Susskind and Swarup [14], which states that a nonelementary geometrically finite subgroup sharing the same limit set with the ambient discrete group is of finite index. So next we suppose that H is geometrically infinite. Then there exist finitely many geometrically infinite ends E_i for the manifold $N := \mathbf{H}^3/H$.

We first claim that $\mathrm{Ax}(H) = \mathrm{Ax}(G)$ implies that H cannot be a virtually fibered subgroup of G. Otherwise, by taking finite-index subgroups of G and H, we can suppose that H is normal in G. Then it follows that every element of the quotient group G/H has finite order by Lemma 2.11. Thus G/H could not be isomorphic to $\mathbf{Z}$. This is a contradiction. So H is not a virtually fibered subgroup of G.

Using Theorem 2.1, we know that for each geometrically infinite end E_i, there exists a neighborhood U_i of E_i such that the covering map $\mathcal{P} : N \to M := \mathbf{H}^3/G$ is finite-to-one on U_i.

Now we argue by way of contradiction. Let $\mathcal{Q}_N : \mathbf{H}^3 \to N$ and let $\mathcal{Q}_M : \mathbf{H}^3 \to M$ be the covering maps and notice that $\mathcal{Q}_M = \mathcal{P} \circ \mathcal{Q}_N$. Suppose that $[G : H]$ is infinite. This implies that $\mathcal{P}$ is an infinite covering map. By the definition of a geometrically infinite end, we can take a point z from the neighborhood U_1 of a geometrically infinite end E_1 such that z also lies in the convex core of N. By lifting the point $\mathcal{P}(z) \in M$ to $\mathbf{H}^3$, it is easy to see that the infinite set $\widetilde{S} := \mathcal{Q}_M^{-1}(\mathcal{P}(z))$ lies in the common convex hull $\widetilde{C}(H) = \widetilde{C}(G) \subset \mathbf{H}^3$, by observing that $\widetilde{C}(G)$ is invariant under G and the preimage $\mathcal{Q}_N^{-1}(z) \subset \widetilde{S}$ lies in $\widetilde{C}(H)$. Since $\mathcal{P}$ is an infinite covering map, the set $S := \mathcal{P}^{-1}(\mathcal{P}(z))$ is infinite. By considering $\mathcal{Q}_M = \mathcal{P} \circ \mathcal{Q}_N$, it follows that $S = \mathcal{Q}_N(\widetilde{S}) \subset N$ and thus S lies in the convex core of N, because $\widetilde{S} \subset \widetilde{C}(H)$.

We claim that we can take a smaller invariant horoball system for H such that infinitely many points of S lie outside all cusp ends of N. Otherwise, we can suppose that infinitely many points of S are contained inside a cusp end E_c of N, since there are only finitely many cusp ends for N. Thus infinitely many points of $\mathcal{Q}_N^{-1}(S)$ lie in the corresponding horoball B for the end E_c. Normalizing the parabolic fixed point for E_c to ∞ in the upper half-space model of $\mathbf{H}^3$, the horoball B at ∞ is precisely invariant under the stabilizer of ∞ in H. On the other hand, we have that infinitely many points of $\mathcal{Q}_N^{-1}(S)$ have the same height, since the covering map $\mathcal{P}$ maps $S \subset N$ to a single point on M, and the horoball B is also precisely invariant under the stabilizer of ∞ in G, which is a Euclidean group preserving the height of points in the horoball B. Then we can take a smaller horoball for E_c such that these infinitely many points of $\mathcal{Q}_N^{-1}(S)$ lie outside the horoball.

Continuing the above process for all cusp ends of N, we can get a new invariant horoball system such that infinitely points of S lie outside these cusp ends of N.

Since S projects to a single point on M, we can conclude that S cannot lie in any compact subset of the convex core of N. Thus, by the second claim above, there exist infinitely many points of S that can only lie in geometrically infinite ends of N. This is in contradiction to Theorem 2.1, which states that the covering map $\mathcal{P}$ restricted to each geometrically infinite end is finite-to-one, if H is not a virtually fibered subgroup of G. □

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