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INVARIANT ANALYTIC HYPERSURFACES IN COMPLEX LIE GROUPS

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Suppose G is a complex Lie group and H is a closed complex subgroup of G. Let G' denote the commutator subgroup of G. If there are no nonconstant holomorphic functions on G/H and H is not contained in any proper parabolic subgroup of G, then Akhiezer [**2**] asked whether every analytic hypersurface in G which is invariant under the right action of H is also invariant under the right action of G'. In this paper we answer a related question in two settings. Under the assumptions stated above we show that the orbits of the radical of G in G/H cannot be Cousin groups, provided G/H is Kähler. We also introduce an intermediate fibration of G/H induced by the holomorphic reduction of the radical orbits and resolve the related question in a situation arising from this fibration.

1. INTRODUCTION

In a study of hypersurfaces in complex nil-manifolds Akhiezer [**2**] asks the following question:

> Let G be a connected complex Lie group and H be a closed complex subgroup of G such that
>
> (a) H is not contained in any proper parabolic subgroup of G and
>
> (b) $\mathcal{O}(G/H) = \mathbb{C}$.
>
> Is every hypersurface in G which is invariant under the right action of H also invariant under the right action of G'?

It is known that if $X = G/H$ is a homogeneous complex manifold and X has at least one hypersurface, then there exists a proper closed complex subgroup J of G containing H such that the bundle map $\pi : G/H \to G/J$ is the hypersurface reduction of G/H, that is, G/J is locally separable by hypersurfaces and every hypersurface in G/H arises as the pull-back via the holomorphic map π of a hypersurface in G/J, see [**12**]. In terms of the hypersurface reduction Akhiezer's question asks whether $G' \subset J$.

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Whenever this question has an affirmative answer, then G/J is an Abelian complex Lie group and since $\mathcal{O}(G/H) = \mathbb{C}$, it is clear that $\mathcal{O}(G/J) = \mathbb{C}$ too. We call a complex Lie group which has no nonconstant holomorphic functions a *Cousin group*, in honour of Cousin, see [**6**]. Thus G/J is a Cousin group that is locally hypersurface separable and so is a *quasi-Abelian variety* in the sense of Andreotti–Gherardelli [**4**]. In particular, G/J is a covering space of an Abelian variety. As a consequence, G/J admits nonconstant meromorphic functions and nondegenerate theta-functions. Intuitively speaking, Akhiezer's question asks whether hypersurfaces occur in a homogeneous complex manifold *only because of* nonconstant holomorphic functions, nonconstant meromorphic functions or as the pull-backs of divisors from the projective manifold G/P, where P is a parabolic subgroup of G containing H.

Akhiezer's question is known to have a positive answer in a number of settings. For $X = G/H$ compact this follows from the results in [**9**]. If G is a semisimple complex Lie group and H is a closed complex subgroup satisfying the above conditions, then H is a Zariski dense discrete subgroup of G. Otherwise, the algebraic closure of H would be either reductive (implying $\mathcal{O}(G/H) \neq \mathbb{C}$) or would be contained in a proper parabolic subgroup of G. It follows that the space G/H has no analytic hypersurfaces [**10**] and thus $G' = G = J$ in this setting. Akhiezer's question has also been answered affirmatively for G nilpotent [**2**] or [**12**], for G solvable [**13**], and for $G = S \times R$ a group theoretic direct product of a maximal semisimple subgroup S of G with the radical R of G [**14**].

The purpose of this short note is to provide an affirmative answer to the above question in some other settings. In [**14**] it was observed that if G and H satisfy conditions (a) and (b) with G acting almost effectively on G/H, then H is discrete and the subgroup $J := R \cdot H$ is closed in G, where R denotes the radical of G. Now if G/H is Kähler and satsifies conditions (a) and (b) with $\mathcal{O}(R \cdot H/H) = \mathbb{C}$, that is, the radical orbits are Cousin groups, then we prove that $S = \{e\}$, see Theorem 3. Hence G is solvable and it follows that G/H itself is a Cousin group by [**13**].

If $\mathcal{O}(R \cdot H/H) \neq \mathbb{C}$, then we make use of the holomorphic reduction $J/H \to J/I$ to construct an *intermediate fibration*

$$G/H \to G/I \to G/J.$$

Here I/H is a Cousin group and J/I is Stein. Now I° is normal in G and is the connected component of the identity of the ineffectivity of the G-action on G/I. Set $\widehat{G} := G/I^\circ$, $\widehat{R} := R/I^\circ$, and $\widehat{I} := I/I^\circ$. We show that if $\widehat{G} = S \times \widehat{R}$ is a group theoretic direct product, then $S \cap \widehat{I}$ is infinite. As a consequence, we prove in Theorem 4 that the assumption $\widehat{G} = S \times \widehat{R}$ yields a contradiction, unless $\dim S = 0$. This rules out $\dim_{\mathbb{C}} \widehat{R} = 1$.

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2. Some preliminaries

Throughout this paper we shall assume G is a connected complex Lie group and H is a closed complex subgroup of G such that G is acting almost effectively on G/H. Without loss of generality we shall also assume that G is simply connected. Further, we suppose that G and H satisfy conditions (a) and (b) stated above. Let $G = S \ltimes R$ denote a Levi–Malcev decomposition of G, where S is a maximal semisimple subgroup of G and R is the radical of G.

The next result was noted in [**14**, pp. 116–117]. The way the result is used in that paper is in the product case. However, its proof is also valid in a more general setting, as we have now stated it.

Theorem 1. *Suppose G is a connected simply connected complex Lie group with Levi decomposition $G = S \ltimes R$. Let H be a closed complex subgroup of G. Assume that G is acting almost effectively on G/H. Further assume that H is not contained in any proper parabolic subgroup of G and that $\mathcal{O}(G/H) = \mathbb{C}$. Then $N_G(H^\circ) = G$, that is, H is a discrete subgroup of G. Moreover, the R-orbits in G/H are closed and thus one has the homogeneous fibration*

$$G/H \to G/R \cdot H = S/\Lambda,$$

where $\Lambda := S \cap R \cdot H$ is a Zariski dense discrete subgroup of the group S.

Corollary 1. *Suppose H is contained in a proper closed complex subgroup I of G and assume that H is not contained in any proper parabolic subgroup of G and $\mathcal{O}(G/H) = \mathbb{C}$. Then $I^\circ \triangleleft G$. As well, $R \cdot I$ is closed in G and $S \cap R \cdot I$ is Zariski dense in S.*

Proof: Since H is not contained in any proper parabolic subgroup of G and $\mathcal{O}(G/H) = \mathbb{C}$, the same statements hold relative to the subgroup I. Thus the Theorem also applies to I. ∎

A hypersurface in a complex manifold X is a 1-codimensional complex analytic subset of X. Now suppose G/H is a homogeneous complex manifold which has at least one hypersurface. Assume that G and H satisfy conditions (a) and (b). Let $G/H \to G/J$ be the hypersurface reduction of G/H, see [**12**]. Then G/J is locally hypersurface separable and the groups G and J satisfy conditions (a) and (b). In order to show that Akhiezer's question has an affirmative answer, it is enough to show that the commutator subgroup G' of G is contained in J, that is, that G/J is a Cousin group. By the above Corollary the subgroup J° is normal in G. Let $p : G \to G_1 := G/J^\circ$ be the natural epimorphism. It is easy to check that $p^{-1}(G_1') = J^\circ \cdot G'$. As a consequence, whenever G_1 is Abelian, it follows that J contains G' and Akhiezer's question has an affirmative answer. We shall show that the group acting on a given Kähler homogeneous manifold is Abelian in certain settings. Throughout the rest of this paper we consider Kähler homogeneous manifolds, since a homogeneous complex manifold which is locally hypersurface separable is known to be Kähler, see [**8**].

Using known techniques one can answer Akhiezer's question when the group G is a group theoretic direct product, at least in the case when the manifold G/H is itself Kähler. This result was proved in [**13**] and the proof given there, as well as the one presented here, both rely on the fact that if G is a product, then the maximal semisimple subgroup S is normal in G. We present our proof here because it uses an important tool: the Tits' alternative for linear groups, see [**15**].

THEOREM 2. *Suppose G is a connected simply connected complex Lie group whose Levi–Malcev decomposition is a direct product $G = S \times R$. Suppose H is a discrete subgroup of G such that G/H satisfy conditions* (a) *and* (b) *and G/H is Kähler. Then $S = \{e\}$, that is, G/H is a Cousin group.*

PROOF: We shall show that the assumption $\dim S > 0$ yields a contradiction. Since G/H is Kähler, $S \cap H$ is algebraic [5] and thus is finite. Using the Tits' alternative for linear groups we showed in [**3**, Lemma 6] that H is contained in a subgroup $L := A \times R$, where A is an algebraic subgroup of S such that its connected component of the identity A° is solvable. As a consequence, L is a proper subgroup of G. Hence $G/L = S/A$, a contradiction, since A is either reductive or contained in a proper parabolic. $\square$

3. THE RADICAL ORBITS CANNOT BE COUSIN GROUPS

We now show that *in the Kähler setting* the radical orbits cannot be Cousin groups.

THEOREM 3. *Suppose G is a connected complex Lie group and H is a discrete subgroup of G such that G/H satisfies* (a) and (b) *and is Kähler. Let R denote the radical of G. Assume the typical radical orbit has no nonconstant holomorphic functions, that is, $\mathcal{O}(R\cdot H/H) = \mathbb{C}$. Then $S = \{e\}$, that is, G is solvable. In particular, G/H is a Cousin group and G is Abelian. Moreover, if G/H is also locally hypersurface separable, then G/H is a quasi-Abelian variety.*

PROOF: We apply an argument similar to [**8**, proof of Lemma 2]. Since Λ is Zariski dense in S, it follows that Λ contains a semisimple element λ. It is well-known that the Zariski closure of the cyclic subgroup generated by λ is an algebraic torus in S. Set $A := (\mathbb{C}^*)^k = \overline{\langle\lambda\rangle_{\mathbf{Z}}}$. It is also known that $A/\langle\lambda\rangle_{\mathbf{Z}}$ is a Cousin group.

Let $S = S_1 \cdot \ldots \cdot S_n$ and let p_i denote the projection of S onto its ith simple factor S_i for $i = 1, \ldots, n$. As noted in [**1**, Lemma p. 328] we may choose A such that $p_i(A) \neq \{e\}$ for all i.

Let $\pi_S : G \to S$ denote the projection map. Pick $h \in H$ such that $\lambda = \pi_S(h)$. This is possible by the definition of $\Lambda = S \cap R\cdot H$. Define $\widetilde{H} := (H \cap R)\cdot\langle h\rangle_{\mathbf{Z}}$ and $\widetilde{G} := A \ltimes R$. Note that $\widetilde{G} \cap H \supset \widetilde{H}$ and thus $\widetilde{G}/\widetilde{H}$ is Kähler. The fibration

$$\widetilde{G}/\widetilde{H} \to \widetilde{G}/R\cdot\widetilde{H} = A/\langle\lambda\rangle_{\mathbf{Z}}$$

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has the Cousin group $R/R \cap \widehat{H}$ as fibre and the Cousin group $A/\langle\lambda\rangle_{\mathbf{Z}}$ as base and thus $\mathcal{O}(\widetilde{G}/\widetilde{H}) = \mathbb{C}$. Since $\widetilde{G}$ is solvable and $\widetilde{G}/\widetilde{H}$ is Kähler, it follows that $\widetilde{G}$ is Abelian [**13**]. Hence A acts trivially on R. But the kernel of the representation of S on the radical R is a normal subgroup of S. Since this kernel contains A and $p_i(A) \neq \{e\}$ for every i, it follows that this kernel is equal to S and thus the representation of S on the radical R is trivial, that is, $G = S \times R$. But G/H is Kähler and $\mathcal{O}(G/H) = \mathbb{C}$. This is a contradiction, see [**14**] or Theorem 2, unless G is solvable. Hence $S = \{e\}$ and the other claims of the theorem follow. □

4. The intermediate fibration

Our next task is to introduce an intermediate fibration of the homogeneous manifold G/H which is induced by the holomorphic reduction of the radical orbit $R \cdot H/H$. We shall assume G is connected and thus R is also connected. For convenience, set $J := R \cdot H$. Let $J/H \to J/I$ be the holomorphic reduction of J/H, see [**7**, Section 1], where I is a closed complex subgroup of J containing H. Applying this along with Theorem 1 we get the intermediate fibration

$$G/H \overset{I/H}{\to} G/I \overset{J/I}{\to} G/J = S/\Lambda, \text{ where } \Lambda := S \cap R \cdot H.$$

Now J need not be solvable. However, $J/I = R/(R \cap I)$ is a holomorphically separable solvmanifold and thus J/I is Stein, see [**11**]. As a consequence, the fibre I/H of the holomorphic reduction of J/H is connected, for example, see [**7**, Proposition 1]. Since J/H is Kähler in the settings in which we are interested, I/H is a Cousin group, see [**13**].

It follows from Corollary 1 that I° is normal in G. We set $\widehat{G} := G/I^\circ$ and $\widehat{R} := R/I^\circ$ and let $\pi : G \to \widehat{G}$ be the canonical epimorphism. Since $I \subset J = R \cdot H$ and H is discrete, $I^\circ \subset R$ and thus $\widehat{R}$ is the radical of $\widehat{G}$. Let $G = S \ltimes R$ be a Levi–Malcev decomposition of G. (Recall G is simply connected.) Note that $\pi(S)$ is isomorphic to S. Thus $\widehat{G} = S \ltimes \widehat{R}$ is a Levi–Malcev decomposition of $\widehat{G}$. Since $I/H = I^\circ/I^\circ \cap H$ is a Cousin group, it follows that I° is Abelian. Set $\widehat{I} := I/I^\circ$. Note that $\widehat{I}$ is a discrete subgroup of $\widehat{G}$.

5. The group $\widehat{G}$ is a product

Our approach will be to assume that $\dim S > 0$ and $\dim \widehat{R} > 0$. We do not know that $\widehat{G}/\widehat{I}$ is Kähler. In general, we can push down the Kähler metric from the total space G/H to G/I only if the fibre I/H of this bundle is compact. We do not wish to make this restriction. But, if $\widehat{G}/\widehat{I}$ were Kähler, then the S-orbit $S/S \cap \widehat{I}$ in $\widehat{G}/\widehat{I}$ would also be Kähler and thus $S \cap \widehat{I}$ would be finite by the results in [**5**]. Hence the "good situation" occurs when $S \cap \widehat{I}$ is finite and the S-orbit is algebraic. We are going to look at the setting where the group $\widehat{G}$ is a group theoretic direct product. It then turns out that $S \cap \widehat{I}$ is infinite in this setting and we shall then show that this leads to a contradiction. Our proof of this uses a technique from [**14**] along with Theorem 3.

Lemma 1. *Let G be a simply connected complex Lie group and H a discrete subgroup of G such that G/H satisfies* (a) *and* (b) *and G/H is Kähler. Assume G acts almost effectively on G/H. Also assume that $\widehat{G} = S \times \widehat{R}$ is a group theoretic direct product, with $\dim S > 0$. Then $S \cap \widehat{I}$ is infinite.*

Proof: Assume to the contrary that the intersection $S \cap \widehat{I}$ is finite. Using the Tits' alternative, we showed in [**3**, Lemma 6] that in the product group $S \times \widehat{R}$ the discrete subgroup $\widehat{I}$ is contained in a subgroup of the form $A \times \widehat{R}$, where A is algebraic in S with A° solvable; that is, $\widehat{I}$ is contained in a proper algebraic subgroup of $\widehat{G}$. But, this contradicts the assumption that G/H satisfies conditions (a) and (b). Hence $S \cap \widehat{I}$ is infinite. ∎

Theorem 4. *Let G be a simply connected complex Lie group and H a discrete subgroup of G such that G/H satisfies* (a) *and* (b) *and G/H is Kähler. Assume G acts almost effectively on G/H. Suppose $\widehat{G} = S \times \widehat{R}$ is a group theoretic direct product. Then $S = \{e\}$, that is, G/H is a Cousin group. Moreover, if G/H is locally hypersurface separable, then G/H is a quasi-Abelian variety.*

Proof: The case $\widehat{R} = \{e\}$ is handled by Theorem 3. So we assume $\dim \widehat{R} > 0$. We claim that the group $\Lambda := S \cap \widehat{I} \cdot \widehat{R}$ normalises $S \cap \widehat{I}$. To see this suppose $g = (s, 0) \in \Lambda$, where $s = ir$ with $i \in \widehat{I}$ and $r \in \widehat{R}$ and $h \in S \cap \widehat{I}$. Then

$$ghg^{-1} = irhr^{-1}i^{-1} = ihi^{-1}$$

since the action of S on $\widehat{R}$ is trivial and thus r and h commute. Clearly, $ihi^{-1} \in S \cap \widehat{I}$. As a consequence, the Zariski closure S of Λ normalises the Zariski closure S^* of $S \cap \widehat{I}$. By the previous lemma $S \cap \widehat{I}$ is infinite. Hence S^* is a positive dimensional semisimple complex Lie subgroup of S. Note that $S \cap \widehat{I} = S^* \cap \widehat{I}$.

Let $Z := S^*/S^* \cap \widehat{I}$ be the orbit of the base point in $\widehat{G}/\widehat{I}$ and let $p : G/H \to \widehat{G}/\widehat{I}$ be the bundle projection. Then $Y := p^{-1}(Z)$ is a Kähler manifold that is homogeneous under the group $S^* \ltimes I^\circ$. Since it fibres via the map $p|_Y : Y \to Z$ with fibre a Cousin group and base Z, it follows that Y satisfies conditions (a) and (b). But this contradicts Theorem 3. Thus $S = \{e\}$ and the remaining statements follow by now applying the results of [**13**] to the solvable group G. ∎

Corollary 2. *Suppose G/H as above and $\dim S > 0$. Then $\dim_{\mathbb{C}} \widehat{R} \neq 1$.*

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