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On Gunning's Prime Form in Genus 2

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Abstract. Using a classical generalization of Jacobi's derivative formula, we give an explicit expression for Gunning's prime form in genus 2 in terms of theta functions and their derivatives.

Let X be a compact Riemann surface of genus $g > 0$. Let $\tilde{X}$ denote the universal cover of X, $\Pi\colon \tilde{X} \to X$ denote the projection, and Γ be the group of covering transformations of $\tilde{X}$ over X.

By a prime form for X we mean a function on $\tilde{X} \times \tilde{X}$ which is an analytic relatively automorphic function for some prescribed factor of automorphy for the action of Γ on each copy of $\tilde{X}$, and which has a simple zero on the diagonal of $\tilde{X} \times \tilde{X}$ and its translates under $\Gamma \times \Gamma$ and has no other zeros. The classic prime form is due to Klein, see [F] and [M]. In [Gu1] Gunning introduced a different prime form, which has a factor of automorphy that is more closely related to that of theta functions. For applications see [Gu1], [Gu2], [Gu3], [Gu4], [P].

Gunning's prime form is only characterized up to a constant factor by its automorphic and vanishing properties. In [Gu5] Gunning gives an implicit normalization for his prime form (see (2) below) that uses his theory of canonical coordinates on $\tilde{X}$ described in [Gu3].

The purpose of this paper is to give for $g = 2$ an explicit expression for Gunning's prime form in terms of genus 2 theta functions and their derivatives. We do so in the Theorem below up to sign: it may well be that the method described below will also suffice to determine the requisite sign, but it seems like a lengthy and perhaps unenlightening exercise to do so. The keys are to use the function theory on the Jacobian of the curve and a generalization of Jacobi's derivative formula due to Rosenhain.

We first recall some basic facts about compact Riemann surfaces and their Jacobians, following the exposition in [Gu1]. A marking on X consists of a fixed point z_0 of $\tilde{X}$, and a canonical basis $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$ of $H_1(X, \mathbb{Z})$. We let $P_0 = \Pi(z_0)$. With this marking we get an identification between Γ and the fundamental group of X based at P_0, through which we can consider $A_1, \ldots, A_g, B_1, \ldots, B_g$ as generators for Γ.

For any holomorphic differential ϕ on X, $\Pi^*(\phi)$ is a holomorphic differential on the simply connected space $\tilde{X}$, hence $\Pi^*(\phi) = dw$, where w is some analytic function on $\tilde{X}$ which we normalize so that $w(z_0) = 0$. Since $\Pi^*(\phi)$ is Γ-invariant, we get a corresponding map $\bar{\phi}\colon \Gamma \to \mathbb{C}$ defined by $\bar{\phi}(\gamma) = w(\gamma z) - w(z)$ for any $z \in \tilde{X}$.

Let $\{\phi_1, \ldots, \phi_g\}$ be the basis for the space of holomorphic differentials on X normalized so that $\bar{\phi}_i(A_j) = \delta_{ij}$. Let $\omega_{ij} = \bar{\phi}_i(B_j)$. Then $\Omega = [\omega_{ij}]_{i,j=1,\ldots,g}$ is the

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period matrix of the marked Riemann surface. A standard calculation shows that Ω is a symmetric $g \times g$ matrix with positive definite imaginary part. Let ${}^t m$ denote the transpose of a matrix m, and set $\Phi = {}^t(\bar{\phi}_1, \ldots, \bar{\phi}_g)$, and $L = \Phi(\Gamma)$. Then $L = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ is a lattice in $\mathbb{C}^g$. The torus $\mathbb{C}^g/L$ is the Jacobian $J(X)$ of X. Let $\Pi^*(\phi_i) = dw_i$ with $w_i(z_0) = 0$. We then define a map $w\colon \tilde{X} \to \mathbb{C}^g$ by setting $w(z) = {}^t\big(w_1(z), \ldots, w_g(z)\big)$. This induces an embedding $X \to J(X)$ by setting $w(P) = w(z) \mod L$, where $z \in \tilde{X}$ is any point such that $\Pi(z) = P$. The image of X under w is denoted W_1, and for $s < g$ we write W_s for the sum of the s terms $W_1 + \cdots + W_1$. We extend w to a map on divisor classes of X by linearity.

For any $v = {}^t(v_1, \ldots, v_g) \in \mathbb{C}^g$, $a, b \in \frac{1}{2}\mathbb{Z}^g$, we define the genus g theta function with characteristic $[{a \atop b}]$ and period matrix Ω as

$$\theta[{a \atop b}](v) = \theta[{a \atop b}](v, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i^t(n+a)\Omega(n+a) + 2\pi i^t(n+a)(v+b)}. \tag{1}$$

Note that $\theta[{a \atop b}](v)$ is analytic in v. We let $\theta(v) = \theta[{0 \atop 0}](v)$. Also $\theta[{a \atop b}](-v) = e^{4\pi i^t ab}\theta[{a \atop b}](v)$, so $\theta[{a \atop b}](v)$ is even or odd depending on whether $e^{4\pi i^t ab}$ is 1 or -1, and the characteristic $[{a \atop b}]$ is called *even* or *odd* accordingly.

For $\gamma \in \Gamma$, any factor of automorphy $\chi(\ell, v)$ for the action of L on $\mathbb{C}^g$ induces the factor of automorphy $\hat{\chi}(\gamma, z) = \chi\big(\Phi(\gamma), w(z)\big)$ for the action of Γ on $\tilde{X}$. For $s \in \mathbb{C}^g$, we define the factor of automorphy ρ_s for the action of L on $\mathbb{C}^g$ by $\rho_s\big(\Phi(A_i)\big) = 1$, $\rho_s\big(\Phi(B_i)\big) = e^{2\pi i s_i}$. Let ζ be the factor of automorphy for the action of Γ on $\tilde{X}$ defined in [Gu2] by $\zeta(A_j, z) = 1$, $\zeta(B_j, z) = e^{-2\pi i(m_j + r_j + w_j(z))/g}$, where $r, m \in \mathbb{C}^g$ are defined by $m_j = \sum_{k=1}^g \omega_{jk}$ and $r_j = \sum_{k=1}^g \int_{z_0}^{A_k z_0} w_j(z)\Pi^*(\phi_k)(z)$. Let $\epsilon \in \mathbb{C}^g$ be defined by $\epsilon_i = \omega_{ii}/2$.

We can now define Gunning's prime form $q(z_1, z_2)$. It is described up to a constant factor as an analytic function on $\tilde{X} \times \tilde{X}$ such that for all $\gamma \in \Gamma$,

$$q(\gamma z_1, z_2) = \hat{\rho}_{w(z_2)}(\gamma)\zeta(\gamma, z_1)q(z_1, z_2),$$

and

$$q(z_1, z_2) = -q(z_2, z_1).$$

To normalize q, Gunning requires that for any $z, z_1, \ldots, z_g \in \tilde{X}$,

$$\begin{aligned} \theta\big(r - \epsilon + m + w(z) - w(z_1) - \cdots - w(z_g)\big) \prod_{1 \le j < k \le g} q(z_j, z_k) \\ = \det\big(w_j'(z_k)\big)_{1 \le j,k \le g} \prod_{1 \le i \le g} q(z, z_i), \end{aligned} \tag{2}$$

where the derivatives are taken with respect to the "canonical coordinates" described in [Gu3]; that is

$$w_j'(z_k) = \lim_{z_k' \to z_k} \frac{w_j(z_k) - w_j(z_k')}{q(z_k, z_k')}.$$

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Since the transformation $q \to \kappa q$ takes $w_j'(z_k)$ to $w_j'(z_k)/\kappa$, (2) determines q up to a $\binom{g}{2}$-th root of unity.

It follows directly from (1) that for any $\mu \in \mathbb{C}^g$, the factor of automorphy of $\theta(v - \mu - \epsilon)$ for the action of L on $\mathbb{C}^g$ is

$$\xi_\mu\big(\Phi(A_i), v\big) = 1, \xi_\mu\big(\Phi(B_i), v\big) = e^{2\pi i(\mu_i - v_i)}. \tag{3}$$

It follows immediately that

$$\hat{\xi}_{-r-m} = \zeta^g, \quad \xi_{\mu+s} = \rho_s \xi_\mu. \tag{4}$$

A fundamental result is Riemann's vanishing theorem, which says that the zeros of θ modulo L are $-W_{g-1} + r - \epsilon$. Since θ is an even function, $-W_{g-1} + r - \epsilon = W_{g-1} - r + \epsilon$, so by the Riemann-Roch theorem, $2(r - \epsilon) = k$, where k is the image under w of any canonical divisor of X.

Now let X be the Riemann surface defined by the complex points of the genus 2 curve

$$C : y^2 = x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5, \quad b_i \in \mathbb{C}.$$

Every genus 2 Riemann surface arises in this way. We first choose an ordering $P_i = (a_i, 0)$, $1 \le i \le 5$, for the affine Weierstrass points of X. Then we choose a marking for X so that $\Pi(z_0) = P_0$ is the point at infinity on the normalization of C, and the canonical homology basis is the traditional one employed for hyperelliptic curves with a given ordering of Weierstrass points [M, p. 3.76].

We will be combining the uniformization of X with that of its Jacobian. Most of what we need is given in [M].

Since P_0 is a Weierstrass point, k is the origin of $J(X)$, so $r - \epsilon + m = \Omega a + b$, for some $a, b \in \frac{1}{2}\mathbb{Z}^2$, and Riemann's vanishing theorem now says that that $\theta[{a \atop b}](v)$ vanishes for any v in W_1 modulo L. With the traditional choice of canonical basis, $a \equiv \binom{1/2}{1/2}$ mod 1, and $b \equiv \binom{1}{1/2}$ mod 1 [M, p. 3.82], and $[{a \atop b}]$ is an odd theta characteristic.

Let σ be the matrix such that $\sigma\begin{pmatrix}\frac{dx}{y}\\ \frac{xdx}{y}\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$. Following [M], we define the differential operators

$$\begin{bmatrix} D_2 \\ D_1 \end{bmatrix} = -{}^t\sigma \begin{bmatrix} \frac{\partial}{\partial v_1} \\ \frac{\partial}{\partial v_2} \end{bmatrix}.$$

Then if $z \in \tilde{X} - \Pi^{-1}(P_0)$,

$$D_{x(z)} = D_2 + x(z) D_1$$

is a differential operator such that if we choose an appropriate local coordinate $z(\rho)$ centered at z, then

$$D_{x(z)} f(v) = \frac{d}{d\rho} f\Big(v + w(z) - w\big(z(\rho)\big)\Big)\Big|_{\rho=0}. \tag{5}$$

Similarly, if $z \in \Pi^{-1}(P_0)$, then $D_\infty = D_1$ has the property corresponding to (5). It follows immediately from Riemann's vanishing theorem that for the correct choice of local coordinate $z_0(\rho)$ centered at z_0, that

$$D_\infty \theta(a\Omega + b) = \frac{d}{d\rho}\theta\Big(a\Omega + b + w(z_0) - w\big(z_0(\rho)\big)\Big)\Big|_{\rho=0} = 0.$$

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And again, since $\theta\big(a\Omega + b + w(z)\big)$ vanishes identically, $D_\infty\Big(\theta\big(a\Omega + b + w(z)\big)\Big)$ has the factor of automorphy $\hat{\xi}_{-r-m} = \zeta^2$ for the action of Γ on $\tilde{X}$. In [Gu1] it is shown that there exists a relatively analytic function h for the factor of automorphy ζ which vanishes simply at $\Pi^{-1}(P_0)$ and has no other zeros, hence $D_\infty\Big(\theta\big(a\Omega+b+w(z)\big)\Big)\Big/h^2$ is a function on X with at most a single, simple pole, so is a constant. Hence $D_\infty\Big(\theta\big(a\Omega + b + w(z)\big)\Big)$ has a double zero at $\Pi^{-1}(P_0)$ and no other zeros, and has a well-defined square root $\psi(z)$. There is an ambiguity of a sign in the definition of $\psi(z)$, but the ambiguity will disappear in the formula (6) below.

We can now calculate Gunning's prime form for X up to constant factor. Let $f(z_1, z_2) = \theta\big(w(z_1) - w(z_2) + \Omega a + b\big)$. We then define

$$Q(z_1, z_2) = \frac{f(z_1, z_2)e^{-4\pi i^t a w(z_2)}}{\psi(z_1)\psi(z_2)} \tag{6}$$

$$= \frac{\theta[\begin{smallmatrix} a \\ b \end{smallmatrix}]\big(w(z_1) - w(z_2)\big)}{\Sigma(z_1)\Sigma(z_2)}, \tag{7}$$

where we set $\Sigma(z) = e^{\pi i^t a\Omega a/2 + \pi i^t ab + 2\pi i^t a w(z)}\psi(z)$, so

$$\Sigma(z)^2 = e^{2\pi i^t a w(z)} D_\infty \theta[\begin{smallmatrix} a \\ b \end{smallmatrix}]\big(w(z)\big). \tag{8}$$

Since $f(z_1, z_0)$ vanishes, $Q(z_1, z_2)$ is analytic. From (3) and (4) we have that the factor of automorphy of $f(z_1, z_2)$ under the action of Γ on z_1 is $\hat{\rho}_{w(z_2)}\zeta^2$. So (6) shows that

$$Q(\gamma z_1, z_2) = \hat{\rho}_{w(z_2)}\zeta(\gamma, z_1)Q(z_1, z_2),$$

and (7) shows that Q is skew-symmetric. Hence $q = CQ$ for some constant C which we now determine up to sign.

Remarks 1) A particular odd theta characteristic was singled out in the definition of Q because we assumed a particular marking for X.

2) Formula (6) is similar to one given in [Gu5], where the derivatives are taken with respect to canonical coordinates.

Theorem

$$q(z_1, z_2) = \pm\frac{e^{\pi i^t a\Omega a + 2\pi i^t ab}\det(\sigma)\theta[\begin{smallmatrix} a \\ b \end{smallmatrix}]\big(w(z_1) - w(z_2)\big)}{D_2\theta[\begin{smallmatrix} a \\ b \end{smallmatrix}](0)\Sigma(z_1)\Sigma(z_2)}.$$

Proof We will use (2) to compute $\pm C$. It follows directly from (1) that changing η' or η'' by an integer vector at most changes the sign of $\theta[\begin{smallmatrix} \eta' \\ \eta'' \end{smallmatrix}](v)$. Since we will only be computing $\pm C$, we will identify theta characteristics modulo 1, and this will not affect any of the formulas that follow. For $1 \le i \le 5$, we define theta characteristics η_i by setting

$$w(P_i) = \Omega\eta_i' + \eta_i'' \mod L,$$

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and $\eta_i = [{\eta_i' \atop \eta_i''}]$. Let $\delta = [{a \atop b}] \mod 1$. It is standard [Gr] that the six odd theta characteristics are δ, $\delta + \eta_i$, $1 \leq i \leq 5$, and the 10 even theta characteristics are $\delta + \eta_i + \eta_j$, $1 \leq i < j \leq 5$. Also $\sum_{i=1}^{5} \eta_i = 0 \mod 1$.

We will use the following generalization of Jacobi's derivative formula. If ν_1, ν_2 are distinct odd theta characteristics, then

$$\det\Big(\frac{\partial}{\partial v_n}\theta[\nu_m](0)\Big)_{1\leq m,n\leq 2} = \pm\pi^2 \prod_{n=1}^{4} \theta[\rho_n](0), \tag{9}$$

for some set $\{\rho_n\}$ of even theta characteristics. This is due to Rosenhain, and was generalized to all hyperelliptic Riemann surfaces by Thomae. For a modern reference and further generalizations, see [I].

It can be shown (see [C]) that if $\nu_1 = \delta, \nu_2 = \delta + \eta_i$, then

$$\{\rho_n\} = \{\delta + \eta_i + \eta_j, \delta + \eta_i + \eta_k, \delta + \eta_i + \eta_\ell, \delta + \eta_i + \eta_m\},$$

where $\{i, j, k, \ell, m\} = \{1, 2, 3, 4, 5\}$. If $\nu_1 = \delta + \eta_i, \nu_2 = \delta + \eta_j$, then

$$\{\rho_n\} = \{\delta + \eta_i + \eta_j, \delta + \eta_k + \eta_\ell, \delta + \eta_k + \eta_m, \delta + \eta_\ell + \eta_m\}.$$

Now plugging $q = CQ$ into (2), we get for any $z, z_1, z_2 \in \tilde{X}$ that

$$\begin{aligned} &\theta\big(\Omega a + b + w(z) - w(z_1) - w(z_2)\big)\,\theta[{a \atop b}]\big(w(z_1) - w(z_2)\big)\,\Sigma(z)^2 \\ &\qquad = C\det\big(w_i'(z_j)\big)_{1\leq i,j\leq 2}\theta[{a \atop b}]\big(w(z) - w(z_1)\big)\,\theta[{a \atop b}]\big(w(z) - w(z_2)\big)\,. \end{aligned} \tag{10}$$

Now

$$\begin{aligned} w_i'(z_j) &= \lim_{z_j'\to z_j} \frac{w_i(z_j) - w_i(z_j')}{q(z_j, z_j')} = \frac{\Sigma(z_j)^2}{C} \lim_{z_j'\to z_j} \frac{w_i(z_j) - w_i(z_j')}{\theta[{a \atop b}]\big(w(z_j) - w(z_j')\big)} \\ &= \frac{\Sigma(z_j)^2}{C} \lim_{z_j'\to z_j} \frac{1}{\frac{\partial}{\partial v_1}\theta[{a \atop b}](0)\frac{w_1(z_j)-w_1(z_j')}{w_i(z_j)-w_i(z_j')} + \frac{\partial}{\partial v_2}\theta[{a \atop b}](0)\frac{w_2(z_j)-w_2(z_j')}{w_i(z_j)-w_i(z_j')}}. \end{aligned} \tag{11}$$

Using

$$\lim_{z_j'\to z_j} \frac{\int_{z_j'}^{z_j} \frac{x\,dx}{y}}{\int_{z_j'}^{z_j} \frac{dx}{y}} = x(z_j),$$

we get

$$\lim_{z_j'\to z_j} \frac{w_1(z_j) - w_1(z_j')}{w_2(z_j) - w_2(z_j')} = \frac{\sigma_{11} + \sigma_{12}x(z_j)}{\sigma_{21} + \sigma_{22}x(z_j)}.$$

So

$$\det\left(\lim_{z_j'\to z_j}\frac{w_i(z_j)-w_i(z_j')}{\theta[{}^a_b]\big(w(z_j)-w(z_j')\big)}\right)_{1\le i,j\le 2}$$
$$=\frac{\det\begin{pmatrix}\sigma_{11}+\sigma_{12}x(z_1) & \sigma_{11}+\sigma_{12}x(z_2)\\ \sigma_{21}+\sigma_{22}x(z_1) & \sigma_{21}+\sigma_{22}x(z_2)\end{pmatrix}}{\prod_{n=1}^2\Big(\frac{\partial}{\partial v_1}\theta[{}^a_b](0)\big(\sigma_{11}+\sigma_{12}x(z_n)\big)+\frac{\partial}{\partial v_2}\theta[{}^a_b](0)\big(\sigma_{21}+\sigma_{22}x(z_n)\big)\Big)}$$
$$=\frac{\det(\sigma)\big(x(z_2)-x(z_1)\big)}{\prod_{n=1}^2\big(-D_2\theta[{}^a_b](0)-x(z_n)D_1\theta[{}^a_b](0)\big)}$$
$$=\det(\sigma)\big(x(z_2)-x(z_1)\big)\big/\big(D_2\theta[{}^a_b](0)\big)^2. \tag{12}$$

Hence putting together (10), (11) and (12), we have

$$\begin{aligned}&C\theta\big(\Omega a+b+w(z)-w(z_1)-w(z_2)\big)\,\theta[{}^a_b]\big(w(z_1)-w(z_2)\big)\,\Sigma(z)^2\big(D_2\theta[{}^a_b](0)\big)^2\\ &\quad=\big(\det(\sigma)\big)\,\big(x(z_2)-x(z_1)\big)\,\Sigma(z_1)^2\Sigma(z_2)^2\theta[{}^a_b]\big(w(z)-w(z_1)\big)\,\theta[{}^a_b]\big(w(z)-w(z_2)\big).\end{aligned}\tag{13}$$

Since from (1)

$$\begin{aligned}&e^{2\pi i^t a(w(z)-w(z_1)-w(z_2))}\theta\big(\Omega a+b+w(z)-w(z_1)-w(z_2)\big)\\ &\qquad\qquad=e^{-\pi i^t a\Omega a-2\pi i^t ab}\theta[{}^a_b]\big(w(z)-w(z_1)-w(z_2)\big),\end{aligned}$$

using (8) repeatedly we get from (13) that

$$\begin{aligned}&C'\frac{\theta[{}^a_b]\big(w(z)-w(z_1)-w(z_2)\big)\,D_\infty\theta[{}^a_b]\big(w(z)\big)}{\theta[{}^a_b]\big(w(z)-w(z_1)\big)\,\theta[{}^a_b]\big(w(z)-w(z_2)\big)}\,\frac{\theta[{}^a_b]\big(w(z_1)-w(z_2)\big)}{D_\infty\theta[{}^a_b]\big(w(z_1)\big)\,D_\infty\theta[{}^a_b]\big(w(z_2)\big)}\\ &\qquad=\big(\det(\sigma)\big)\,\big(x(z_2)-x(z_1)\big)\big/\big(D_2\theta[{}^a_b](0)\big)^2,\end{aligned}\tag{14}$$

where $C'=Ce^{-\pi i^t a\Omega a-2\pi i^t ab}$.

At this point we square (14), and let z, z_1, z_2 be any points such that $\Pi(z)=P_k$, $\Pi(z_1)=P_i$, $\Pi(z_2)=P_j$, for distinct i, j, $k\in\{1,2,3,4,5\}$. Then using (1) repeatedly, from (14) we have

$$\begin{aligned}&(C')^2\frac{\theta[\delta+\eta_\ell+\eta_m](0)^2D_\infty\theta[\delta+\eta_k](0)^2}{\theta[\delta+\eta_i+\eta_k](0)^2\theta[\delta+\eta_j+\eta_k](0)^2}\,\frac{\theta[\delta+\eta_i+\eta_j](0)^2}{D_\infty\theta[\delta+\eta_i](0)^2D_\infty\theta[\delta+\eta_j](0)^2}\\ &\qquad=\big(\det(\sigma)\big)^2(a_i-a_j)^2\big/\big(D_2\theta[\delta](0)\big)^4,\end{aligned}\tag{15}$$

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where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. We will now apply Rosenhain's formula (9). Since $D_\infty \theta[\delta](0) = 0$, we have

$$\begin{aligned} D_2\theta[\delta](0)^2 D_\infty \theta[\delta + \eta_k](0)^2 &= \det \begin{pmatrix} D_2\theta[\delta](0) & D_2\theta[\delta + \eta_k](0) \\ D_1\theta[\delta](0) & D_1\theta[\delta + \eta_k](0) \end{pmatrix}^2 \\ &= \big(\det(\sigma)\big)^2 \det \begin{pmatrix} \frac{\partial}{\partial v_1}\theta[\delta](0) & \frac{\partial}{\partial v_1}\theta[\delta + \eta_k](0) \\ \frac{\partial}{\partial v_2}\theta[\delta](0) & \frac{\partial}{\partial v_2}\theta[\delta + \eta_k](0) \end{pmatrix}^2 \\ &= \big(\det(\sigma)\big)^2 \pi^4 \theta[\delta + \eta_k + \eta_i](0)^2 \theta[\delta + \eta_k + \eta_j](0)^2 \\ &\qquad \theta[\delta + \eta_k + \eta_\ell](0)^2 \theta[\delta + \eta_k + \eta_m](0)^2. \end{aligned} \tag{16}$$

Similarly, (5) and Riemann's vanishing theorem imply that $D_{a_i}\theta[\delta + \eta_i](0) = 0$, so $D_{a_j}\theta[\delta + \eta_i](0) = (a_j - a_i) D_\infty \theta[\delta + \eta_i](0)$. Hence, reasoning as in (16), by (9),

$$\begin{aligned} &D_\infty \theta[\delta + \eta_i](0)^2 D_\infty \theta[\delta + \eta_j](0)^2 \\ &\quad = (a_i - a_j)^{-4} D_{a_j}\theta[\delta + \eta_i](0)^2 D_{a_i}\theta[\delta + \eta_j](0)^2 \\ &\quad = (a_i - a_j)^{-2} \det \begin{pmatrix} D_2\theta[\delta + \eta_i](0) & D_2\theta[\delta + \eta_j](0) \\ D_1\theta[\delta + \eta_i](0) & D_1\theta[\delta + \eta_j](0) \end{pmatrix}^2 \\ &\quad = (a_i - a_j)^{-2} \big(\det(\sigma)\big)^2 \det \begin{pmatrix} \frac{\partial}{\partial v_1}\theta[\delta + \eta_i](0) & \frac{\partial}{\partial v_1}\theta[\delta + \eta_j](0) \\ \frac{\partial}{\partial v_2}\theta[\delta + \eta_i](0) & \frac{\partial}{\partial v_2}\theta[\delta + \eta_j](0) \end{pmatrix}^2 \\ &\quad = (a_i - a_j)^{-2} \big(\det(\sigma)\big)^2 \pi^4 \theta[\delta + \eta_i + \eta_j](0)^2 \\ &\qquad\quad \theta[\delta + \eta_k + \eta_\ell](0)^2 \theta[\delta + \eta_k + \eta_m](0)^2 \theta[\delta + \eta_\ell + \eta_m](0)^2. \end{aligned} \tag{17}$$

Combining (15), (16), and (17) we get

$$(C')^2 = \big(\det(\sigma)\big)^2 / \big(D_2\theta[\delta](0)\big)^2,$$

so $C = \pm e^{\pi i^t a \Omega a + 2\pi i^t ab} \det(\sigma)/D_2\theta[\delta](0)$, which gives us our theorem.

Remarks 1) Although affine transformations $(x, y) \to (\alpha^2 x + \beta, \alpha^5 y)$ of our curve affect the differential operators D_1, D_2, they leave $\det(\sigma)/D_2\theta[\delta](0)\Sigma(z_1)\Sigma(z_2)$ invariant.

2) The constant $D_2\theta[\delta](0)$ is related to the discriminant of our curve: see [Gr].

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