# SOME REMARKS ON A CLASS OF DISTRIBUTIVE LATTICES

T. P. SPEED

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## 1. Introduction

Distributive pseudo-complemented lattices form an extensively studied class of distributive lattices. Examples are the lattice of all open sets of a topological space, the lattice of all ideals of a distributive lattice with zero and the lattice of all congruences of an arbitrary lattice. Lattices which are just pseudo-complemented have been studied in detail by J. Varlet [6], [7] where, however, the most interesting results require at least the assumption of modularity, sometimes distributivity.

In this note we introduce a new class of distributive lattices which includes the class of pseudo-complemented distributive lattices. We call these lattices distributive *-lattices and denote the set of all such lattices by $\Delta^*$. Several characterisations of $\Delta^*$ are given, an example of a non-pseudo-complemented lattice in $\Delta^*$ is given and some properties of the congruence $R$ defined below are studied.

## 2. Definitions, Notation and Preliminary results

We refer to G. Birkhoff [1] for the elementary properties of distributive lattices. For $A \subseteqq L$ in a distributive lattice $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ with zero we define $A^* = \{t \in L : t \wedge a = 0 \text{ for all } a \in A\}$. The principal ideal generated by $a \in L$ is written (a) and the principal dual ideal generated by $a$ is written $[a]$. The congruence $R$ is defined in a distributive lattice with zero by

$$(a, b) \in R \text{ if and only if } (a)^* = (b)^*$$

An element $d \in L$ is called *dense* if $(d)^* = (0)$.

In a distributive lattice with zero the existence of minimal prime ideals can be readily proved and we denote the set of all such ideals by $\mathscr{M} = \mathscr{M}(\mathscr{L})$. For a subset $\mathscr{A} \subseteqq \mathscr{M}$ we write $k(\mathscr{A}) = \bigcap \{A : A \in \mathscr{A}\}$ and for a subset $A \subseteqq L$ we write $h(A) = \{M \in \mathscr{M} : A \subseteqq M\}$. Also let $\mathscr{M}_x = \mathscr{M} \setminus h(x)$. Then if the family $\{\mathscr{M}_x : x \in L\}$ of subsets of $\mathscr{M}$ is taken as an open basis the resulting topology is called the *hull-kernel* topology. If the family

$\{h(x) : x \in L\}$ is taken as an open basis the resulting topology is called the *dual hull-kernel* topology. These ideas have been discussed in commutative semigroups by J. Kist [3] and in commutative rings by M. Henriksen & M. Jerison [2]. A detailed discussion of spaces of minimal prime ideals in distributive lattices will appear in the author's forthcoming thesis; for the present only those results which are used in the study of $\Delta^*$ will be stated.

We now give our basic definition, noting that the idea was suggested by the work of M. Henriksen & M. Jerison [2].

DEFINITION. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ *be a distributive lattice with zero. Then* $\mathscr{L} \in \Delta^*$ *if and only if for all* $x \in L$, $(x)^{**} = (x')^*$ *for some* $x' \in L$.

## 3. Characterisation of $\Delta^*$

The results of this section begin with a topological characterisation of $\Delta^*$. We shall need some preliminary results which are distributive lattice analogues of results from [2] and [3]. For this reason we state them without proof.

LEMMA 3.1. *For a distributive with zero* $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ *the following hold.*

(i) *A prime ideal* $M$ *is minimal if and only if* $(x)^* \setminus M \neq \square$ *for any* $x \in M$.

(ii) $\mathscr{M}_x = h((x)^*)$

(iii) $h(x) = h((x)^{**})$

(iv) $(z)^* = (x)^* \cap (y)^* \Leftrightarrow h(z) = h(x) \cap h(y)$

(v) $(x \wedge y)^{**} = (x)^{**} \cap (y)^{**}$

(vi) $(x)^{**} = (y)^* \Leftrightarrow h(x) = h((y)^*)$

If we write $\mathscr{T}_h$ for the hull-kernel topology on $\mathscr{M}$ and $\mathscr{T}_d$ for the dual hull-kernel topology on $\mathscr{M}$ we have

PROPOSITION 3.2. *If* $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ *is a distributive lattice with zero, the following are equivalent*:

I $\mathscr{L} \in \Delta^*$ *i.e. for any* $x \in L$, $(x)^{**} = (x')^*$ *for some* $x' \in L$.

II $\mathscr{T}_h = \mathscr{T}_d$ *i.e. the two topologies on* $\mathscr{M}$ *coincide.*

III $\mathscr{M}(\mathscr{L})$ *is compact in the hull-kernel topology.*

PROOF. I $\Rightarrow$ II. Assume I and take an arbitrary $x \in L$. Then

$$\begin{aligned} \mathscr{M}_x &= h((x)^*) && \text{by 3.1 (ii)} \\ &= h((x')^{**}) && \text{since } (x)^{***} = (x)^* \text{ and by I} \\ &= h(x') && \text{by 3.1 (iii).} \end{aligned}$$

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Similarly

$$\begin{aligned} h(y) &= h((y)^{**}) && \text{by 3.1 (iii)} \\ &= h((y')^{*}) && \text{by I} \\ &= \mathscr{M}_{y'} && \text{by 3.1 (ii).} \end{aligned}$$

Thus $\{\mathscr{M}_x : x \in L\} = \{h(x) : x \in L\}$ and so the two topologies coincide.

II $\Rightarrow$ III. Assume II and consider a centred family of closed sets in $\mathscr{M}$ (with the hull-kernel topology). Since the family $\{\mathscr{M}_x : x \in L\}$ is a closed basis when the two topologies coincide, the centred family may be taken to be of the form $\{\mathscr{M}_t : t \in T\}$. Hence we have

$$\bigcap_{i=1}^{n} \mathscr{M}_{t_i} \neq \square \text{ for all finite } \{t_1, t_2, \cdots, t_n\} \subseteqq T.$$

This implies that $T$ is a subset of $L$ with the property

$$t_1 \wedge t_2 \wedge \cdots \wedge t_n \neq 0 \text{ for any finite } \{t_1, \cdots, t_n\} \subseteqq T.$$

From this fact, we may enclose the dual ideal $[T]$ generated by $T$ in a prime dual ideal $F$ whose complement $L \setminus F$ in $L$ is a minimal prime ideal not meeting $T$. Now $T \cap (L \setminus F) = \square$ implies that $t \notin L \setminus F$ for all $t \in T$ i.e. $L \setminus F \in \mathscr{M}_t$ for all $t \in T$.

Thus $L \setminus F \in \bigcap_{t \in T} \mathscr{M}_t$ and the compactness of $\mathscr{M}$ in the hull-kernel topology is proved.

III $\Rightarrow$ I. Assume that $\mathscr{M}$ is compact in the hull-kernel topology. Then $h(x)$ is a closed subset of $\mathscr{M}$ and so is compact in the relative topology. Now

$$\square = h(x) \cap h((x)^*) = h(x) \cap \bigcap \{h(t) : t \in (x)^*\}$$

and so, by the compactness of $h(x)$, there is $\{t_1, t_2, \cdots, t_n\} \subseteqq (x)^*$ such that

$$\square = h(x) \cap h(t_1) \cap \cdots \cap h(t_n).$$

On taking complements in $\mathscr{M}$ we find that

$$\mathscr{M} = \mathscr{M}_x \cup \mathscr{M}_{t_1} \cup \cdots \cup \mathscr{M}_{t_n}$$

But the map $\mu : x \to \mathscr{M}_x$ is an homomorphism and so we have

$$\mathscr{M} = \mathscr{M}_x \cup \mathscr{M}_{\vee t_i}$$

and

$$\mathscr{M}_x \cap \mathscr{M}_{\vee t_i} = \mathscr{M}_{x \wedge \vee t_i} = \square.$$

Thus putting $x' = \bigvee_{i=1}^{n} t_i$ we have

$$\mathscr{M}_{x'} = \mathscr{M} \setminus \mathscr{M}_x = h(x)$$

i.e. $h((x')^*) = h(x) = h((x)^{**})$ which, by 3.1 (vi) gives us the required result $(x)^{**} = (x')^*$.

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REMARK. The final implication is exactly as in the commutative ring case and is thus due to M. Henriksen & M. Jerison [2].

The congruence $R$ was defined in § 2 and we will now state how it relates to $\Delta^*$.

PROPOSITION 3.3. *Let $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ be a distributive lattice with zero. Then $\mathscr{L} \in \Delta^*$ if and only if $\mathscr{L}/R$ is a Boolean lattice.*

PROOF. This result can be proved algebraically by the methods of [4] but we prefer to give an alternative proof here using the topological ideas. J. Kist [3] has proved that in a commutative semigroup $S$, the semi-lattice formed by $\{\mathscr{M}_x : x \in S\}$ is isomorphic to $S/R$. His proof carries over to the distributive lattice case and so $\mathscr{L}/R \cong \mu(\mathscr{L}) = \langle\{\mathscr{M}_x : x \in L\}; \cup, \cap, \Box\rangle$.

Firstly assume $\mathscr{L} \in \Delta^*$. The for any $x \in L$

$$\mathscr{M}\setminus\mathscr{M}_x = h(x) = h((x)^{**}) = h((x')^*) = \mathscr{M}_{x'}$$

Thus $\mu(\mathscr{L})$ is complemented and so is a Boolean lattice.

For the converse assume that $\mathscr{L}/R$, and so $\mu(\mathscr{L})$, is a Boolean lattice. I.e. for any $x \in L$, $\mathscr{M}\setminus\mathscr{M}_x = \mathscr{M}_{x'}$ for some $x' \in L$. Then

$$h(x) = h((x)^{**}) = h((x')^*)$$

and so, by 3.1 (vi) $(x)^{**} = (x')^*$ follows.

The proposition is thus proved.

We now give two algebraic conditions on $\mathscr{L}$ which are equivalent to membership of $\Delta^*$. Condition II was kindly supplied to me by J. Varlet. Let the set of all dense elements of $\mathscr{L}$ be denoted by $D$.

PROPOSITION 3.4. *If $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ is a distributive lattice with zero, the following are equivalent:*

I *$\mathscr{L} \in \Delta^*$ i.e. for any $x \in L$, $(x)^{**} = (x')^*$ for some $x' \in L$.*

II *For any $x \in L$ there is $x' \in L$ such that $x \wedge x' = 0$, $x \vee x' \in D$.*

III *For any ideal $I$ of $\mathscr{L}$ such that $I \cap D = \Box$, there is a minimal prime ideal $M \supseteqq I$.*

PROOF. I $\Rightarrow$ II. Assume $\mathscr{L} \in \Delta^*$. Then clearly $x \wedge x' = 0$. We shall see that $x \vee x' \in D$.

$$(x \wedge x')^* = (x)^* \cap (x')^* = (x)^* \cap (x)^{**} = (0)$$

and so the result follows.

II $\Rightarrow$ III. Assume II and observe that since $I \cap D = \Box$, $D$ can be extended to a dual ideal $F$ maximal with respect to not meeting $I$. By well known results of M. H. Stone $F$ is a prime dual ideal and also $L\setminus F$ is a prime ideal of $\mathscr{L}$. For any $x \in L\setminus F$ we note that $x' \notin L\setminus F$ since $x \wedge x' \in D$.

Thus $(x)^*\setminus(L\setminus F) \neq \square$ and so, by 3.1 (i), $L\setminus F$ is a minimal prime ideal containing $I$.

III $\Rightarrow$ I. Assume $\mathscr{L}$ satisfies III. Then since $(x) \vee (x)^*$ cannot be contained in any minimal prime ideal $M$, for otherwise $(x)^*\setminus M$ would be empty, we deduce that $((x) \vee (x)^*) \cap D \neq \square$. Suppose $d \in D$ is an element of $(x) \vee (x)^*$ — i.e. $d = a \vee b$ where $a \in (x)$ and $b \in (x)^*$, then $x \vee b \in D$ also. We shall show that taking $b = x'$ will satisfy I. Clearly $b \wedge x = 0$ and so $(b) \subseteqq (x)^*$ or $(b)^* \supseteqq (x)^{**}$. Let $t \in (b)^*$ and $s \in (x)^*$, and observe that $t \wedge s \wedge b = 0$ and $t \wedge s \wedge x = 0$. Thus $t \wedge s \wedge (b \vee x) = 0$ whence $t \wedge s = 0$ since $b \vee x \in D$. The reverse inclusion $(b)^* \subseteqq (x)^{**}$ is now proved and so $(x)^{**} = (b)^*$.

All of these results are collected in the following theorem.

THEOREM 1. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ *be a distributive lattice with zero. Then the following are equivalent*:

I $\mathscr{L} \in \Delta^*$ *i.e. for any* $x \in L$, $(x)^{**} = (x')^*$ *for some* $x' \in L$.

II $\mathscr{T}_h = \mathscr{T}_d$ *i.e. the two topologies on* $\mathscr{M}$ *coincide.*

III $\mathscr{M}$ *is compact in the hull-kernel topology.*

IV $\mathscr{L}/R$ *is a Boolean lattice.*

V *For any* $x \in L$ *there is* $x' \in L$ *such that* $x \wedge x' = 0$, $x \vee x' \in D$.

VI *For any ideal* $I$ *of* $\mathscr{L}$ *with* $I \cap D = \square$, *there is a minimal prime ideal* $M \supseteqq I$.

## 4. An Example

We give an example of a distributive lattice belonging to $\Delta^*$ which is not pseudo-complemented. Let $I^+ = \{0, 1, 2, 3, \cdots, n, \cdots\}$ denote the chain of non-negative integers, and **2** denote the two element chain. Consider the lattice (cardinal product)

$$L = (\mathbf{2} \times I^+) \cup \{u\}$$

with a unit $u$ adjoined. The lattice has a zero $(0, 0)$ and we shall see that for any $x \in L$, $(x)^{**} = (x')^*$ for some $x' \in L$.

| | |
|---|---|
| Elements of Type $(0, n)$: | $\{(0, 0)\}^* = L = \{u\}^{**}$ |
| | $\{(0, 1)\}^* = \{(0, 0), (1, 0)\}$ |
| | $\{(0, 2)\}^* = \{(0, 0), (1, 0)\}$ |
| and, generally, | $\{(0, n)\}^* = \{(0, 0), (1, 0)\}$ |
| Elements of Type $(1, n)$: | $\{(1, 0)\}^* = \{(0, 0), (0, 1), \cdots, (0, n), \cdots\}$ |
| and, generally, | $\{(1, n)\}^* \quad \{(0, 0)\}$. |

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Thus the elements of type $(0, n)$ for $n = 1, 2, \cdots$ all have

$$\{(0, n)\}^{**} = \{(1, 0)\}^{*}.$$

In fact, they are all pseudo-complemented with

$$(0, n)^{*} = (1, 0)$$

The elements of type $(1, n)$ for $n = 1, 2, \cdots$ all have

$$\{(1, n)\}^{**} = \{(0, 0)\}^{*}.$$

They also are pseudo-complemented with $(1, n)^{*} = (0, 0)$ i.e. they are all dense.

Finally $(0, 0)^{*} = u$ exists but $(1, 0)$ is not pseudo-complemented, since the join of all elements $(0, n)$ does not exist. However

$$\{(1, 0)\}^{**} = \{(0, 0), (1, 0)\} = \{(0, 1)\}^{*}.$$

## 5. Some Properties of $\Delta^*$

We begin this section with a simple result which determines when elements of $\Delta^*$ are Boolean. A distributive lattice with zero is said to be *disjunctive* if for any pair $x$ and $y$ with $x < y$ there is $z$ such that

$$0 = x \wedge z \neq y \wedge z.$$

THEOREM 2. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle \in \Delta^*$. *Then the following are equivalent*:

I $\mathscr{L}$ *is a Boolean lattice*

II $\mathscr{L}$ *is a disjunctive lattice*

III 1 *is the only dense element of* $\mathscr{L}$.

PROOF. I $\Rightarrow$ II. II is a known property of Boolean lattices.

II $\Rightarrow$ III. Since any $x \in L$ must satisfy $x < 1$ or $x = 1$ II implies that if $x \neq 1$ there is at least one element $z \neq 0$ with $x \wedge z = 0$. Thus 1 remains as the only dense element.

III $\Rightarrow$ I. If 1 is the only dense element, then $V$ of Theorem 1 tells us that for any $x \in L$ there is an $x' \in L$ with

$$x \wedge x' = 0 \quad x \vee x' = 1$$

i.e. $\mathscr{L}$ is complemented and hence a Boolean lattice.

Our remaining results concern the congruence $R$ in lattices $\mathscr{L} \in \Delta^*$. Recall that for any dual ideal $F$ of $\mathscr{L}$, the least congruence with $F$ as a congruence class is the congruence $\Theta[F]$ defined by

$$(x, y) \in \Theta[F] \text{ if and only if } (x \vee y) \wedge f = x \wedge y \text{ for some } f \in F.$$

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The following Proposition can be considered as an extension of Theorem 16 p. 148 of G. Birkhoff [1].

PROPOSITION 5.1. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle \in \Delta^*$. *Then for* $a, b \in L$

(i) *If* $a \wedge d = b \wedge d$ *for some* $d \in D$, *we have* $(a, b) \in R$

(ii) *If* $(a, b) \in R$ *there is* $d \in D$ *such that* $a \wedge d = b \wedge d$.

PROOF. (i) If $a \wedge d = b \wedge d$ we may use **3.1** (v) to obtain

$$(a)^{**} = (a)^{**} \cap (d)^{**} = (a \wedge d)^{**} = (b \wedge d)^{**} = (b)^{**} \cap (d)^{**} = (b)^{**}$$

since $(d)^{**} = L$. Thus $(a)^{***} = (b)^{***}$ or $(a)^* = (b)^*$ which gives us the result $(a, b) \in R$.

(ii) Suppose $(a, b) \in R$ i.e. $(a)^* = (b)^*$. Then since $\mathscr{L} \in \Delta^*$ we know $a'$ and $b'$ exist and have the required properties. In fact we may take $a' = b'$. Now consider $d = (a \wedge b) \vee a'$.

$$(d)^* = (a \wedge b)^* \cap (a')^* = (a)^* \cap (a)^{**} = (0) \text{ and so } d \in D.$$

Also

$$\begin{aligned} a \wedge d &= a \wedge ((a \wedge b) \vee a') \\ &= (a \wedge b) \vee (a \wedge a') \\ &= a \wedge b \end{aligned}$$

and

$$\begin{aligned} b \wedge d &= b \wedge ((a \wedge b) \vee a') \\ &= (b \wedge a) \vee (b \wedge a') \\ &= a \wedge b \end{aligned}$$

Thus $a \wedge d = b \wedge d$ for some $d \in D$ and our proposition is proved.

REMARK. Proposition **5.1** can be considered as saying: For $\mathscr{L} \in \Delta^*$ $R = \Theta[D]$. This is easily seen, for if $(a \vee b) \wedge d = a \wedge b$ for $d \in D$, $(a, b) \in R$ follows readily. Similarly if $(a, b) \in R$ take $d = (a \wedge b) \vee a'$ and we find $(a \vee b) \wedge d = a \wedge b$.

It is known that the largest congruence on $\mathscr{L}$ with a given dual ideal $F$ as congruence class is the congruence $R'_F$ defined by

$$(x, y) \in R'_F \text{ if and only if } (x; F)^\dagger = (y; F)^\dagger$$

where $(a; F)^\dagger = \{t \in L : a \vee t \in F\}$. Our next proposition relates this congruence to $R$ when $F = D$.

PROPOSITION 5.2. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle \in \Delta^*$, *then* $R = R'_D$.

PROOF. We must show that $(a)^* = (b)^*$ if and only if $(a; D)^\dagger = (b; D)^\dagger$. But since it is known that $R \leqq R'_D$ in the partial ordering of congruences we need only prove the if assertion.

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Suppose $t \vee a \in D$ when and only when $t \vee b \in D$. I.e.

$$(t \vee a)^* = (t)^* \cap (a)^* = (0) \Leftrightarrow (t)^* \cap (b)^* = (0)$$

This gives us, since $\mathscr{L} \in \Delta^*$

$$(t')^{**} \cap (a')^{**} = (0) \Leftrightarrow (t')^{**} \cap (b')^{**} = (0)$$

By 3.1 (v) this is equivalent to

$$(t' \wedge a')^{**} = (0) \Leftrightarrow (t' \wedge b')^{**} = (0)$$

or

$$t' \wedge a' = 0 \quad \Leftrightarrow t' \wedge b' = 0$$

Now any element of $L$ is of the form $t'$ and so this last result tells us that $(a')^* = (b')^*$, and finally

$$(a)^* = (b)^*.$$

Hence $(a, b) \in R$ and $R'_D \leqq R$. The result is now proved.

We can combine the last two results in

THEOREM 3. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle \in \Delta^*$. *Then* $\Theta[D] = R = R'_D$ *i.e. $R$ is the unique congruence with the dual ideal of dense elements as a congruence class.*

It should be remarked that Theorem 3 was suggested by a result of J. Varlet [7] for pseudo-complemented modular lattices. The method of proof however, is entirely different from that used in [7].

Our final result is also an analogue of a result of J. Varlet [6] first proved for pseudo-complemented modular lattices. Since the proof in this case is similar to that of [6] we omit it.

PROPOSITION 5.3. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle \in \Delta^*$ *and suppose $D$ is principal i.e. $D = [d]$ where $d \in D$. Then*

$$(a, b) \in R \textit{ if and only if } a \wedge d = b \wedge d \textit{ and } \mathscr{L}/R \cong (d).$$

For further results about lattices in $\Delta^*$, especially ones involving Stone lattices, we refer to [5].

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