# 6
# Fermions on the lattice

In this chapter we introduce the path integral for Fermi fields. We shall discuss the species-doubling phenomenon – the fact that a naively discretized Dirac fermion field leads to more particle excitations than expected and desired, two remedies for this, which go under the names 'Wilson fermions' and 'staggered fermions', the interpretation of the path integral in Hilbert space, and the construction of the transfer operator. Integration over 'anticommuting numbers', the 'Grassmann variables' and the relation with creation and annihilation operators in fermionic Hilbert space is reviewed in appendix C.

## 6.1 Naive discretization of the Dirac action

In continuous Minkowski space–time the action for a free fermion field can be written as (see appendix D for an introduction)

$$S = -\int d^4x \big[\tfrac{1}{2}(\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \partial_\mu\bar{\psi}(x)\gamma^\mu\psi(x)) + m\bar{\psi}(x)\psi(x)\big], \quad (6.1)$$

or, exhibiting the Dirac indices $\alpha$, $\beta$, ... (but suppressing the label $x$ for brevity),

$$S = -\int d^4x \,\big[(\gamma^\mu)_{\alpha\beta}\tfrac{1}{2}(\bar{\psi}_\alpha\partial_\mu\psi_\beta - \partial_\mu\bar{\psi}_\alpha\psi_\beta) + m\bar{\psi}_\alpha\psi_\alpha\big]. \quad (6.2)$$

The $\psi$ and $\bar{\psi}$ are anticommuting objects, so-called Grassmann variables, e.g. $\psi_\alpha(x)\bar{\psi}_\beta(y) = -\bar{\psi}_\beta(y)\psi_\alpha(x)$. The integrand in (6.1) is Hermitian, treating $\psi$ and $\psi^+$,

$$\psi^+ = \bar{\psi}\beta, \;\; \beta = i\gamma^0, \quad (6.3)$$

as Hermitian conjugates, e.g. $(\psi_\alpha(x)\psi^+_\beta(y))^\dagger = \psi^+_\beta(y)\psi_\alpha(x)$. Note, however, that $\psi$ and $\psi^+$ are independent 'variables' (which is why we use



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the superscript + instead of †). The Dirac matrices have the following properties:

$$\gamma^0 = -\gamma_0, \ \ \gamma_0 = -\gamma_0^\dagger, \ \ \gamma_0^2 = -1, \tag{6.4}$$

$$\gamma^k = \gamma_k = \gamma_k^\dagger, \ \ \gamma_k^2 = 1, \ \ k = 1, 2, 3. \tag{6.5}$$

implying that $\beta = \beta^\dagger$ and $\beta^2 = 1$.† Replacing the derivative operators by discrete differences,

$$\partial_\mu \psi(x) \to \frac{1}{a_\mu}[\, \psi(x + a_\mu \hat{\mu}) - \psi(x)\,], \tag{6.6}$$

we obtain from (6.1) a lattice version

$$S = -\sum_{x,\mu} \frac{1}{2a_\mu}[\bar{\psi}(x)\gamma^\mu \psi(x+a_\mu\hat{\mu}) - \bar{\psi}(x+a_\mu\hat{\mu})\gamma^\mu\psi(x)] - m\sum_x \bar{\psi}(x)\psi(x). \tag{6.7}$$

Recall that $a_\mu$ is the lattice spacing in the $\mu$ direction. We shall occasionally only need the spacing in the time direction, $a_0$, to be different from the spatial lattice spacing $a_k = a$, $k = 1, 2, 3$.

The path integral for free fermions with anticommuting external sources $\eta$ and $\bar{\eta}$ is now tentatively defined by

$$Z(\eta, \bar{\eta}) = \int D\bar{\psi}\, D\psi\, e^{i[S + \sum_x (\bar{\eta}\psi + \bar{\psi}\eta)]}, \tag{6.8}$$

where

$$D\bar{\psi}\, D\psi = \prod_{x,\alpha} d\bar{\psi}_{x\alpha}\, d\psi_{x\alpha} = \prod_{x\alpha} d\psi^+_{x\alpha}\, d\psi_{x\alpha}. \tag{6.9}$$

We assumed the action to be rewritten in terms of dimensionless $\psi_x$ and $\bar{\psi}_x$,

$$\psi_x = a^{3/2}\psi(x), \quad \bar{\psi}_x = a^{3/2}\bar{\psi}(x), \tag{6.10}$$

and similarly the symbols $d\psi^+_{x\alpha}$ and $d\psi_{x\alpha}$ are dimensionless. The last equality in (6.9) follows from the rule $d(T\psi) = (\det T)^{-1}\, d\psi$ (cf. appendix C) and $\det \beta = 1$. The $\psi_{x\alpha}$ and $\psi^+_{x\alpha}$ are independent generators of a Grassmann algebra. We recall also the definition of fermionic integration (cf. appendix C),

$$\int db = 0, \quad \int db\, b = 1, \tag{6.11}$$

where $b$ is any of the $\psi_{x\alpha}$ or $\psi^+_{x\alpha}$. Before making the transition to imaginary time we need to make the dependence on $a_0$ explicit. So let $n_\mu$

† We usually write just 1 for the unit matrix $\mathbb{1}$.

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be the integers specifying the lattice site $x$, $x^0 = n_0 a_t$, $\mathbf{x} = \mathbf{n}a$, and let $\psi_n \equiv \psi_x$ and $\bar\psi_n \equiv \bar\psi_x$. Recalling that $\sum_x = a_0 a^3 \sum_n$ in our notational convention, the lattice action reads more explicitly

$$S = -\sum_n \left[ \frac{1}{2}(\bar\psi_n \gamma^0 \psi_{n+\hat 0} - \bar\psi_{n+\hat 0}\gamma^0 \psi_n) \right.$$
$$\left. + \sum_{k=1}^{3} \frac{a_0}{2a}(\bar\psi_n \gamma^k \psi_{n+\hat k} - \bar\psi_{n+\hat k}\gamma^k \psi_n) + (a_0 m)\bar\psi_n \psi_n \right]. \quad (6.12)$$

Furthermore

$$\sum_x (\bar\eta\psi + \bar\psi\eta) \equiv \frac{a_0}{a} \sum_n (\bar\eta_{\alpha n}\psi_{\alpha n} + \bar\psi_{\alpha n}\eta_{\alpha n}), \quad (6.13)$$

with dimensionless $\eta_{\alpha n}$ and $\bar\eta_{\alpha n}$.

It follows from the rules of fermionic integration that the path integral for a finite space–time volume is a polynomial in $a_0 m$ and $a_0/a$. Hence, an analytic continuation to ‘imaginary time’ poses no problem:

$$a_0 = |a_0| \exp(-i\varphi), \quad \varphi\colon 0 \to \pi/2, \quad a_0 \to -i a_4, \quad (6.14)$$

with $a_4 = |a_0|$. This transforms the path integral into its Euclidean version ($iS \to S_{\Im}$, dropping the $\Im$),

$$Z = \int D\bar\psi D\psi \, e^{S + \sum_n (\bar\eta\psi + \bar\psi\eta)}, \quad (6.15)$$
$$S = -\sum_n \left[ \sum_\mu \frac{a_4}{2a_\mu}(\bar\psi_n \gamma_\mu \psi_{n+\hat\mu} - \bar\psi_{n+\hat\mu}\gamma_\mu \psi_n) + a_4 m \, \bar\psi_n \psi_n \right],$$

where $\mu$ now runs from 1 to 4 (with $n_4 \equiv n_0$, $\hat 4 \equiv \hat 0$), and

$$\gamma_4 = i\gamma^0 = \beta. \quad (6.16)$$

## 6.2 Species doubling

It turns out that the model described by the action in (6.15) yields $2^4 = 16$ Dirac particles (fermions with two charge and two spin states) instead of one. This is the species-doubling phenomenon. We shall infer it in this section from inspection of the fermion propagator and the excitation energy spectrum.

Using a matrix notation, writing

$$Z(\eta, \bar\eta) = \int D\bar\psi D\psi \, e^{-\bar\psi A \psi + \bar\eta\psi + \bar\psi\eta}, \quad (6.17)$$

where (in lattice units, $a = a_4 = 1$)

$$A_{xy} = \sum_{z\mu} \gamma_\mu \frac{1}{2} (\bar{\delta}_{x,z}\bar{\delta}_{y,z+\hat{\mu}} - \bar{\delta}_{x,z+\hat{\mu}}\bar{\delta}_{y,z}) + m \sum_z \bar{\delta}_{x,z}\bar{\delta}_{y,z}, \tag{6.18}$$

the path integral is easily integrated (appendix C) to give

$$Z(\eta, \bar{\eta}) = \det A \, e^{\bar{\eta}A^{-1}\eta}. \tag{6.19}$$

Here $A^{-1}_{xy} \equiv S_{xy}$ is the fermion propagator. It can be evaluated in momentum space, assuming infinite space–time,

$$A(k, -l) = \sum_{xy} e^{-ikx+ily} A_{xy} = S(k)^{-1}\bar{\delta}(k-l), \tag{6.20}$$

$$S(k)^{-1} = \sum_\mu i\gamma_\mu \sin k_\mu + m, \tag{6.21}$$

$$S(k) = \frac{m - i\gamma_\mu s_\mu}{m^2 + s^2}, \quad s_\mu = \sin k_\mu. \tag{6.22}$$

Reverting to non-lattice units the propagator becomes

$$S(k) = \frac{m - i\sum_\mu \gamma_\mu \sin(ak_\mu)/a}{m^2 + \sum_\mu \sin^2(ak_\mu)/a^2}, \tag{6.23}$$

for which the limit $a \to 0$ gives the continuum result

$$S(k) = \frac{m - i\gamma k}{m^2 + k^2} + O(a^2). \tag{6.24}$$

The propagator has a pole at $k_4 = i\omega = i\sqrt{\mathbf{k}^2 + m^2}$ corresponding to a Dirac particle. The pole is near the zeros of the sine functions at the origin $ak_\mu = 0$. However, there are 15 more regions in the four dimensional torus $-\pi < ak_\mu \leq \pi$ where the sine functions vanish, 16 in total:

$$S(k) = \frac{m - i\gamma^{(A)}_\mu p_\mu}{m^2 + p^2} + O(a), \quad k = k_A + p \tag{6.25}$$

where the $k_A$ is one of the 16 four-vectors

$$k_A = \frac{\pi_A}{a}, \quad \bmod 2\pi \tag{6.26}$$

with

$$\begin{aligned}
\pi_0 &= (0,0,0,0), \quad \pi_{1234} = (\pi,\pi,\pi,\pi), \\
\pi_1 &= (\pi,0,0,0), \quad \pi_2 = (0,\pi,0,0), \ldots, \quad \pi_4 = (0,0,0,\pi), \\
\pi_{12} &= (\pi,\pi,0,0), \ldots, \quad \pi_{34} = (0,0,\pi,\pi), \\
\pi_{123} &= (\pi,\pi,\pi,0), \ldots, \quad \pi_{234} = (0,\pi,\pi,\pi),
\end{aligned} \tag{6.27}$$

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and

$$\gamma_\mu^{(A)} = \gamma_\mu \cos \pi_{A\mu} = \pm\gamma_\mu. \tag{6.28}$$

Since the $\gamma_\mu^A$ differ only by a sign from the original $\gamma_\mu$, they are equivalent to these by a unitary transformation. This transformation is easy to build up out of products of $\gamma_\rho\gamma_5$, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma_1\gamma_2\gamma_3\gamma_4$ is the Hermitian and unitary matrix which anticommutes with the $\gamma_\mu$: $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$. So let

$$\{S_A\} = \{\mathbb{1}, S_\rho, S_\rho S_\sigma, S_\rho S_\sigma S_\tau, S_1 S_2 S_3 S_4\}, \quad S_\rho = i\gamma_\rho\gamma_5, \tag{6.29}$$

where $\rho \neq \sigma \neq \tau \neq \rho$ and $A \leftrightarrow \pi_A \leftrightarrow S_A$, e.g. $\pi_{23} \leftrightarrow S_{23} = S_2 S_3$. Then

$$\gamma_\mu^{(A)} = S_A^\dagger \gamma_\mu S_A, \tag{6.30}$$

and we have

$$S(k_A + p) = S_A^\dagger \frac{m - i\gamma_\mu p_\mu}{m^2 + p^2} S_A + O(a^2). \tag{6.31}$$

The transformations $S_A$ are useful for the detailed interpretation of the zeros of the sine functions near $k_A \neq 0$ in terms of genuine particles [70]. Here we shall support the interpretation of the 15 additional particles – the species doublers – by deriving the spectrum of excitation energies above the energy of the ground state.

The excitation-energy spectrum is conveniently obtained from the time dependence of the propagator, analogously to the boson case:

$$S(\mathbf{x}, t) = \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{kx}} \int_{-\pi}^{\pi} \frac{dk_4}{2\pi}\, e^{ik_4 t}\, \frac{m - i\gamma\mathbf{s} - i\gamma_4 \sin k_4}{m^2 + \mathbf{s}^2 + \sin^2 k_4}, \tag{6.32}$$

where we reverted to lattice units and used the notation $s_\mu = \sin k_\mu$. The $k_4$ integral can be performed by changing variables to

$$z = e^{ik_4}, \tag{6.33}$$

in terms of which $s_4^2 = 1 - (z^2 + z^{-2} + 2)/4$, and

$$\begin{aligned} S(\mathbf{x}, t) &= -4 \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{kx}} \int \frac{dz}{2\pi i}\, z^t\, \frac{z(m - i\gamma\mathbf{s}) - \gamma_4(z^2 - 1)/2}{z^4 - 2fz^2 + 1}, \\ f &= 1 + 2(m^2 + \mathbf{s}^2). \end{aligned} \tag{6.34}$$

The integral over $z$ is over the unit circle in the complex plane, as shown in figure 6.1. The denominator of the integrand has four zeros, at $\pm z_+$ and $\pm z_-$, where $z_\pm$ are given by

$$(z_\pm)^2 = f \pm \sqrt{f^2 - 1}, \quad z_\pm = e^{\pm\omega}, \tag{6.35}$$

$$\cosh(2\omega) = f, \quad \sinh\omega = \sqrt{m^2 + \mathbf{s}^2}. \tag{6.36}$$

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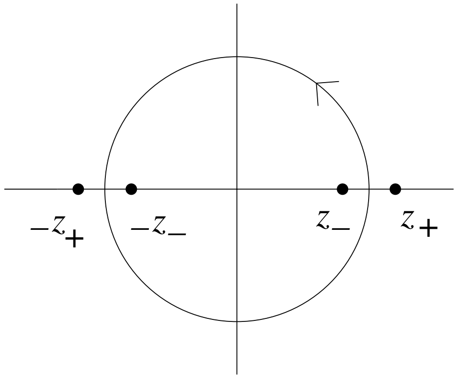


Fig. 6.1. Contour integration in the complex-$z$ plane.

For $t > 0$ ($t =$ integer) the two poles at $z = \pm z_-$ contribute, giving

$$
\begin{aligned}
S(\mathbf{x}, t) = &\int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\mathbf{kx}-\omega t}}{\sinh(2\omega)}\, (m - i\gamma\mathbf{s} + \gamma_4 \sinh\omega) \\
&+ (-1)^t \int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\mathbf{kx}-\omega t}}{\sinh(2\omega)}\, (m - i\gamma\mathbf{s} - \gamma_4 \sinh\omega). \qquad (6.37)
\end{aligned}
$$

Before interpreting this result we want to summarize it in terms of the variable $k_4$, for later use. In terms of $k_4$ the zeros of the denominator $m^2 + \mathbf{s}^2 + \sin^4 k_4$ at $z = z_\pm$ are at $k_4 = \mp i\omega$, and for $z = -z_\pm$ at $k_4 = \mp i\omega + \pi$ (mod $2\pi$). The $k_4 = -i\omega$, $-i\omega + \pi$ poles are relevant for $t < 0$. The residues of the other poles are given by

$$
\begin{aligned}
&e^{ik_4 t}\, (m - i\gamma\mathbf{s} - i\gamma_4 s_4) \\
&\quad = e^{-\omega t}\, (m - i\gamma\mathbf{s} + \gamma_4 \sinh\omega), && k_4 = i\omega, \\
&\quad = (-1)^t e^{-\omega t}\, (m - i\gamma\mathbf{s} - \gamma_4 \sinh\omega), && k_4 = i\omega + \pi, \\
&\quad = e^{\omega t}\, (m - i\gamma\mathbf{s} - \gamma_4 \sinh\omega), && k_4 = -i\omega, \\
&\quad = (-1)^t e^{\omega t}\, (m - i\gamma\mathbf{s} + \gamma_4 \sinh\omega), && k_4 = -i\omega + \pi. \qquad (6.38)
\end{aligned}
$$

We see that we cannot blindly perform the inverse Wick rotation on the lattice $k_4 \to ik^0$ and look for particle poles at $k^0 = \pm\omega$. We have to let $k_4 \to ik^0 + \varphi$, $\varphi \in [0, 2\pi)$: then $k^0 = \pm\omega$ corresponds to $e^{\mp\omega t}\, e^{i\varphi t}$, $t \gtrless 0$. In this case we have have poles at $\varphi = 0$ and $\varphi = \pi$. Recall that the Bose-field denominator $m^2 + 2\sum_\mu (1 - \cos k_\mu)$ gives only a pole for $\varphi = 0$.

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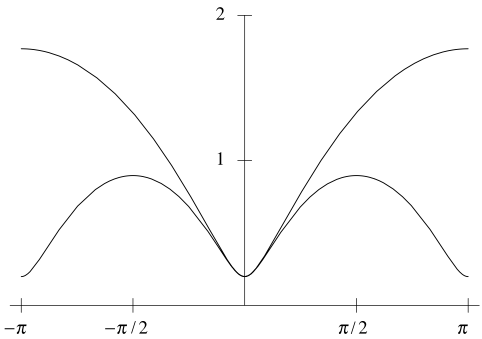


Fig. 6.2. Excitation-energy spectra for bosons (upper curve) and fermions (lower curve) on the lattice, in lattice units ($m = 0.2$).

We now interpret the result (6.37). From the time dependence of the propagator we identify the energy spectrum $\omega(\mathbf{k})$. Since there are two poles contributing for $t > 0$, there must be two fermion particles for every $\mathbf{k}$. One of them (the pole at $z = z_-$) has the usual $e^{-\omega t}$ factor. The other (at $-z_-$) has in addition the rapidly oscillating factor $(-1)^t$. Apparently, to obtain smooth behavior at large times (in lattice units) we have to take two lattice units as our basic time step. This is in accordance with the transfer operator interpretation of the path integral, in which in general *two* adjacent time slices are identified with the fermion Hilbert space [78, 79, 89], in which *two independent* operator Dirac fields $\hat{\psi}_{1,2}$ act, corresponding to the two particle poles. An exception is Wilson's fermion method [86, 87], which has no fermion doubling (for $r = 1$, see below).

So there is a doubling of fermion species due to the discretization of time. There is a further proliferation of particles due to the discretization of space. In figure 6.2 we compare the boson and fermion excitation-energy spectra

$$\cosh\omega = 1 + \tfrac{1}{2}\left[m^2 + 2\sum_{j=1}^{3}(1 - \cos k_j)\right], \quad \text{boson}; \tag{6.39}$$

$$\sinh\omega = \sqrt{m^2 + \sum_{j=1}^{3}\sin^2 k_j}, \quad \text{fermion}. \tag{6.40}$$

We define a particle state to correspond to a local minimum of the energy surface $\omega(\mathbf{k})$. The minima are at $\mathbf{k} = \mathbf{k}_A$, $\mathbf{k}_A = \mathbf{0}$, $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$, $\boldsymbol{\pi}_3$, $\boldsymbol{\pi}_{12}$, $\boldsymbol{\pi}_{23}$, $\boldsymbol{\pi}_{31}$, $\boldsymbol{\pi}_{123}$, with rest energy given by $\omega_A \equiv \omega(\mathbf{k}_A)$, $\sinh \omega_A = m$. For $m \to 0$ (in lattice units) the spectrum is relativistic near $\mathbf{k} = \mathbf{k}_A$,

$$\omega \to \sqrt{m^2 + \mathbf{p}^2}, \quad m \to 0, \quad \mathbf{p} = \mathbf{k} - \mathbf{k}_A \to 0, \tag{6.41}$$

and $\mathbf{p}$ can be interpreted as the momentum of the particle. From the time and space 'doubling' we count $2^4 = 16$ particles. Note that the wave vector $\mathbf{k}$ is just a label to identify the states and that the physical momentum interpretation has to be supplied separately.

One may wish to ignore the $k_A \neq 0$ particles. However, in an interacting theory this is not possible, because $k_\mu$ is conserved only modulo $2\pi$. For example, two $k_A = 0$ particles may collide and produce two $k_A = \pi_1 = (\pi, 0, 0, 0)$ particles: $p_1 + p_2 = p_3 + \pi_1 + p_4 + \pi_1 = p_3 + p_4$ (mod $2\pi$).

The phenomena related to fermions on a lattice touch on deep issues involving anomalies and topology. This is a vast and technically difficult subject and we shall give only a brief review in sections 8.4 and 8.6. In a first exploration we shall describe two important methods used for ameliorating the effects of species doubling in QCD-like theories: Wilson's method [71] and the method of Kogut–Susskind [72, 40] (in the Hamiltonian formulation). The latter is also known as the staggered-fermion method, in its generalization to Euclidean space–time (see for example [79, 74, 80]). For the hypercubic lattice the staggered-fermion method is equivalent to the 'geometrical' or Dirac–Kähler fermion method of Becher and Joos [81], provided that an appropriate choice is made of the couplings to the gauge fields.

We shall first describe Wilson's method and then briefly introduce the staggered-fermion method.

## 6.3 Wilson's fermion method

Wilson's method can be viewed as adding a momentum-dependent 'mass term' to the fermion action, which raises the masses of the unwanted doublers to values of the order of the cutoff, thereby decoupling them from continuum physics. For free fermions we replace the mass term in

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the action as follows,

$$\begin{aligned} m\sum_x \bar{\psi}_x\psi_x \to m\sum_x \bar{\psi}_x\psi_x + \frac{ar}{2}\sum_{x\mu} \partial_\mu\bar{\psi}_x\partial_\mu\psi_x \\ = m\sum_x \bar{\psi}_x\psi_x + \frac{ar}{2}\sum_{x\mu} \frac{1}{a^2}(\bar{\psi}_{x+a\hat{\mu}} - \bar{\psi}_x)(\psi_{x+a\hat{\mu}} - \psi_x) \\ = \left(m + \frac{4r}{a}\right)\sum_x \bar{\psi}_x\psi_x - \frac{r}{2a}\sum_{x\mu}(\bar{\psi}_{x+a\hat{\mu}}\psi_x + \bar{\psi}_x\psi_{x+a\hat{\mu}}). \end{aligned} \tag{6.42}$$

The has the effect of replacing the mass $m$ in the inverse propagator in momentum space by

$$m + r\sum_\mu(1 - \cos k_\mu) \equiv \mathcal{M}(k), \tag{6.43}$$

in lattice units. The propagator is then given by

$$S(k) = \frac{\mathcal{M}(k) - i\gamma_\mu \sin k_\mu}{\mathcal{M}^2(k) + \sum_\mu \sin^2 k_\mu}. \tag{6.44}$$

For $k = k_A + p$ and small $p$ in lattice units this takes the form

$$S(p) = \frac{m_A - i\gamma_\mu^{(A)} p_\mu}{m_A^2 + p^2}, \tag{6.45}$$

$$m_A = m + 2n_A r, \quad n_A = 0, 1, \ldots, 4, \tag{6.46}$$

where $n_A$ is the number of $\pi$'s in $k_A$.

Hence, the mass parameters of the doubler ($n_A > 0$) fermions are of order one in lattice units as long as $r \neq 0$. These mass parameters $m_A$ may be identified with the fermion masses if they are small in lattice units, i.e. for small $m$ and $r$. For general $r$ and momenta $\mathbf{p}$ the fermion energies differ from $\sqrt{m_A^2 + \mathbf{p}^2}$ and it is interesting to see what they actually are. We therefore look for the poles of the propagator as a function of $k_4$ and identify the energy $\omega$ from $k_4 = i\omega$ or $k_4 = i\omega + \pi$, as explained below (6.37). For simplicity we shall use the notation

$$s_\mu = \sin k_\mu, \quad c_\mu = \cos k_\mu, \quad s^2 = s_\mu s_\mu. \tag{6.47}$$

Separating the $k_4$ dependence, the denominator of the propagator can be written as

$$\mathcal{M}^2 + s^2 = 1 + \mathbf{s}^2 + \Sigma^2 - 2r\Sigma c_4 - (1 - r^2)c_4^2, \tag{6.48}$$

$$\Sigma = m + r + r\sum_{j=1}^{3}(1 - c_j), \tag{6.49}$$

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which denominator vanishes for

$$\cosh\omega^{\pm} = \frac{\sqrt{\Sigma^2 + (1-r^2)(1+\mathbf{s}^2)} \pm r\Sigma}{1-r^2}. \tag{6.50}$$

Here the plus sign corresponds to $k_4 = i\omega + \pi$ and the minus sign to $k_4 = i\omega$. The rest energies of the particles at $\mathbf{k} = \mathbf{k}_A$ follow from $\mathbf{s} = 0$ and

$$\Sigma = m + r + 2nr, \quad n = 0, 1, 2, 3 \text{ for } \mathbf{k}_A = 0,\, \pi_j,\, \pi_{jk},\, \pi_{123}. \tag{6.51}$$

For $m = 0$ the particles have rest energy $\omega_n$ given by

$$\cosh\omega_n^{\pm} = \frac{\sqrt{r^2(1+2n)^2 + 1 - r^2} \pm r^2(1+2n)}{1-r^2}. \tag{6.52}$$

Hence, only the wanted ($n = 0$, $-$ sign) fermion has rest energy zero and the doubler fermions have rest energies of order 1 in lattice units (energies of order of the cutoff). For $r \to 1$ the rest energies of the time-doublers (for which the $+$ sign applies) become infinite, $\omega^+ \to \infty$. The non-time-doubler rest energies become $\omega_n^- = \ln(1+2n)$ at $r = 1$. Actually, as $r$ increases from 0 to 1 the doublers disappear before reaching $r = 1$ in the sense that the local minima of the energy surface at $k_A \neq 0$ disappear.

Wilson's choice is $r = 1$. It can be seen directly from (6.48) that in this case there is no species doubling because the inverse propagator is linear in $\cos k_4$. Re-installing the lattice spacing $a$, the particle energy can be found to contain errors of order $a$, to be compared with $O(a^2)$ for naive/staggered fermions or bosons,

$$\omega = \omega_0^- = \sqrt{m^2 + \mathbf{p}^2} + O(a). \tag{6.53}$$

The special significance of $r = 1$ can be seen in another way from the complete action, which has the form

$$S = \sum_{x\mu}\left(\bar\psi_x \frac{r-\gamma_\mu}{2}\psi_{x+\hat\mu} + \bar\psi_{x+\hat\mu}\frac{r+\gamma_\mu}{2}\psi_x\right) - M\sum_x \bar\psi_x\psi_x,$$
$$M = m + 4r. \tag{6.54}$$

The combinations

$$P_\mu^{\pm} = \frac{r \pm \gamma_\mu}{2}, \tag{6.55}$$

become orthogonal projectors for $r = 1$,

$$(P_\mu^{\pm})^2 = P_\mu^{\pm}, \quad P_\mu^+ P_\mu^- = 0, \quad P_\mu^+ + P_\mu^- = 1. \tag{6.56}$$

Replacing derivatives by covariant derivatives we obtain the expression for the fermion action coupled to a lattice gauge field $U_{\mu x}$,

$$S_{\mathrm{F}} = \sum_{x\mu} (\bar{\psi}_x P_\mu^- U_{\mu x} \psi_{x+\hat{\mu}} + \bar{\psi}_{x+\hat{\mu}} P_\mu^+ U_{\mu x}^\dagger \psi_x) - \sum_x \bar{\psi}_x M \psi_x, \tag{6.57}$$

or, temporarily reintroducing the lattice spacing $a$,

$$S_{\mathrm{F}} = -\sum_x \left[ \frac{1}{2} (\bar{\psi}\gamma_\mu D_\mu \psi - D_\mu \bar{\psi} \gamma_\mu \psi) + \bar{\psi} m \psi + a \frac{r}{2} D_\mu \bar{\psi} D_\mu \psi \right], \tag{6.58}$$

where $D_\mu \bar{\psi}(x) = [\bar{\psi}(x + \hat{\mu}a) U_{\mu x}^\dagger - \bar{\psi}(x)]/a$, etc., we rearranged the summation over $x$, and $m = M - 4r/a$ is sometimes called the bare fermion mass.

In the QCD case $M$ is a diagonal matrix in flavor space and $r$ is usually chosen flavor-independent, mostly $r = 1$. A parameterization introduced by Wilson follows from rescaling $\psi \to M^{-1/2}\psi$, $\bar{\psi} \to \bar{\psi} M^{-1/2}$. For one flavor this gives the form

$$S = -\sum_x \bar{\psi}_x \psi_x + \kappa \sum_{x\mu} [\bar{\psi}_x (r - \gamma_\mu) U_{\mu x} \psi_{x+\hat{\mu}} + \bar{\psi}_{x+\hat{\mu}} (r + \gamma_\mu) U_{\mu x}^\dagger \psi_x], \tag{6.59}$$

where

$$\kappa = \frac{1}{2M}, \tag{6.60}$$

is Wilson's hopping parameter (it is flavor dependent). This $\kappa$ is analogous to the hopping parameter in the scalar field models. We may interpret $-\sum_x \bar{\psi}_x \psi_x$ as belonging to the integration measure in the path integral.

For free fermions the continuum limit means $m \to 0$ in lattice units, which implies a critical value for the hopping parameter

$$\kappa \to \kappa_{\mathrm{c}} = 1/8r, \quad M \to M_{\mathrm{c}} = 4r. \tag{6.61}$$

At this critical value there is somehow a cancellation of the $\bar{\psi}\psi$-like terms, such that the fermions acquire zero mass. With the gauge field present the effective strength of the hopping term is reduced by the 'fluctuating' unitary $U_{\mu x}$. We then expect $M_{\mathrm{c}} < 4$ and $\kappa_{\mathrm{c}} > 1/8r$, for given gauge coupling $g$. However, in the QCD case we know already that $g$ itself should go to zero in the continuum limit, because of asymptotic freedom, implying $U_{\mu x} \to 1$ in a suitable gauge and (6.61) should still be valid ($\kappa_{\mathrm{c}}$ is of course gauge independent). However, at gauge coupling of order one we can be deep in the scaling region of QCD and we may expect an effective $\kappa_{\mathrm{c}}$ substantially larger than $1/8r$. Since there are no free quarks in QCD we cannot define $\kappa_c$ as the value at which the quark mass vanishes. We shall see later that it may be defined as the value at which the pion mass vanishes.

## 6.4 Staggered fermions

Starting with the naive fermion action we make the unitary transformation of variables (in lattice units)

$$\psi_x = \gamma^x \chi_x, \quad \bar{\psi}_x = \bar{\chi}_x (\gamma^x)^\dagger, \tag{6.62}$$

$$\gamma^x \equiv (\gamma_1)^{x_1} (\gamma_2)^{x_2} (\gamma_3)^{x_3} (\gamma_4)^{x_4}. \tag{6.63}$$

Because

$$\left[(\gamma^x)^\dagger \gamma_\mu \gamma^{x+\hat{\mu}}\right]_{\alpha\beta} = \left[\gamma^x \gamma_\mu (\gamma^{x+\hat{\mu}})^\dagger\right]_{\alpha\beta} = \eta_{\mu\, x}\, \delta_{\alpha\beta}, \tag{6.64}$$

where

$$\eta_{1\,x} = 1, \;\; \eta_{2\,x} = (-1)^{x_1}, \;\; \eta_{3\,x} = (-1)^{x_1+x_2}, \;\; \eta_{4\,x} = (-1)^{x_1+x_2+x_3}, \tag{6.65}$$

this transformation has the effect of removing the gamma matrices from the naive fermion action, which acquires the form

$$S = -\sum_{\alpha=1}^{4} \left[ \sum_{x\mu} \eta_{\mu\, x} \frac{1}{2} (\bar{\chi}_x^\alpha \chi_{x+\hat{\mu}}^\alpha - \bar{\chi}_{x+\hat{\mu}}^\alpha \chi_x^\alpha) + m \sum_x \bar{\chi}_x^\alpha \chi_x^\alpha \right]. \tag{6.66}$$

In this representation the Dirac spinor labels $\alpha$ on $\bar{\chi}$ and $\chi$ are like internal symmetry labels and the action is just a sum of four identical terms, one for each value of the Dirac index. Hence, one of these should suffice in describing fermion particles. It can indeed be shown that taking $\chi$ and $\bar{\chi}$ as one-component fields leads to $16/4 = 4$ Dirac particles in the continuum limit. In QCD all these fermions are interpreted as quark flavors. Inserting the 'parallel transporters' $U_{\mu x}$ then leads to a gauge-invariant staggered-fermion action

$$S_{\rm F} = -\sum_{x\mu} \eta_{\mu x} \frac{1}{2} \left[\bar{\chi}_{ax} (U_{\mu x})_{ab} \chi_{bx+\hat{\mu}} - \bar{\chi}_{ax+\mu} (U_{\mu x}^\dagger)_{ab} \chi_{bx}\right] - \sum_x m \bar{\chi}_{ax} \chi_{ax}, \tag{6.67}$$

where we have made all indices on $\chi$ and $\bar{\chi}$ explicit ($a$ and $b$ are color indices) – there are e.g. no spin or flavor indices for $\overline{\chi}$ and $\bar{\chi}$. Analysis in weak-coupling perturbation theory leads to the conclusion that this action describes QCD with four mass-degenerate flavors in the continuum limit [73, 74] (the mass degeneracy of the quarks can be lifted by adding

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other terms to the action). The action has an interesting symmetry group [76], which is important for the construction of composite fields with the quantum numbers of hadrons [75, 77]. In the scaling region this symmetry group enlarges to the group in the continuum (including 'anomalies').

A further reduction by a factor of two is possible by assigning $\bar{\chi}_x$ only to the even sites and $\chi_x$ only to the odd sites [78, 79, 80]. Even and odd sites are defined by $\epsilon_x = 1$ and $-1$, respectively, with

$$\epsilon_x = (-1)^{x_1+x_2+x_3+x_4}. \tag{6.68}$$

In this formulation we may as well omit the bar on $\bar{\chi}_x$ since no confusion between even and odd sites is possible. Then a minimal action with only one Grassmann variable per site is given by

$$S = -\sum_{x\mu} \eta_{\mu\, x} \tfrac{1}{2} \chi_x \chi_{x+\hat{\mu}}, \tag{6.69}$$

in case of zero fermion mass. This method leads essentially to four Majorana fermions, which are equivalent to two Dirac fermions or eight Weyl fermions. Non-zero mass requires one-link or multilink couplings, since $\chi_x^2 = 0$.

Staggered fermions are technically rather specialized and we shall not emphasize them in this book. For an application of the method (6.69) to numerical simulations of the Higgs–Yukawa sector of the Standard Model see [36].

## 6.5 Transfer operator for Wilson fermions

It will now be shown that the fermion partition function with Wilson fermions can for $r = 1$ be written in the form

$$Z = \mathrm{Tr}\, \hat{T}^N, \tag{6.70}$$

where $\hat{T}$ is a positive transfer operator in Hilbert space and $N$ is the number of time slices. A transfer operator was first given by Wilson [86] and a study of its properties was presented in [87]. The construction below is slightly different. (A general construction for $r \neq 1$ is sketched in [89], which is easily adapted to naive or staggered fermions. See also [79, 88, 90] and references therein.) To identify $\hat{T}$ we first assume that the gauge field is external and and write $\mathrm{Tr}\, \hat{T}^N$ in the Grassmann representation,

$$\mathrm{Tr}\,\hat{T}^N = \int da_1^+\, da_1 \cdots da_N^+\, da_N\, e^{-a_N^+ a_N}\, T(a_N^+, a_{N-1})\, e^{-a_{N-1}^+ a_{N-1}} \times T(a_{N-1}^+, a_{N-2}) \cdots T(a_{k+1}^+, a_k)\, e^{-a_k^+ a_k}\, T(a_k^+, a_{k-1}) \cdots \times e^{-a_2^+ a_2}\, T(a_2^+, a_1)\, e^{-a_1^+ a_1}\, T(a_1^+, -a_N). \tag{6.71}$$

The minus sign in the last factor corresponds to the same sign in (C.68) in appendix C. It implies that there are antiperiodic boundary conditions in the path integral, i.e. there is a change of sign in the couplings in the action between time slices 0 and $N-1$. The expression above is to be compared with

$$Z_F = \int D\bar{\psi} D\psi\, \exp S_{\mathrm{F}}, \tag{6.72}$$

where $S_{\mathrm{F}}$ is the fermion part of the action. We have seen in the pure-gauge case that the integration over the timelike links $U_{4\mathbf{x}}$ leads to the projector on the gauge-invariant subspace of Hilbert space, together with a transfer operator in the temporal 'gauge' $U_{4x} = 1$. We therefore set $U_{4x} = 1$ and write the fermion action in the form (using lattice units, $t \equiv x_4$ and $\mathbf{x}$ are integers)

$$S_{\mathrm{F}} = \sum_t \left( -\psi_t^+ \frac{1-\beta}{2}\, \psi_{t+1} + \psi_{t+1}^+ \frac{1+\beta}{2}\, \psi_t \right) - \sum_t \psi_t^+ \beta A_t \psi_t - \epsilon \sum_t \psi_t^+ D_t \psi_t. \tag{6.73}$$

Here a matrix notation is used with

$$A_{\mathbf{xy},t} = M\delta_{\mathbf{x},\mathbf{y}} - \epsilon \sum_{j=1}^{3} \frac{1}{2} (U_{\mathbf{xy},t}\, \delta_{\mathbf{x}+\hat{\jmath},\mathbf{y}} + \mathbf{x} \leftrightarrow \mathbf{y}), \tag{6.74}$$

$$D_{\mathbf{xy},t} = \sum_{j=1}^{3} \alpha_j\, \frac{1}{2i}\, (U_{\mathbf{xy},t}\, \delta_{\mathbf{x}+\hat{\jmath},\mathbf{y}} - \mathbf{x} \leftrightarrow \mathbf{y}), \tag{6.75}$$

and $\alpha_j = i\gamma_4\gamma_j$ and $\beta = \gamma_4$ are Dirac's matrices, and furthermore

$$\epsilon = a_4/a. \tag{6.76}$$

We recognize the projectors

$$P^{\pm} \equiv P_4^{\pm} = (1 \pm \beta)/2 \tag{6.77}$$

for Wilson parameter $r = 1$. They reduce the number of $\psi_{t\pm1}^+ \psi_t$ couplings by a factor of two compared with the naive fermion action.

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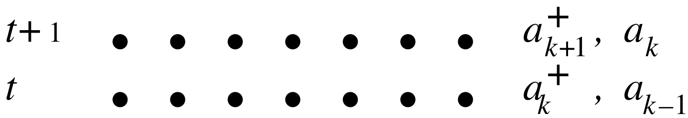


Fig. 6.3. The association of time slices with Hilbert space for Wilson fermions ($r = 1$).

The association of time slices $t$ with Hilbert-space slices $k$ is as follows (for each $\mathbf{x}$, T denotes transposition):

$$P^+\psi_t = (a_k^+ P^+)^{\mathrm{T}}, \quad \psi_t^+ P^- = a_k^+ P^-, \tag{6.78}$$

$$P^-\psi_{t+1} = P^- a_k, \quad \psi_{t+1}^+ P^+ = (P^+ a_k)^{\mathrm{T}}, \tag{6.79}$$

$$A_t = A_k, \quad D_t = D_k \tag{6.80}$$

as illustrated in figure 6.3. With this notation the action can be written as

$$\begin{aligned} S_{\mathrm{F}} = &- \sum_k a_k^+ a_k + \sum_k a_k^+ A_k a_{k-1} \\ &- \epsilon \sum_k (a_{k-1} P^+ D_k P^- a_{k-1} + a_k^+ P^- D_k P^+ a_k^+). \end{aligned} \tag{6.81}$$

Here we have used $\beta D = -D\beta$, such that

$$D = (P^+ + P^-) D (P^+ + P^-) = P^+ D P^- + P^- D P^+, \tag{6.82}$$

and abused the notation by leaving out the transposition symbol T. Comparison with $\mathrm{Tr}\, \hat{T}^N$ in the form (6.71) gives the (Grassmannian) transfer-matrix elements

$$\begin{aligned} T_{\mathrm{F}}(a_k^+, a_{k-1}) = &\exp(-\epsilon a_k^+ P^- D_k P^+ a_k^+) \ \exp(a_k^+ A_k a_{k-1}) \\ &\times \exp(-\epsilon a_{k-1} P^+ D_k P^- a_{k-1}). \end{aligned} \tag{6.83}$$

Using the rules listed above (C.68) in appendix C this translates into operator form as

$$\hat{T}_{\mathrm{F}} = e^{-\epsilon \hat{a}^\dagger P^- D P^+ \hat{a}^\dagger} \, e^{\hat{a}^\dagger \ln(A) \hat{a}} \, e^{-\epsilon \hat{a} P^+ D P^- \hat{a}}. \tag{6.84}$$

Here $D$ and $A$ depend in general on the gauge-field configuration in a time slice.

Consider now first the case of free fermions, $U_{\mathbf{xy}} = 1$. Then $\hat{T}$ is clearly a positive operator provided that $A$ is positive, i.e. a Hermitian

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matrix with only positive eigenvalues. In momentum space we get the eigenvalues

$$A(\mathbf{p}) = M - \epsilon \sum_{j=1}^{3} \cos p_j, \tag{6.85}$$

which shows that $A > 0$ for

$$M > 3\epsilon. \tag{6.86}$$

With dynamical gauge fields we have to take into account in (6.84) the transfer operator for the gauge field $\hat{T}_U$. The complete transfer operator can be taken as

$$\hat{T} = \hat{T}_{\mathrm{F}}^{1/2}\, \hat{T}_U\, \hat{T}_{\mathrm{F}}^{1/2}\, \hat{P}_0, \tag{6.87}$$

$$\hat{T}_{\mathrm{F}} = e^{-\epsilon \hat{a}^\dagger P^- \hat{D} P^+ \hat{a}^\dagger}\, e^{\hat{a}^\dagger \ln \hat{A}\, \hat{a}}\, e^{-\epsilon \hat{a} P^+ \hat{D} P^- \hat{a}} \tag{6.88}$$

where we have also put in the projector $P_0$ on the gauge-invariant subspace. Since $A$ has lowest eigenvalues when the link variables are unity, the condition (6.86) remains sufficient in general for positivity of $\hat{T}_{\mathrm{F}}$.

We can now use (6.80) in reverse and define operator fields $\hat{\psi}$ and $\hat{\psi}^\dagger$, for each spatial site $\mathbf{x}$, by

$$P^+ \hat{\psi} = (\hat{a}^\dagger P^+)^{\mathrm{T}}, \quad \hat{\psi}^\dagger P^- = \hat{a}^\dagger P^-, \tag{6.89}$$

$$P^- \hat{\psi} = P^- \hat{a}, \quad \hat{\psi}^\dagger = (P^+ \hat{a})^{\mathrm{T}}. \tag{6.90}$$

In terms of these fields the fermion transfer operator takes the explicitly charge-conserving form

$$\hat{T}_{\mathrm{F}} = e^{-\epsilon \hat{\psi}^\dagger P^- \hat{D} P^+ \hat{\psi}}\, e^{-\hat{\psi}^\dagger \beta \ln \hat{A}\, \hat{\psi}}\, e^{\mathrm{Tr}\, P^+ \ln \hat{A}}\, e^{-\epsilon \hat{\psi}^\dagger P^+ D P^- \hat{\psi}}. \tag{6.91}$$

Notice the Dirac-sea factor $\exp(\mathrm{Tr}\, P^+ \ln A)$.

The continuous time limit $\hat{T} = 1 - \epsilon \hat{H} + O(\epsilon^2)$ can be taken if we let $M$ depend on $\epsilon \to 0$ according to

$$M = 1 + \epsilon M_3, \tag{6.92}$$

such that $A$ takes the form

$$A(U) = 1 + \epsilon \mathcal{M}_3(U) \tag{6.93}$$

$$\mathcal{M}_3 = M_3 - \sum_{j=1}^{3} \tfrac{1}{2} (U_{\mathbf{x},\mathbf{y}}\, \delta_{\mathbf{x}+\hat{j},\mathbf{y}} + \mathbf{x} \leftrightarrow \mathbf{y}), \tag{6.94}$$

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and we get the fermion Hamiltonian

$$\hat{H}_{\rm F} = \hat{\psi}^{\dagger}[\mathcal{M}_3(\hat{U})\beta + D(\hat{U})]\hat{\psi}. \tag{6.95}$$

This may be called a Wilson–Dirac Hamiltonian on a spatial lattice.

In summary, we conclude that the Euclidean-lattice formulation of QCD using Wilson's fermion method has a good Hilbert-space interpretation, with a positive transfer operator.

## 6.6 Problems

The following exercises serve to clarify the continuum limit in QED and the phenomenon of species doubling by calculation of the photon self-energy at one loop in the weak-coupling expansion.

(i) *Vertex functions*

Consider the naive fermion action in QED

$$S_{\rm F} = -\sum_{x\mu} \tfrac{1}{2}(\bar{\psi}_x \gamma_\mu e^{-igA_{\mu x}}\psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}}\gamma_\mu e^{igA_{\mu x}}\psi_x). \tag{6.96}$$

The bare fermion–photon vertex functions are the derivatives of the action with respect to the fields. Taking out a minus sign and the momentum-conserving delta functions, let the vertex function $V_{\mu_1\cdots\mu_n}(p,q;k_1\cdots k_n)$ be defined by

$$S_{\rm F} = -\sum_{uvx_1\cdots x_n} \frac{1}{n!}\,\bar{\psi}_u V_{\mu_1\cdots\mu_n}(u,v;x_1\cdots x_n)\psi_v A_{\mu_1 x_1}\cdots A_{\mu_n x_n}, \tag{6.97}$$

$$\begin{aligned}\sum_{uvx_1\cdots x_n} & e^{-ipu+iqv-ik_1x_1\cdots-ik_nx_n}\,V_{\mu_1\cdots\mu_n}(u,v;x_1\cdots x_n) \\ &= V_{\mu_1\cdots\mu_n}(p,q;k_1\cdots k_n)\,\bar{\delta}(p-q+k_1+\cdots k_n).\end{aligned} \tag{6.98}$$

Show that $(p-q+k_1+\cdots k_n=0)$

$$\begin{aligned}V_{\mu_1\cdots\mu_n}(p,q;k_1\cdots k_n) = \sum_\mu \gamma_\mu \tfrac{1}{2}[(-ig)^n e^{iq_\mu} - (ig)^n e^{-ip_\mu}] \\ \times \delta_{\mu\mu_1}\cdots\delta_{\mu\mu_n},\end{aligned} \tag{6.99}$$

as illustrated in figure 6.4. The fermion propagator is given by

$$S(p)^{-1} = V(p,p), \quad S(p) = \frac{m - i\gamma_\mu \sin p_\mu}{m^2 + \sin^2 p}. \tag{6.100}$$

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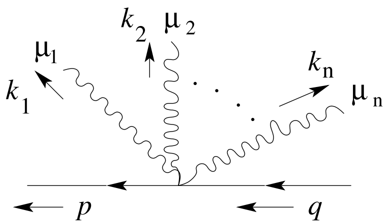


Fig. 6.4. Fermion vertex function $-V_{\mu_1\cdots\mu_n}(p,q;k_1,\ldots,k_n)$.

(ii) *Ward–Takahashi identities*

The gauge invariance of $S_{\rm F}$ implies certain properties of the vertex functions, called Ward–Takahashi identities. Consider a small gauge transformation $\psi'_x = (1+i\omega_x+O(\omega^2))\psi_x$, $\bar\psi'_x = (1-i\omega_x+O(\omega^2))\bar\psi_x$, $A'_{\mu x} = A_{\mu x}+(1/g)\partial_\mu\omega_x$ (recall the definition of the forward and backward lattice derivatives, $\partial_\mu\omega_x = \omega_{x+\hat\mu}-\omega_x$ and $\partial'_\mu\omega_x = \omega_x-\omega_{x-\hat\mu}$). Collect the linear terms in $\omega_x$ in the invariance relation $0 = S_{\rm F}(\psi',\bar\psi',A') - S_{\rm F}(\psi,\bar\psi,A)$ and derive the Ward identities

$$\begin{aligned}0 = \frac{1}{g}\, &i\partial'_\mu V_{\nu\mu_1\cdots\mu_n}(u,v;x,x_1,\ldots,x_n)\\ &+\delta_{ux}V_{\mu_1\cdots\mu_n}(u,v;x_1,\ldots,x_n)\\ &-\delta_{vx}V_{\mu_1\cdots\mu_n}(u,v;x_1,\ldots,x_n),\end{aligned}\tag{6.101}$$

and the momentum-space version

$$\begin{aligned}0 = \frac{1}{g}\, &K^*_\mu(k)V_{\mu\mu_1\cdots\mu_n}(p,q;k,k_1,\ldots,k_n)\\ &+V_{\mu_1\cdots\mu_n}(p+k,q;k_1,\ldots,k_n)\\ &-V_{\mu_1\cdots\mu_n}(p,q-k;k_1,\ldots,k_n),\end{aligned}\tag{6.102}$$

where $K_\mu(k) = (e^{ik_\mu}-1)/i$. In particular, for $n=0$ and 1,

$$K^*_\mu(k)V_\mu(p,q;k) = S(p)^{-1}-S(q)^{-1},\tag{6.103}$$
$$K^*_\mu(k)V_{\mu\nu}(p,q;k,l) = V_\mu(p,p+l;l)-V_\nu(q-l,q;l)$$

$(p-q+k=0,\; p-q+k+l=0)$.

(iii) *Photon self-energy*

We study the 'vacuum-polarization' diagrams in figure 6.5, which describe the photon self-energy vertex function $\Pi = \Pi^{(a)}+\Pi^{(b)}$

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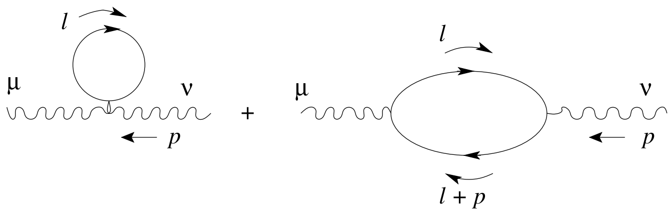


Fig. 6.5. Vacuum-polarization diagrams for $-\Pi_{\mu\nu}(p)$.

given by

$$\Pi^{(a)}_{\mu\nu}(p) = -g^2 \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \, \mathrm{Tr}\,[V_{\mu\nu}(l, -l, p, -p) S(l)], \tag{6.104}$$

$$\Pi^{(b)}_{\mu\nu}(p) = g^2 \int_{-\pi}^{\pi} \frac{d^4 l}{(2\pi)^4} \, \mathrm{Tr}\,[V_\mu(l, l+p) S(l+p) V_\nu(l+p, l) S(l)].$$

Use the identities (6.103) to show that the sum $\Pi_{\mu\nu} = \Pi^{(a)}_{\mu\nu} + \Pi^{(b)}_{\mu\nu}$ satisfies the Ward identity

$$K^*_\mu(p)\Pi_{\mu\nu}(p) = 0. \tag{6.105}$$

(Note that the loop integrals in the lattice regularization are invariant under translation of the integration variable.)

(iv) *Continuum region and lattice-artifact region*

The calculation of the continuum limit of $\Pi_{\mu\nu}(p)$ can be done in the same way as for the scalar field in section 3.4 We split the integration region into a ball of radius $\delta$ around the origin $l = 0$ and the rest, where $\delta$ is so small that we may use the continuum form of the propagators and vertex functions.

Going over to physical units, $p \to ap$, $m \to am$, $\Pi \to \Pi a^{-2}$, show that in the scaling region limit $a \to 0$, $\delta \to 0$, $am/\delta \to 0$, $ap/\delta \to 0$ the contribution of this ball can be written as

$$-\frac{g^2}{2\pi^2}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) \int_0^1 dx\, x(1-x) \ln[a^2(m^2 + x(1-x)p^2)] \tag{6.106}$$

up to a second-degree polynomial in $p$.

Verify that the 15 fermion doublers in similar balls around non-zero $l = \pi_A$ give identical contributions, up to possible arbitrariness in the polynomials.

The region outside the 16 balls can contribute only a second-degree polynomial $T_{\mu\nu}(p)$ in $a^{-1}$, $m$ and $p$ in the continuum

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limit, because possible infrared divergences cannot develop in the outside regions.

Note that $\Pi^{(a)}_{\mu\nu}$ contributes only to the polynomial part of $\Pi_{\mu\nu}$ in the continuum limit (it is just a constant $\propto \delta_{\mu\nu}$). The reason is that there is no logarithmic contribution from the balls around $l = \pi_A$ because the vertex function $V_{\mu\nu}$ vanishes in the classical continuum limit.

(v) *Lattice symmetries*
The polynomial has to comply with the symmetries of the model, in particular cubic rotations $R^{(\rho\sigma)}$ in a plane $(\rho, \sigma)$,

$$\begin{aligned}
&(R^{(\rho\sigma)}p)_\mu \equiv R^{(\rho\sigma)}_{\mu\nu} p_\nu, \\
&(R^{(\rho\sigma)}p)_\rho = p_\sigma, \quad (R^{(\rho\sigma)}p)_\sigma = -p_\rho, \\
&(R^{(\rho\sigma)}p)_\mu = p_\mu, \quad \mu \neq \{\rho, \sigma\}
\end{aligned} \tag{6.107}$$

and inversions $I^{(\rho)}$,

$$(I^{(\rho)}p)_\rho = -p_\rho, \quad (I^{(\rho)}p)_\mu = p_\mu, \quad \mu \neq \rho. \tag{6.108}$$

The polynomial $T_{\mu\nu}(p)$ has to be a tensor under these transformations,

$$T_{\mu\nu}(R^{(\rho\sigma)}p) = R^{(\rho\sigma)}_{\mu\mu'} R^{(\rho\sigma)}_{\nu\nu'} T_{\mu'\nu'}(p), \tag{6.109}$$

$$T_{\mu\nu}(I^{(\rho)}p) = I^{(\rho)}_{\mu\mu'} I^{(\rho)}_{\nu\nu'} T_{\mu'\nu'}(p). \tag{6.110}$$

Show using the lattice symmetries that the form of the polynomial is limited to

$$c_1 a^{-2}\delta_{\mu\nu} + c_2 m^2 + c_3 p_\mu^2 \delta_{\mu\nu} + c_4 p_\mu p_\nu + c_5 p^2 \delta_{\mu\nu}. \tag{6.111}$$

(In the third term there is of course no summation over $\mu$.)

(vi) *Constraints from the Ward identity*
Use the continuum limit of the Ward identity (6.105) to show finally that $\Pi_{\mu\nu}(p)$ has the continuum covariant form,

$$\begin{aligned}
\Pi_{\mu\nu}(p) = {} & -16\frac{g^2}{2\pi^2}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) \\
& \times \int_0^1 dx\, x(1-x)\ln[a^2(m^2 + x(1-x)p^2)] \\
& + c(p^2\delta_{\mu\nu} - p_\mu p_\nu).
\end{aligned} \tag{6.112}$$

Note that the coefficient $c_1$ of the quadratic divergence is zero. This can of course also be verified by an explicit calculation, e.g.

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for $p = 0$. In non-Abelian gauge theory such quadratic divergences are also absent, provided that the contribution from the integration (Haar) measure in the path integral is not forgotten. Note also that the coefficient $c_3$, of the term that is lattice covariant but not covariant under continuous rotations, is zero. Such cancellations will not happens in models in which vector fields are not gauge fields (no Ward identities). Then counterterms are needed in order to ensure covariance.

The numerical constant $c$ can be obtained by a further careful analysis and numerical integration. It determines e.g. the ratio of lambda scales $\Lambda_{\overline{\mathrm{MS}}}/\Lambda_{\mathrm{L}}$ in the theory with (naive) dynamical fermions. On dividing by a factor of four we get the analogous result for four-flavor staggered fermions described by the $U(1)$ version of the action (6.67).

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