On the use of the Hyperbolic Sine and Cosine in connection with the Hyperbola.

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The excentric angle notation in the ellipse is extremely useful, and in part we can replace it by the hyperbolic sine and cosine in connection with the hyperbola.

Take the hyperbola $x^2/a^2 - y^2/b^2 = 1$, then the coordinates of any point on it may be written $a\cosh\phi$, $b\sinh\phi$, for $\cosh^2\phi - \sinh^2\phi = 1$. The objection to its use in all cases is that the hyperbolic cosine of an angle is always positive, so that $(a\cosh\phi, b\sinh\phi)$ can only represent any point on the branch on the positive side of the axis of y, for any point on the other branch we must take its coordinates as $(-a\cosh\phi, b\sinh\phi)$.

FIGURE 28.

Take up the discussion of conjugate diameters in this notation.

Take a series of parallel chords, joining, first, points on the same branch, and let QQ′ be one of them, to find the locus of their middle points.

The line joining the points $(x_1y_1)(x_2y_2)$ on the hyperbola has the equation

$$\frac{x(x_1+x_2)}{a^2} - \frac{y(y_1+y_2)}{b^2} = 1 + \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}$$

∴ if we take Q as the point $(a\cosh\alpha, b\sinh\alpha)$, Q′ as $(a\cosh\beta, b\sinh\beta)$ the equation of QQ′ is

$$\frac{x}{a}(\cosh\alpha + \cosh\beta) - \frac{y}{b}(\sinh\alpha + \sinh\beta) = 1 + \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

i.e. $$2\frac{x}{a}\cosh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} - 2\frac{y}{b}\sinh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} = 1 + \cosh(\alpha-\beta)$$
$$= 2\cosh^2\frac{\alpha-\beta}{2}$$

i.e. $$\frac{x}{a}\cosh\frac{\alpha+\beta}{2} - \frac{y}{b}\sinh\frac{\alpha+\beta}{2} = \cosh\frac{\alpha-\beta}{2}\,.$$

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If then we are to have a series of parallel chords we must join points, parameters α and β, such that

$$\frac{b}{a}\coth\frac{\alpha+\beta}{2} \text{ is constant.}$$

i.e. such that $\alpha+\beta$ is constant.

Join then points, parameters $\lambda+\mu$ and $\lambda-\mu$, keeping λ constant and varying μ, then we shall get a series of parallel chords, the gradient of which is

$$\frac{b}{a}\coth\frac{\lambda+\mu+\lambda-\mu}{2} \qquad i.e. \ \frac{b}{a}\coth\lambda.$$

Now the middle point of the line joining the points,

$$(a\cosh\overline{\lambda+\mu},\ b\sinh\overline{\lambda+\mu}) \text{ and } (a\cosh\overline{\lambda-\mu},\ b\sinh\overline{\lambda-\mu})$$

$$\text{is } \{\tfrac{1}{2}a(\cosh\overline{\lambda+\mu}+\cosh\overline{\lambda-\mu}),\ \tfrac{1}{2}b(\sinh\overline{\lambda+\mu}+\sinh\overline{\lambda-\mu})\}$$

i.e.
$$(a\cosh\lambda\cosh\mu,\ b\sinh\lambda\cosh\mu)$$

$\therefore$ this point lies on the line $y=\dfrac{b}{a}\tanh\lambda\,.\,x$

$\therefore$ the locus of the middle points of this series of parallel chords, gradient $\dfrac{b}{a}\coth\lambda$, is the line $y=\dfrac{b}{a}\tanh\lambda\,.\,x$

Let us draw now a series of chords parallel to this latter line, $y=\dfrac{b}{a}\tanh\lambda\,.\,x$, and find locus of middle points of them.

Take a line joining two points on opposite branches,

$$(a\cosh\gamma,\ b\sinh\gamma) \text{ and } (-a\cosh\delta,\ b\sinh\delta),$$

its equation is

$$\frac{x}{a}(\cosh\gamma-\cosh\delta)-\frac{y}{b}(\sinh\gamma+\sinh\delta)=1-\cosh\gamma\cosh\delta-\sinh\gamma\sinh\delta\,,$$

using the form we already quoted,

$$i.e. \ \ 2\frac{x}{a}\sinh\frac{\gamma+\delta}{2}\sinh\frac{\gamma-\delta}{2}-2\frac{y}{b}\sinh\frac{\gamma+\delta}{2}\cosh\frac{\gamma-\delta}{2}=1-\cosh(\gamma+\delta)$$

$$=2\sinh^2\frac{\gamma+\delta}{2}$$

i.e.
$$\frac{x}{a}\sinh\frac{\gamma-\delta}{2}-\frac{y}{b}\cosh\frac{\gamma-\delta}{2}=\sinh\frac{\gamma+\delta}{2}$$

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The gradient of this line is $\frac{b}{a}\tanh\frac{\gamma-\delta}{2}$, and we wish it to be $\frac{b}{a}\tanh\lambda$, so join points, parameters γ and δ, so that $\gamma=\nu+\lambda$, $\delta=\nu-\lambda$ and keep λ constant but vary ν, then we get a series of parallel chords, gradient $\frac{b}{a}\tanh\lambda$.

The middle point of the line joining the points

$$(a\cosh\overline{\nu+\lambda},\ b\sinh\overline{\nu+\lambda})\quad(-a\cosh\overline{\nu-\lambda},\ b\sinh\overline{\nu-\lambda})$$

is
$$(a\sinh\nu\sinh\lambda,\ b\sinh\nu\cosh\lambda)$$

$\therefore$ this point lies on the line $y=\frac{b}{a}\coth\lambda\,.\,x$, which is therefore the locus of the middle points of this series of parallel chords, but this line is parallel to our original series of parallel chords,

$\therefore$ we have the lines $y=\frac{b}{a}\tanh\lambda\,.\,x$, $y=\frac{b}{a}\coth\lambda\,.\,x$ bisect each chords parallel to the other.

Thus the product of the gradients of two conjugate diameters is

$$\frac{b^2}{a^2}\,.$$

The line $y=\frac{b}{a}\tanh\lambda\,.\,x$ cuts the original hyperbola in P, which is the point $(a\cosh\lambda,\ b\sinh\lambda)$, while $y=\frac{b}{a}\coth\lambda\,.\,x$ cuts the conjugate hyperbola, $x^2/a^2-y^2/b^2=-1$ in the point $(a\sinh\lambda,\ b\cosh\lambda)$, say D.

These simple expressions for the coordinates of P and D give readily the theorems for example that $CP^2-CD^2=a^2-b^2$, that PD is bisected by the asymptote, and that tangents at P and D to the original and conjugate hyperbola respectively meet on the same asymptote.

Taking also again Q as the point $(a\cosh\overline{\lambda+\mu},\ b\sinh\overline{\lambda+\mu})$, and Q′ as $(a\cosh\overline{\lambda-\mu},\ b\sinh\overline{\lambda-\mu})$, and V as the middle point of QQ′, we easily prove

$$QV^2 : CV^2-CP^2 :: CD^2 : CP^2$$

which gives the equation of the hyperbola referred to the two conjugate diameters CP, CD as axes.

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