Canad. Math. Bull. Vol. **56** (1), 2013 pp. 39–43
http://dx.doi.org/10.4153/CMB-2011-159-6

Comparison Theorem for Conjugate Points of a Fourth-order Linear Differential Equation

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Abstract. In 1961, J. Barrett showed that if the first conjugate point $\eta_1(a)$ exists for the differential equation $(r(x)y'')'' = p(x)y$, where $r(x) > 0$ and $p(x) > 0$, then so does the first systems-conjugate point $\widehat{\eta}_1(a)$. The aim of this note is to extend this result to the general equation with middle term $(q(x)y')'$ without further restriction on $q(x)$, other than continuity.

1 Introduction

This note is concerned with the fourth-order differential equation

$$(r(x)y'')'' - (q(x)y')' = p(x)y, \tag{1.1}$$

where $r(x) > 0$, $p(x) \geq 0$, and $q(x)$ are continuous functions on $[a, \infty)$, $a \geq 0$. In the case $q(x) \equiv 0$, Leighton and Nehari [8] introduced the double-zero conjugate point concept of which the first conjugate point $\eta_1(a)$ of a is defined as the smallest number $b \in (a, \infty)$ for which the two point boundary conditions

$$y(a) = y'(a) = y(b) = y'(b) = 0 \tag{1.2}$$

are satisfied by a nontrivial solution of equation (1.1).

Later, Barrett [2] introduced and defined the first systems-conjugate point $\widehat{\eta}_1(a)$ of a as the smallest number $b \in (a, \infty)$ for which the two point boundary conditions

$$y(a) = y_1(a) = y(b) = y_1(b) = 0$$

$(y_1(x) = r(x)y'')$ are satisfied by a nontrivial solution of equation (1.1) for $q(x) \equiv 0$. The notation $y_1(x)$ will be used throughout the paper. In [2, Th. 4.1] it is shown that if $\eta_1(a)$ exists for (1.1) with $q(x) \equiv 0$, then $\widehat{\eta}_1(a)$ exists.

In this paper, these results are extended to equation (1.1) without further restrictions on the coefficients $r(x)$, $q(x)$ and $p(x)$. Furthermore, the following relation is established $0 < \widehat{\eta}_1(a) < \eta_1(a)$. Note that, in view of the extension of Barrett's theorem, all the criteria for the existence of $\eta_1(a)$ (*e.g.*, see [4, 7, 9, 10]) also ensure the existence of $\widehat{\eta}_1(a)$.

Received by the editors May 24, 2010; revised January 26, 2011.
Published electronically September 19, 2011.
AMS subject classification: **47E05**, 34B05, 34C10.
Keywords: fourth-order linear differential equation, conjugate points, system-conjugate points, subwronskians.

2 Main Results and Preliminaries

The main result of this paper is stated as follows:

Theorem 2.1 *If $\eta_1(a)$ exists for (1.1), then so does $\widehat{\eta}_1(a)$, and $a < \widehat{\eta}_1(a) < \eta_1(a)$.*

For the proof of Theorem 2.1, we need some preliminary results. The following comparison theorem was stated in [3, Theorem 5.1] for $q(x) \le 0$, and later was extended in [7, Corollary 2.2] to the general case. Here, we propose a simpler proof of this theorem, which is based on an extension of the Leighton–Nehary transformation [8, Theorem 12.1].

Lemma 2.2 *Let $p_0(x) > 0$ be a continuous function on $[a, \infty)$ such that, in comparison with the coefficient $p(x) > 0$ in (1.1), $p_0 \le p$. If the first conjugate point exists, say $\eta_1^0(a)$, for the equation*

$$(r(x)y'')'' - (q(x)y')' = p_0(x)y, \tag{2.1}$$

then $\eta_1(a)$ exists for the original equation (1.1) and $a < \eta_1(a) \le \eta_1^0(a)$, with equality holding if and only if $p_0 \equiv p$

Proof According to [8, Theorem 12.1], if the following initial value problem

$$(ry')' - q(x)y = 0, \quad a \le x \le b, \tag{2.2}$$

$$y'(a) = 0, \quad y(a) = 1 \tag{2.3}$$

has a positive solution $h(x)$ on the interval $[a, b]$, then the change of variables $t(x) := \int_a^x h(s)ds$ transforms $[a, b]$ into the t-interval $[0, \widetilde{b}]$ (where $\widetilde{b} = \int_a^b h(s)ds$) and equation (1.1) into

$$\left(\widetilde{r}(t)\widetilde{h}^3(t)\ddot{y}\right)^{\cdot\cdot} = \widetilde{h}^{-1}\widetilde{p}(t)y. \tag{2.4}$$

Here, $\widetilde{p}(t) = p(x(t))$, $\widetilde{h}(t) = h(x(t))$, and $\dot{}$ is $\frac{d}{dt}$. If y is a nontrivial solution of (1.1), then $\widetilde{y}(t) \equiv y(x(t))$ is a nontrivial solution of (2.4) and $\dot{\widetilde{y}} = y'h^{-1}$. Assume now that problem (2.2)–(2.3) has a solution $h(x)$ that changes sign in (a, b). In this case, it is easily seen that the first eigenvalue μ_1 of the boundary problem

$$-(ry')' + q(x)y = \lambda y, \quad a \le x \le b, \quad y'(a) = y'(b) = 0$$

is negative. Let $h(x) := y(x, \mu_1)$ be the corresponding eigenfunction. It is known from the Sturm oscillation theory that it has constant sign on $[a, b]$. Without loss of generality, we suppose that $h(x) > 0$. By the use of the same change of variables $t(x) := \int_a^x h(s)ds$, equations (1.1) and (2.1) are rewritten in the forms

$$\left(\widetilde{r}(t)\widetilde{h}^3(t)\ddot{y}\right)^{\cdot\cdot} - \mu_1\left(\widetilde{h}(t)\dot{y}\right)^{\cdot} = \widetilde{h}^{-1}\widetilde{p}(t)y \tag{2.5}$$

and

$$(\widetilde{r}(t)\widetilde{h}^3(t)\ddot{y})^{\cdot\cdot} - \mu_1\left(\widetilde{h}(t)\dot{y}\right)^{\cdot} = \widetilde{h}^{-1}\widetilde{p}_0(t)y, \tag{2.6}$$

respectively. Obviously, if $p_0(x) \le p(x)$ on $[a,b]$ then $\widetilde{h}^{-1}\widetilde{p}_0(t) \le \widetilde{h}^{-1}\widetilde{p}(t)$ on $[0,\widetilde{b}]$. If we put $b = \eta_1^0(a)$, then $\widetilde{\eta}_1^0(a)$ exists for (2.6) and $\widetilde{b} = \widetilde{\eta}_1^0(a)$. Consequently, since the coefficient in the middle term of (2.6) is negative, in view of [3, Theorem 5.1], there follows the existence of $\widetilde{\eta}_1(a)$ for (2.5), and $\widetilde{\eta}_1(a) \le \widetilde{\eta_1^0}(a)$. Therefore, $\eta_1(a)$ exists for (1.1), and $\eta_1(a) \le \eta_1^0(a)$. The lemma is proved. ■

For convenience, we introduce the following equation similar to (1.1), but depending on a parameter $\lambda \in \mathbf{R}$:

$$(r(x)y'')'' - (q(x)y')' = \lambda p(x)y. \tag{2.7}$$

The following lemma is similar to that of Greenberg [6] stated for the first eigenvalue of problem (1.1)–(1.2).

Lemma 2.3 *Let $\lambda_1(b)$ denote the first eigenvalue of the eigenvalue problem determined by equation* (2.7) *and the boundary conditions*

$$y(a) = y_1(a) = y(b) = y'(b) = 0. \tag{2.8}$$

If $b \to a$, then $\lambda_1(b) \to +\infty$.

Proof It is well known (see for example [1]) that the spectrum of problem (2.7)–(2.8) is discrete; it consists of a sequence of real eigenvalues tending to $+\infty$. Let $b_0 > a$; then from the minimax principle, we have, for every $b \in (a, b_0]$:

$$\lambda_1(b) = \min_{y\in \mathrm{H}} \frac{I(y)}{\int_a^b p(y)^2 dx},$$

where $I(y) = \int_a^b r(y'')^2 + q(y')^2 dx$ and H is a set of nontrivial admissible function y (*i.e.*, $y(x) \in C^1[a,b]$, y' is absolutely continuous and $y'' \in L_2[a,b]$) for which $y(a) = y(b) = y'(b) = 0$. The following expressions follows from the Cauchy–Schwarz inequality:

$$\int_a^b (y)^2 dx \le (b-a)\int_a^b (y')^2 dx, \qquad \int_a^b (y')^2 dx \le (b-a)\int_a^b (y'')^2 dx.$$

Therefore,

$$I(y) \ge \frac{p^* \int_a^b (y)^2 dx}{(b-a)^2} + \frac{q^* \int_a^b (y)^2 dx}{(b-a)},$$

where $f^* = \min_{x\in[a,b_0]} f(x)$, so that

$$\frac{I(y)}{\int_a^b p(y)^2 dx} \ge \frac{1}{r_*}\Big(\frac{p^*}{(b-a)^2} + \frac{q^*}{(b-a)}\Big),$$

where $r_* = \max_{x\in[a,b_0]} r(x)$. Thus,

$$\lambda_1(b) \ge \frac{1}{r_*}\Big(\frac{p^*}{(b-a)^2} + \frac{q^*}{(b-a)}\Big),$$

and, hence $\lim_{b\to a} \lambda_1(b) = +\infty$. ■

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Lemma 2.4 *Assume that $\eta_1(a)$ exists for equation* (1.1). *Then $\lambda = 1$ is the first eigenvalue of problem* (2.7)–(1.2) *for $b = \eta_1(a)$.*

Proof Obviously if $\eta_1(a)$ exists, then $\lambda = 1$ is an eigenvalue of problem (2.7)–(1.2) for $b = \eta_1(a)$. Thus the lowest eigenvalue λ_1 of this problem will satisfy $\lambda_1 \le 1$. Let $\eta_1(a, \lambda)$ denote the fist conjugate point related to equation (2.7). If $\lambda_1 < 1$, then in view of Lemma 2.2, we have $\eta_1(a, 1) < \eta_1(a, \lambda_1) = \eta_1(a)$. Therefore, $\eta_1(a, 1)$ is the first conjugate point of (1.1) which is less than $\eta_1(a)$, a contradiction. ∎

Lemma 2.5 *Assume that $\eta_1(a)$ exists for equation* (1.1)*, and $\lambda_1(b)$ is the first eigenvalue of problem* (2.7)–(2.8). *If $\lambda_1(b) < 1$ for $b \le \eta_1(a)$, then $\lambda_1(b)$ is a simple eigenvalue.*

Proof Suppose that there exists $b_0 \in (a, \eta_1(a)]$ such that $\lambda_1(b_0) < 1$, and it is an eigenvalue of multiplicity 2. Let y_1 and y_2 be the corresponding eigenfunctions. It is known that any solution y of equation (2.7) (for $\lambda = \lambda_1(b_0)$) that satisfies the initial conditions

$$y(b_0) = y'(b_0) = 0, \tag{2.9}$$

can be expressed as a linear combination of y_1 and y_2. Let y be a nontrivial solution of problem (2.7)–(2.9). Then $y(x) = \alpha y_1(x) + \beta y_2(x)$, where $(\alpha, \beta) \in \mathbf{R}^2$. Since $y_1(a) = y_2(a) = 0$, $y(a) = 0$ and

$$\det \begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix} = 0.$$

Therefore, for some α, β we have $y'(a) = 0$, and this implies that $\lambda_1(b_0)$ is also an eigenvalue of the problem determined by equation (2.7) and the boundary conditions

$$y(a) = y'(a) = y(b_0) = y'(b_0) = 0.$$

This means that the first conjugate point $\eta_1(a, \lambda_1(b_0))$ exists for equation (2.7) (for $\lambda = \lambda_1(b_0)$) and satisfies $\eta_1(a, \lambda_1(b_0)) \le \eta_1(a)$. This yields a contradiction in view of Lemma 2.2. ∎

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 As noted above, any solution of equation (2.7) that satisfies the initial conditions $y(a) = y_1(a) = 0$ may be written as a linear combination of $u(x)$ and $v(x)$, which are the fundamental solutions of equation (2.7), whose initial conditions are

$$u(a) = u_1(a) = Tu(a) = 0, \; u'(a) = 1,$$

$$v(a) = v'(a) = v_1(a) = 0, \; Tv(a) = 1,$$

where, $Ty(x) = (r(x)y'')' - q(x)y'$. We introduce the following subwronskians:

$$\widehat{\sigma}(x) := \widehat{\sigma}(\lambda, x) = uv'(x) - vu'(x),$$

$$r\widehat{\sigma}'(x) := r\widehat{\sigma}'(\lambda, x) = uv_1(x) - vu_1(x),$$

https://doi.org/10.4153/CMB-2011-159-6 Published online by Cambridge University Press

which satisfy the initial conditions

$$\widehat{\sigma}(a) = \widehat{\sigma}'(a) = 0. \tag{2.10}$$

It is easy to see that $\widehat{\eta}_1(a)$ is the first zero on (a, ∞) of $\widehat{\sigma}'$. It is well known (*e.g.,* see [5]) that the spectra of problems (2.7)–(1.2) and (2.7)–(2.8) consist of a sequence of real eigenvalues tending to $+\infty$. Let $\lambda_1'(b)$ and $\lambda_1(b)$ be the first eigenvalues of Problems (2.7)-(1.2) and (2.7)-(2.8), respectively. In view of Lemma 2.4, if $\eta_1(a)$ exists for (1.1), then $\lambda_1'(\eta_1(a)) = 1$. From the minimax principle (*e.g.,* see [11]), we have $\lambda_1(\eta_1(a)) \leq 1$. If $\lambda_1(\eta_1(a)) = 1$, then $\widehat{\sigma}(\eta_1(a)) = 0$, and hence, from the initial conditions (2.10) and Rolle's theorem, there exists a first systems-conjugate point $\widehat{\eta}_1(a) \in (a, \eta_1(a))$. Suppose now $\lambda_1(\eta_1(a)) < 1$. According to Lemma 2.5, if $\lambda_1(b) < 1$ for $b \in (a, \eta_1(a)]$, then it is simple, and hence

$$\widehat{\sigma}(\lambda_1(b), b) = 0, \quad \frac{\partial \widehat{\sigma}}{\partial \lambda}(\lambda, b)_{|\lambda=\lambda_1(b)} \neq 0.$$

Thus, by the implicit function theorem, if $\lambda_1(b) < 1$, then it is a continuous function of $b \in (a, \eta_1(a))$. From this and Lemma 2.3, it follows that, as b varies from $\eta_1(a)$ to a, $\lambda_1(b) \to +\infty$, and for some $b^* \in (a, \eta_1(a))$ we have $\lambda_1(b^*) = 1$. Therefore, $\widehat{\sigma}(1, b^*) = 0$, and again by the initial conditions (2.10) and Rolle's theorem, there exists the first systems-conjugate point $\widehat{\eta}_1(a) \in (a, \eta_1(a))$ for (1.1). The proof of the theorem is complete. ∎

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