11

Locally Finitely Presented Categories

Contents

We study additive categories that are locally finitely presented. This means that every object is the filtered colimit of finitely presented objects. The categorical notion of being finitely presented means for an object X that the functor $\operatorname{Hom}(X, -)$ preserves filtered colimits. Of particular interest is the case of an abelian category. Every locally finitely presented abelian category is a Grothen-

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dieck category; so it is a category with injective envelopes and we can study its injective objects.

The theory of locally finitely presented categories applies in particular to locally noetherian Grothendieck categories, that is, Grothendieck categories having a generating set of noetherian objects. Then finitely presented and noetherian objects coincide. Also, in that case every injective object decomposes into a direct sum of indecomposable objects. We include a discussion of Gröbner categories and provide criteria for when a functor category is locally noetherian; this can be thought of as a generalisation of Hilbert's basis theorem.

11.1 Locally Finitely Presented Categories

We introduce the concept of a locally finitely presented additive category. Any locally finitely presented category $\mathcal{A}$ is completely determined by its subcategory $\operatorname{fp}\mathcal{A}$ of finitely presented objects, because $\mathcal{A}$ identifies with the category of left exact functors $(\operatorname{fp}\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$.

Filtered Colimits

A category $\mathcal{J}$ is called *filtered* if

(Fil1) the category is non-empty,
(Fil2) given objects i, i' there is an object j with morphisms $i \to j \leftarrow i'$, and
(Fil3) given morphisms $\alpha, \alpha' : i \to j$ there is a morphism $\beta : j \to k$ such that $\beta\alpha = \beta\alpha'$.

For a functor $F : \mathcal{J} \to \mathcal{C}$, we denote by $\operatorname{colim} F$ or $\operatorname{colim}_{i \in \mathcal{J}} F(i)$ its colimit, provided it exists in $\mathcal{C}$. The term *filtered colimit* is used for the colimit of a functor $F : \mathcal{J} \to \mathcal{C}$ such that the category $\mathcal{J}$ is filtered.

Example 11.1.1. (1) A partially ordered set $(I, \leq)$ can be viewed as a category. The objects are the elements of I and there is a unique morphism $i \to j$ whenever $i \leq j$. This category is filtered if and only if $(I, \leq)$ is non-empty and directed. A colimit $\operatorname{colim}_{i \in \mathcal{J}} F(i)$ is called a *directed colimit* if $\mathcal{J}$ is given by a directed set.

(2) The coproduct of a family of objects $(X_i)_{i \in I}$ can be written as

$$\coprod_{i \in I} X_i = \operatorname*{colim}_{J \in \mathcal{J}} \Big(\coprod_{i \in J} X_i \Big)$$

where $\mathcal{J}$ denotes the filtered category of finite subsets $J \subseteq I$.

(3) Let $\mathcal{A}$ be an additive category and $\mathcal{C} \subseteq \mathcal{A}$ a full additive subcategory that

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is essentially small. For any $X \in \mathcal{A}$ let $\mathcal{C}/X$ denote the *slice category* consisting of pairs (C, ϕ) given by a morphism $\phi \colon C \to X$ with $C \in \mathcal{C}$. A morphism $(C, \phi) \to (C', \phi')$ is given by a morphism $\alpha \colon C \to C'$ in $\mathcal{C}$ such that $\phi'\alpha = \phi$. Then $\mathcal{C}/X$ is filtered, provided that each morphism in $\mathcal{C}$ admits a cokernel in $\mathcal{A}$ that lies in $\mathcal{C}$. In fact, having weak cokernels is sufficient.

Locally Finitely Presented Categories

Let $\mathcal{A}$ be an additive category and suppose that $\mathcal{A}$ is cocomplete. Thus each functor $F \colon \mathcal{I} \to \mathcal{A}$ from an essentially small category $\mathcal{I}$ admits a colimit. Let us recall the construction of the colimit because it is very explicit. For a morphism $\alpha \colon i \to j$ in $\mathcal{I}$ we set $s(\alpha) = i$ and $t(\alpha) = j$. For $j \in \mathcal{I}$ we write $\iota_j \colon F(j) \to \coprod_{i \in \mathcal{I}} F(i)$ for the canonical inclusion and set $\phi_\alpha = \iota_{s(\alpha)} - \iota_{t(\alpha)} \circ F(\alpha)$. Then $\operatorname{colim} F$ is computed as the cokernel of $\phi = (\phi_\alpha)_{\alpha \in \mathcal{I}}$ and fits into an exact sequence

$$\coprod_{\alpha \in \mathcal{I}} F(s(\alpha)) \xrightarrow{\phi} \coprod_{i \in \mathcal{I}} F(i) \longrightarrow \operatorname{colim} F \longrightarrow 0.$$

Often we write $F_i = F(i)$ for $i \in \mathcal{I}$ and then $\operatorname{colim}_i F_i = \operatorname{colim} F$. A consequence of this construction is the fact that an additive category is cocomplete if and only if it has coproducts and every morphism admits a cokernel.

An object $X \in \mathcal{A}$ is *finitely presented* if the functor $\operatorname{Hom}_{\mathcal{A}}(X, -)$ preserves filtered colimits. This means that for every filtered colimit $\operatorname{colim}_i Y_i$ in $\mathcal{A}$ the canonical map

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathcal{A}}(X, Y_i) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X, \operatorname*{colim}_i Y_i)$$

is bijective. Let $\operatorname{fp} \mathcal{A}$ denote the full subcategory of finitely presented objects. We record the following elementary facts.

Lemma 11.1.2. *The subcategory* $\operatorname{fp} \mathcal{A}$ *is closed under finite coproducts, direct summands, and cokernels. If* $X \in \operatorname{fp} \mathcal{A}$ *is written as a filtered colimit* $X = \operatorname{colim} X_i$*, then for some index* i_0 *the canonical morphism* $X_{i_0} \to X$ *is a split epimorphism.* □

The category $\mathcal{A}$ is called *locally finitely presented* if $\operatorname{fp} \mathcal{A}$ is essentially small and every object in $\mathcal{A}$ is a filtered colimit of finitely presented objects. In that case any object $X \in \mathcal{A}$ can be written canonically as a filtered colimit

$$X = \operatorname*{colim}_{(C,\phi) \in \operatorname{fp} \mathcal{A}/X} C$$

of the forgetful functor $\operatorname{fp} \mathcal{A}/X \to \mathcal{A}$ that takes (C, ϕ) to C, as we will see in Corollary 11.1.16.

From now on the term ‘locally finitely presented’ for a category $\mathcal{A}$ includes the properties that $\mathcal{A}$ is additive and cocomplete.

Remark 11.1.3. Let $\mathcal{A}^{\mathbf{2}}$ denote the category of morphisms in $\mathcal{A}$. If $\mathcal{A}$ is locally finitely presented, then $\mathcal{A}^{\mathbf{2}}$ is locally finitely presented and $(\operatorname{fp}\mathcal{A})^{\mathbf{2}} \xrightarrow{\sim} \operatorname{fp}(\mathcal{A}^{\mathbf{2}})$. This means that each morphism in $\mathcal{A}$ can be written canonically as a filtered colimit of morphisms in $\operatorname{fp}\mathcal{A}$.

Example 11.1.4. (1) Let Λ be any ring. Then the category of Λ-modules is locally finitely presented. The finitely presented objects are precisely the modules M that admit a presentation $\Lambda^r \to \Lambda^s \to M \to 0$ for some integers $r, s \geq 0$.

(2) Let $\mathbb{X}$ be a *scheme* and suppose it is quasi-compact and quasi-separated. Then the category $\operatorname{Qcoh}\mathbb{X}$ of quasi-coherent $\mathcal{O}_{\mathbb{X}}$-modules is locally finitely presented. The finitely presented objects are precisely the finitely presented $\mathcal{O}_{\mathbb{X}}$-modules [97, I.6.9.12]. When $\mathbb{X}$ is noetherian, then the category of finitely presented objects identifies with the category $\operatorname{coh}\mathbb{X}$ of coherent sheaves.

Cofinal Subcategories

For the computation of filtered colimits it is often useful to vary the index category. We consider an essentially small filtered category $\mathcal{I}$ and a fully faithful functor $\phi\colon \mathcal{J} \to \mathcal{I}$. Then ϕ is called *cofinal* if it satisfies the equivalent conditions of the following lemma. When ϕ is an inclusion we call $\mathcal{J}$ a *cofinal subcategory* of $\mathcal{I}$.

Lemma 11.1.5. *Let $\mathcal{I}$ be an essentially small filtered category. For a fully faithful functor $\phi\colon \mathcal{J} \to \mathcal{I}$ the following are equivalent.*

(1) *For every object $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ and a morphism $i \to \phi(j)$.*
(2) *Every functor $F\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Set}$ induces an isomorphism $\lim F \xrightarrow{\sim} \lim(F \circ \phi)$.*
(3) *For every category $\mathcal{C}$ which admits filtered colimits, every functor $F\colon \mathcal{I} \to \mathcal{C}$ induces an isomorphism $\operatorname{colim}(F \circ \phi) \xrightarrow{\sim} \operatorname{colim} F$.*

Moreover, in this case the category $\mathcal{J}$ is filtered.

Proof (1) $\Rightarrow$ (2): Limits in the category of sets can be calculated explicitly. Thus the condition (1) implies that $\lim F \to \lim(F \circ \phi)$ is injective. Combined with the fact that $\mathcal{I}$ is filtered, the map is also bijective.

(2) $\Rightarrow$ (3): We have for each $X \in \mathcal{C}$ a canonical bijection

$$\operatorname{Hom}(\operatorname*{colim}_i F(i), X) \xrightarrow{\sim} \lim_i \operatorname{Hom}(F(i), X).$$

Thus we can use the functor $F_X\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Set}$ given by $i \mapsto \operatorname{Hom}(F(i), X)$. Then

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the isomorphism $\lim F_X \xrightarrow{\sim} \lim(F_X \circ \phi)$ for all X implies that $\operatorname{colim}(F \circ \phi) \xrightarrow{\sim} \operatorname{colim} F$.

(3) ⇒ (1): Consider the Yoneda functor $F\colon \mathcal{I} \to \operatorname{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Set})$. Colimits in $\operatorname{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Set})$ are computed pointwise. Thus we have for each $x \in \mathcal{I}$ a bijection

$$\operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Hom}(x, \phi(j)) \xrightarrow{\sim} \operatorname*{colim}_{i \in \mathcal{I}} \operatorname{Hom}(x, i).$$

Choosing $x = i$, we find $j \in \mathcal{J}$ and a morphism $i \to \phi(j)$.

Using condition (1), the fact that $\mathcal{I}$ is filtered implies that $\mathcal{J}$ is filtered. □

Let $\mathcal{A}$ be a locally finitely presented category. For a full additive subcategory $\mathcal{C} \subseteq \operatorname{fp}\mathcal{A}$ let $\vec{\mathcal{C}}$ denote the full subcategory of $\mathcal{A}$ consisting of the filtered colimits of objects in $\mathcal{C}$.

Lemma 11.1.6. *An object $X \in \mathcal{A}$ belongs to $\vec{\mathcal{C}}$ if and only if every morphism $C \to X$ with $C \in \operatorname{fp}\mathcal{A}$ factors through an object in $\mathcal{C}$.*

Proof Let $X = \operatorname{colim} X_i$ be written as a filtered colimit of objects in $\mathcal{C}$. Then every morphism $C \to X$ with $C \in \operatorname{fp}\mathcal{A}$ factors through $X_i \to X$ for some i. Conversely, let $X = \operatorname{colim}_{(C,\phi) \in \operatorname{fp}\mathcal{A}/X} C$ and suppose that each $\phi\colon C \to X$ factors through an object in $\mathcal{C}$. This means that the inclusion $\mathcal{C}/X \to (\operatorname{fp}\mathcal{A})/X$ is cofinal, so $\operatorname{colim}_{(C,\phi)\in\mathcal{C}/X} C \xrightarrow{\sim} X$ by Lemma 11.1.5. Thus $X \in \vec{\mathcal{C}}$. □

Example 11.1.7. Let Λ be a ring and set $\mathcal{C} = \operatorname{proj}\Lambda$. Then $\vec{\mathcal{C}}$ equals the category of flat Λ-modules.

Categories of Additive Functors

Let $\mathcal{C}$ be an essentially small additive category and let $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ denote the category of additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. This functor category inherits (co)kernels and (co)products from Ab, because these are computed 'pointwise'. In particular, $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an abelian category. Also, filtered colimits of exact sequences are exact.

For an additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ let $\mathcal{C}/F$ denote the category consisting of pairs (C, f) with $C \in \mathcal{C}$ and $f \in F(C)$. A morphism $(C, f) \to (C', f')$ is given by a morphism $\alpha\colon C \to C'$ in $\mathcal{C}$ such that $F(\alpha)(f') = f$.

Lemma 11.1.8. *An additive functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ equals the colimit of the functor*

$$\Phi_{\mathcal{C}}\colon \mathcal{C}/F \longrightarrow \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad (C, f) \mapsto \operatorname{Hom}_{\mathcal{C}}(-, C).$$

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Proof Each pair (C, f) in $\mathcal{C}/F$ yields a morphism $\operatorname{Hom}_{\mathcal{C}}(-, C) \to F$ and these induce a morphism

$$\operatorname*{colim}_{(C,f)\in\mathcal{C}/F} \operatorname{Hom}_{\mathcal{C}}(-, C) \longrightarrow F.$$

We obtain an inverse by giving for each $X \in \mathcal{C}$ a morphism

$$F(X) \longrightarrow \operatorname*{colim}_{(C,f)\in\mathcal{C}/F} \operatorname{Hom}_{\mathcal{C}}(X, C)$$

as follows. An element $x \in F(X)$ is sent to the image of id_X under the canonical map

$$\operatorname{Hom}_{\mathcal{C}}(X, X) \longrightarrow \operatorname*{colim}_{(C,f)\in\mathcal{C}/F} \operatorname{Hom}_{\mathcal{C}}(X, C)$$

corresponding to (X, x) in $\mathcal{C}/F$. □

We write $\operatorname{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for the category of functors $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that admit a presentation

$$\operatorname{Hom}_{\mathcal{C}}(-, C) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-, D) \longrightarrow F \longrightarrow 0.$$

It follows from Yoneda's lemma that each representable functor is a finitely presented object in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Thus a cokernel of a morphism between representable functors is a finitely presented object.

We obtain another presentation of an additive functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ using the slice category $\operatorname{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})/F$ which is filtered; see Example 11.1.1.

Proposition 11.1.9. *An additive functor* $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *equals the filtered colimit of the forgetful functor*

$$\Psi \colon \operatorname{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})/F \longrightarrow \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}).$$

Therefore the additive category $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ *is locally finitely presented and*

$$\operatorname{fp} \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) = \operatorname{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}).$$

Proof We consider the Yoneda functor $h \colon \mathcal{C} \to \mathcal{D} := \operatorname{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and set $\bar{F} = \operatorname{Hom}(-, F)|_{\mathcal{D}}$. Then $\bar{F} = \operatorname{colim} \Phi_{\mathcal{D}}$ by Lemma 11.1.8. We have $\Psi = h^* \circ \Phi_{\mathcal{D}}$ and therefore

$$\operatorname{colim} \Psi = h^*(\operatorname{colim} \Phi_{\mathcal{D}}) = h^*(\bar{F}) = F.$$

The second assertion is an immediate consequence of the first. □

Let us add another useful presentation of an additive functor as a colimit which uses a directed set.

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Proposition 11.1.10. *Every additive functor* $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *is a directed colimit of functors in* $\mathrm{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Proof An additive functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ admits a presentation

$$\coprod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(-, C_i) \longrightarrow \coprod_{j \in J} \mathrm{Hom}_{\mathcal{C}}(-, D_j) \longrightarrow F \longrightarrow 0$$

because $\mathcal{C}$ is essentially small. Let U denote the set of pairs $u = (I', J')$ consisting of finite subsets $I' \subseteq I$ and $J' \subseteq J$ making the following square commutative

$$\begin{array}{ccc} \coprod_{i \in I'} \mathrm{Hom}_{\mathcal{C}}(-, C_i) & \longrightarrow & \coprod_{j \in J'} \mathrm{Hom}_{\mathcal{C}}(-, D_j) \\ \big\downarrow & & \big\downarrow \\ \coprod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(-, C_i) & \longrightarrow & \coprod_{j \in J} \mathrm{Hom}_{\mathcal{C}}(-, D_j) \end{array}$$

and denote by $F_u \to F$ the induced morphism between the cokernels of the horizontal morphisms. The set U is partially ordered by inclusion, and we have $\sup(u_1, u_2) \in U$ for $u_1, u_2 \in U$. Thus U is directed and it is easily checked that $\mathrm{colim}_{u \in U} F_u \xrightarrow{\sim} F$. □

Linear Representations

A category $\mathcal{C}$ is *preadditive* if each morphism set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group, and the composition maps

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

are biadditive. An additive category carries an intrinsic structure of a preadditive category, but in general this is an additional structure. It is often convenient to consider functor categories $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ when $\mathcal{C}$ is preadditive, and the above results generalise with same proofs.

The *centre* $Z(\mathcal{C})$ of a preadditive category $\mathcal{C}$ is the ring of all natural transformations $\mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ of the identity functor on $\mathcal{C}$. For a commutative ring k the structure of a *k-linear category* on $\mathcal{C}$ is given by a ring homomorphism $k \to Z(\mathcal{C})$.

Let $\mathcal{C}$ be a k-linear category $\mathcal{C}$. Then for any additive functor $F \colon \mathcal{C} \to \mathrm{Ab}$ there is a canonical k-module structure on FX for each $X \in \mathcal{C}$ via the homomorphism $k \to \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathbb{Z}}(FX)$. Thus we may view F as a *k-linear functor* $\mathcal{C} \to \mathrm{Mod}\, k$.

Example 11.1.11. A ring Λ may be viewed as a preadditive category with a single object, and then $Z(\Lambda)$ identifies with the usual centre given by all elements $x \in \Lambda$ satisfying $xy = yx$ for all $y \in \Lambda$. Moreover, $Z(\Lambda) \xrightarrow{\sim} Z(\mathrm{Mod}\, \Lambda)$.

Let $\mathcal{C}$ be an essentially small category and k a commutative ring. The forgetful functor $\operatorname{Mod} k \to \operatorname{Set}$ admits a left adjoint which sends a set S to a free k-module $k[S]$ with basis S. Thus there is a natural bijection

$$\operatorname{Hom}_k(k[S], X) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Set}}(S, X)$$

for any k-module X. The *k-linearisation* $k\mathcal{C}$ of $\mathcal{C}$ is the k-linear category obtained by setting $\operatorname{Ob} k\mathcal{C} = \operatorname{Ob} \mathcal{C}$ and

$$\operatorname{Hom}_{k\mathcal{C}}(X, Y) = k[\operatorname{Hom}_{\mathcal{C}}(X, Y)]$$

for each pair of objects X, Y.

Consider the category $\operatorname{Fun}(\mathcal{C}, \operatorname{Mod} k)$ of all functors $\mathcal{C} \to \operatorname{Mod} k$. We think of a functor $\mathcal{C} \to \operatorname{Mod} k$ as a *k-linear representation* of $\mathcal{C}$.

Lemma 11.1.12. *Restriction via the inclusion $i\colon \mathcal{C} \to k\mathcal{C}$ gives an equivalence*

$$\operatorname{Add}(k\mathcal{C}, \operatorname{Ab}) \xrightarrow{\sim} \operatorname{Fun}(\mathcal{C}, \operatorname{Mod} k).$$

Proof The quasi-inverse functor $\operatorname{Fun}(\mathcal{C}, \operatorname{Mod} k) \to \operatorname{Add}(k\mathcal{C}, \operatorname{Ab})$ is obtained by applying the left adjoint of the forgetful functor $\operatorname{Mod} k \to \operatorname{Set}$. Thus any functor $F\colon \mathcal{C} \to \operatorname{Mod} k$ extends uniquely to a k-linear functor $F'\colon k\mathcal{C} \to \operatorname{Mod} k$ such that $F' \circ i = F$. □

Example 11.1.13. (1) Let Λ be a ring. Then evaluation at Λ yields an equivalence

$$\operatorname{Add}((\operatorname{proj} \Lambda)^{\operatorname{op}}, \operatorname{Ab}) \xrightarrow{\sim} \operatorname{Mod} \Lambda.$$

Taking a Λ-module X to $\operatorname{Hom}(-, X)|_{\operatorname{proj} \Lambda}$ gives a quasi-inverse.

(2) Let Q be a quiver, k a commutative ring, and $\operatorname{Rep}(Q, k)$ the category of *k-linear representations* of Q. The *path category* is the k-linearisation kQ of the category of paths in Q. Then restriction to Q yields an equivalence

$$\operatorname{Add}(kQ, \operatorname{Ab}) \xrightarrow{\sim} \operatorname{Rep}(Q, k).$$

(3) Let G be a group, k a commutative ring, and $\operatorname{Rep}(G, k)$ the category of *k-linear representations* of G. We view the group as a category with a single object, and then its k-linearisation identifies with the *group algebra* kG. This yields an equivalence

$$\operatorname{Mod}(kG^{\operatorname{op}}) \xrightarrow{\sim} \operatorname{Rep}(G, k).$$

Categories of Left Exact Functors

Let $\mathcal{C}$ be an essentially small additive category and suppose that $\mathcal{C}$ has cokernels. We consider the functor category $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and denote by $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the full subcategory of additive functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that are *left exact*, so taking an exact sequence $X \to Y \to Z \to 0$ in $\mathcal{C}$ to an exact sequence $0 \to FZ \to FY \to FX$.[1] This category has filtered colimits, kernels, and products, because left exact functors are closed under these operations. Note that every representable functor is a finitely presented object in $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Lemma 11.1.14. *Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ be an additive functor. Then the category $\mathcal{C}/F$ is filtered if and only if F is left exact.*

Proof When $\mathcal{C}/F$ is filtered then F is a filtered colimit of left exact functors since each representable functor is left exact; see Lemma 11.1.8. Thus F is left exact.

Now suppose that F is left exact. We need to show that $\mathcal{C}/F$ is filtered. Given pairs (C, f) and (C', f'), we have canonical morphisms

$$(C, f) \to (C \oplus C', f + f') \leftarrow (C', f')$$

since F is additive. Given morphisms $\alpha_1, \alpha_2\colon (C, f) \to (C', f')$, we obtain a morphism $\beta\colon (C', f') \to (C'', f'')$ by taking $C'' = \mathrm{Coker}(\alpha_1 - \alpha_2)$. Because F is left exact, there is $f'' \in F(C'')$ which is sent to $f' \in F(C')$ since $F(\alpha_1 - \alpha_2)(f') = 0$. Thus $\beta\alpha_1 = \beta\alpha_2$. □

The following correspondence provides a useful description of locally finitely presented categories.

Theorem 11.1.15. *We have a correspondence between locally finitely presented categories and essentially small additive categories with cokernels.*

(1) *Let $\mathcal{C}$ be an essentially small additive category that admits cokernels and set $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Then $\mathcal{A}$ is locally finitely presented with $\mathcal{C} \xrightarrow{\sim} \mathrm{fp}\,\mathcal{A}$.*
(2) *Let $\mathcal{A}$ be a locally finitely presented category and set $\mathcal{C} = \mathrm{fp}\,\mathcal{A}$. Then*

$$\mathcal{A} \longrightarrow \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad X \longmapsto h_X := \mathrm{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{C}}$$

is an equivalence.

Proof (1) We consider the category $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Clearly, each representable functor is a finitely presented object in $\mathcal{A}$, by Yoneda's lemma. Then it follows from Lemma 11.1.8 and Lemma 11.1.14 that every object in $\mathcal{A}$ is

[1] When $\mathcal{C}$ is an exact category with cokernels, there are two notions of a left exact functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. In general, these are different.

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a filtered colimit of finitely presented objects. Any finitely presented object is isomorphic to a representable functor by Lemma 11.1.2. Thus $\mathcal{C} \xrightarrow{\sim} \operatorname{fp}\mathcal{A}$. A simple calculation shows that the Yoneda embedding $\mathcal{C} \to \mathcal{A}$ is right exact, so it takes cokernels to cokernels.

Any morphism $\phi\colon X \to Y$ in $\mathcal{A}$ can be written as a filtered colimit $\phi = \operatorname{colim}\phi_i$ of morphisms in $\operatorname{fp}\mathcal{A}$. Then $\operatorname{Coker}\phi = \operatorname{colim}\operatorname{Coker}\phi_i$. Thus $\mathcal{A}$ has cokernels and is therefore cocomplete, since $\mathcal{A}$ has coproducts.

(2) We show that the assignment $X \mapsto h_X$ is fully faithful and essentially surjective. Let X, Y be objects in $\mathcal{A}$ and $X = \operatorname{colim} X_i$ written as a filtered colimit of objects in $\operatorname{fp}\mathcal{A}$. Then

$$\begin{aligned}\operatorname{Hom}(\operatorname*{colim}_i X_i, Y) &\cong \lim_i \operatorname{Hom}(X_i, Y)\\ &\cong \lim_i \operatorname{Hom}(h_{X_i}, h_Y)\\ &\cong \operatorname{Hom}(\operatorname*{colim}_i h_{X_i}, h_Y)\\ &\cong \operatorname{Hom}(h_X, h_Y),\end{aligned}$$

where we use Yoneda's lemma and the fact that $X \mapsto h_X$ preserves filtered colimits. Any object $F \in \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ can be written as a filtered colimit

$$F = \operatorname*{colim}_{(C,f)\in\mathcal{C}/F} h_C$$

by Lemma 11.1.8. Thus for $X = \operatorname{colim}_{(C,f)\in\mathcal{C}/F} C$ in $\mathcal{A}$ we have $h_X \cong F$.

We conclude that a quasi-inverse $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ sends F to $\operatorname{colim}\tilde{F}$ where $\tilde{F}\colon \mathcal{C}/F \to \mathcal{A}$ is the functor that sends (C, f) to C. □

Let us collect some consequences.

Corollary 11.1.16. *An object X in a locally finitely presented category can be written canonically as a filtered colimit*

$$X = \operatorname*{colim}_{(C,\phi)\in\operatorname{fp}\mathcal{A}/X} C \qquad (11.1.17)$$

of the forgetful functor $\operatorname{fp}\mathcal{A}/X \to \mathcal{A}$ that takes (C, ϕ) to C. □

Corollary 11.1.18. *A locally finitely presented category is complete.*

Proof A limit of left exact functors is again left exact. □

Corollary 11.1.19. *Let $\mathcal{A}$ be a locally finitely presented category.*

(1) *If $\mathcal{A}$ is abelian, then filtered colimits in $\mathcal{A}$ are exact, and therefore $\mathcal{A}$ is a Grothendieck category. In particular, $\mathcal{A}$ has injective envelopes.*
(2) *If $\operatorname{fp}\mathcal{A}$ is abelian, then $\mathcal{A}$ is abelian and the inclusion $\operatorname{fp}\mathcal{A} \to \mathcal{A}$ is exact.*

Proof Set $\mathcal{C} = \operatorname{fp} \mathcal{A}$. We can identify $\mathcal{A}$ with $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and can compute filtered colimits in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, where they are exact, keeping in mind that filtered colimits of left exact functors are left exact.

Now suppose that $\mathcal{C}$ is abelian. Given a morphism $\phi = \operatorname{colim} \phi_i$, written as a filtered colimit of morphisms in $\operatorname{fp} \mathcal{A}$, we have $\operatorname{Ker} \phi = \operatorname{colim} \operatorname{Ker} \phi_i$, since kernels are computed in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and filtered colimits in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ are exact. Thus $\mathcal{A}$ has kernels. The Yoneda embedding $\mathcal{C} \to \mathcal{A}$ is left exact since the embedding $\mathcal{C} \to \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is left exact. On the other hand, $\mathcal{C}$ is closed under cokernels. Thus the inclusion $\mathcal{C} \to \mathcal{A}$ is exact. □

Remark 11.1.20. Let $\mathcal{A}$ be locally finitely presented and $\operatorname{fp} \mathcal{A}$ abelian. Then every exact sequence $\eta\colon 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ in $\mathcal{A}$ can be written as a filtered colimit of exact sequences in $\operatorname{fp} \mathcal{A}$. To see this, write $\alpha = \operatorname{colim} \alpha_i$ with $\alpha_i\colon X_i \to Y_i$ in $\operatorname{fp} \mathcal{A}$ for all i. Let $\beta_i\colon Y_i \to Z_i$ denote the cokernel of each α_i, and let $\alpha'_i\colon X'_i \to Y_i$ denote the kernel of β_i. Then η is the filtered colimit of the exact sequences $0 \to X'_i \xrightarrow{\alpha'_i} Y_i \xrightarrow{\beta_i} Z_i \to 0$.

Lemma 11.1.21. *The inclusion* $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \hookrightarrow \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ *admits a left adjoint.*

Proof The adjoint maps a finitely presented functor $\operatorname{Coker} \operatorname{Hom}_{\mathcal{C}}(-, \phi)$ (given by a morphism ϕ in $\mathcal{C}$) to $\operatorname{Hom}_{\mathcal{C}}(-, \operatorname{Coker} \phi)$; see Example 1.1.4. This extends to

$$\operatorname*{colim}_{i \in \mathcal{I}} \operatorname{Coker} \operatorname{Hom}_{\mathcal{C}}(-, \phi_i) \longmapsto \operatorname*{colim}_{i \in \mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(-, \operatorname{Coker} \phi_i).$$

Alternatively, take $F \in \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ to

$$\operatorname*{colim}_{(C,f) \in \mathcal{C}/F} \operatorname{Hom}_{\mathcal{C}}(-, C)$$

in $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$; see Lemma 11.1.8. □

Corollary 11.1.22. *In a locally finitely presented category every object can be written as a directed colimit of finitely presented objects.*

Proof Any locally finitely presented category is equivalent to one of the form $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Write $F \in \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ as a directed colimit $F = \operatorname{colim} F_i$ of objects $F_i \in \operatorname{Fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$; see Proposition 11.1.10. Let $Q\colon \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ denote the left adjoint of the inclusion; see Lemma 11.1.21. Then $F = Q(F) = \operatorname{colim} Q(F_i)$ is a directed colimit of finitely presented objects. □

Example 11.1.23. Let $\mathcal{A}$ be a locally finitely presented category and $\mathcal{C} \subseteq \operatorname{fp} \mathcal{A}$ a full additive subcategory. Suppose the category $\mathcal{C}$ admits cokernels (not necessarily the same as in $\mathcal{A}$). Then $\vec{\mathcal{C}}$ is locally finitely presented with $\mathcal{C} \xrightarrow{\sim} \operatorname{fp} \vec{\mathcal{C}}$.

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Proof Clearly, $\vec{\mathcal{C}}$ is a category with filtered colimits and $\mathcal{C} \subseteq \operatorname{fp} \vec{\mathcal{C}}$. On the other hand, when $X \in \operatorname{fp} \vec{\mathcal{C}}$ is written as a filtered colimit $X = \operatorname{colim} X_i$ of objects in $\mathcal{C}$, then id_X factors through some X_i, so X is a direct summand of an object in $\mathcal{C}$. Thus $\mathcal{C} \xrightarrow{\sim} \operatorname{fp} \vec{\mathcal{C}}$, and $X \mapsto \operatorname{Hom}(-, X)|_{\mathcal{C}}$ yields an equivalence $\vec{\mathcal{C}} \xrightarrow{\sim} \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. □

Recall that a full subcategory $\mathcal{C} \subseteq \mathcal{D}$ of some category $\mathcal{D}$ is *covariantly finite* if for every object $X \in \mathcal{D}$ there is a morphism $X \to C^X$ (called a *left* $\mathcal{C}$*-approximation*) such that $C^X \in \mathcal{C}$ and every morphism $X \to C$ with $C \in \mathcal{C}$ factors through $X \to C^X$. For example, $\mathcal{C}$ is covariantly finite if the inclusion $\mathcal{C} \to \mathcal{D}$ admits a left adjoint. Then a left approximation $X \to C^X$ is given by the unit of the adjunction..

Example 11.1.24. Let $\mathcal{A}$ be a locally finitely presented category and $\mathcal{C} \subseteq \operatorname{fp} \mathcal{A}$ a full additive subcategory. Then $\vec{\mathcal{C}}$ is closed under products in $\mathcal{A}$ if and only if $\mathcal{C}$ is covariantly finite in $\operatorname{fp} \mathcal{A}$.

Proof We apply the criterion of Lemma 11.1.6. Suppose first that $\mathcal{C}$ is covariantly finite in $\operatorname{fp} \mathcal{A}$. If $X := \prod_{i \in I} X_i$ is a product of objects in $\vec{\mathcal{C}}$, then every morphism $F \to X$ with $F \in \operatorname{fp} \mathcal{A}$ factors through a product $\prod_{i \in I} C_i$ of objects in $\mathcal{C}$, and this factors through $F \to C^F$. Thus $X \in \vec{\mathcal{C}}$. Conversely, suppose that $\vec{\mathcal{C}}$ is closed under products. Fix $X \in \operatorname{fp} \mathcal{A}$ and consider the product $X_{\mathcal{C}} := \prod_{X \to C} C$ where $X \to C$ runs through all morphisms with $C \in \mathcal{C}$. This product belongs to $\vec{\mathcal{C}}$, and therefore the canonical morphism $X \to X_{\mathcal{C}}$ factors through an object in $\mathcal{C}$ via a morphism $X \to C^X$. Clearly, this is a left $\mathcal{C}$-approximation. □

Example 11.1.25. Let $\mathcal{A}$ be a locally finitely presented category and suppose that $\mathcal{A}$ is abelian. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\operatorname{fp} \mathcal{A}$, then $(\vec{\mathcal{T}}, \vec{\mathcal{F}})$ is a torsion pair for $\mathcal{A}$.

Proof Each object $X \in \operatorname{fp} \mathcal{A}$ fits into an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$. If $X = \operatorname{colim} X_i$ is written as a filtered colimit of finitely presented objects, then $0 \to \operatorname{colim} X_i' \to X \to \operatorname{colim} X_i'' \to 0$ is the desired exact sequence in $\mathcal{A}$, using that filtered colimits in $\mathcal{A}$ are exact. The formula

$$\operatorname{Hom}(\operatorname*{colim}_i X_i, \operatorname*{colim}_j Y_j) \cong \lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j)$$

then shows that $\operatorname{Hom}(X, Y) = 0$ for $X \in \vec{\mathcal{T}}$ and $Y \in \vec{\mathcal{F}}$. □

Categories of Exact Functors

Let $\mathcal{C}$ be an essentially small additive category and consider $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Suppose that $\mathcal{C}$ is abelian. Then we denote by $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the full subcategory of additive functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that are exact. This category has filtered colimits and products, because exact functors are closed under these operations.

The following lemma identifies the exact functors in the category of left exact functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$.

Lemma 11.1.26. *Let $\mathcal{C}$ be an essentially small abelian category and consider $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Then $X \in \mathcal{A}$ is exact if and only if $\mathrm{Ext}^1_{\mathcal{A}}(C, X) = 0$ for all $C \in \mathrm{fp}\,\mathcal{A}$.*

Proof Using the identification $\mathcal{C} \xrightarrow{\sim} \mathrm{fp}\,\mathcal{A}$, the functor X is exact if and only if for every exact sequence $\eta\colon 0 \to A \to B \to C \to 0$ in $\mathrm{fp}\,\mathcal{A}$ the induced sequence

$$\mathrm{Hom}_{\mathcal{A}}(\eta, X)\colon 0 \to \mathrm{Hom}_{\mathcal{A}}(C, X) \to \mathrm{Hom}_{\mathcal{A}}(B, X) \to \mathrm{Hom}_{\mathcal{A}}(A, X) \to 0$$

is exact.

Now suppose $\mathrm{Ext}^1_{\mathcal{A}}(C, X) = 0$. This implies the exactness of $\mathrm{Hom}_{\mathcal{A}}(\eta, X)$ for any exact $\eta\colon 0 \to A \to B \to C \to 0$ in $\mathrm{fp}\,\mathcal{A}$. Conversely, let $\mu\colon 0 \to X \to Y \to C \to 0$ be exact in $\mathcal{A}$ and write $Y = \mathrm{colim}\, Y_i$ as a filtered colimit of finitely presented objects. This yields an exact sequence $\mu_j\colon 0 \to X_j \to Y_j \to C \to 0$ in $\mathrm{fp}\,\mathcal{A}$ for some j. Now exactness of $\mathrm{Hom}_{\mathcal{A}}(\mu_j, X)$ implies that μ splits. □

The next proposition provides an explicit construction that turns every left exact functor into an exact functor.

Proposition 11.1.27. *Let $\mathcal{C}$ be an essentially small abelian category. Then $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a covariantly finite subcategory of $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.*

Proof Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ be a left exact functor and choose a representative set of monomorphisms $\alpha\colon A \to B$ in $\mathcal{C}$. We construct inductively a sequence

$$F = F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

such that $\mathrm{colim}\, F_n$ is exact and $F \to \mathrm{colim}\, F_n$ is the left approximation of F. Set

$$\Gamma_n = \coprod_{\alpha\colon A \to B} F_n A \setminus \mathrm{Im}\, F_n\alpha.$$

Then Yoneda's lemma yields a morphism $\coprod_{i \in \Gamma_n} \mathrm{Hom}(-, A_i) \to F_n$ and we

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can form the pushout

$$\begin{array}{ccc} \coprod_{i\in\Gamma_n} \operatorname{Hom}(-,A_i) & \xrightarrow{\coprod_i(-,\alpha_i)} & \coprod_{i\in\Gamma_n} \operatorname{Hom}(-,B_i) \\ \downarrow & & \downarrow \\ F_n & \longrightarrow & F_{n+1} \end{array}$$

in $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. It is clear from the construction that $F' = \operatorname{colim} F_n$ is exact, because for any monomorphism $\alpha\colon A \to B$ each element in $F'A = \operatorname{colim} F_n A$ lies in the image of $F_n A \to F'A$ for some n, and therefore also in the image of $F_{n+1}B \to F'B \to F'A$. Now let $F \to G$ be a morphism such that G is exact. Then in each step $F_n \to G$ factors through $F_n \to F_{n+1}$. Thus $F \to G$ factors through $F \to F'$. □

Corollary 11.1.28. *Let* $\mathcal{C}$ *be an essentially small abelian category. Then* $\operatorname{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ *is a covariantly finite subcategory of* $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Proof Observe that $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \subseteq \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is covariantly finite, since the inclusion admits a left adjoint by Lemma 11.1.21. Then for each F in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the unit $F \to F^{\mathrm{Lex}}$ yields a left approximation. This approximation one composes with a left approximation $F^{\mathrm{Lex}} \to F^{\mathrm{Ex}}$ from the preceding proposition. □

Change of Categories

Let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between essentially small additive categories. Then

$$f^*\colon \operatorname{Add}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}) \longrightarrow \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto X \circ f$$

admits a left adjoint $f_!$ that is defined by

$$f_!(X) = \operatorname*{colim}_{(C,x)\in\mathcal{C}/X} \operatorname{Hom}_{\mathcal{D}}(-, f(C))$$

for $X \in \operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. In particular, for $C \in \mathcal{C}$ one has

$$f_!(\operatorname{Hom}_{\mathcal{C}}(-,C)) = \operatorname{Hom}_{\mathcal{D}}(-, f(C)).$$

When $\mathcal{C}$ and $\mathcal{D}$ admit cokernels and $f\colon \mathcal{C} \to \mathcal{D}$ is right exact, this yields an adjoint pair

$$\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \underset{f^*}{\overset{f_!}{\rightleftarrows}} \operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}).$$

We collect some basic properties of f^* and $f_!$.

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Lemma 11.1.29. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor that inverts universally a class of morphisms in $\mathcal{C}$. Then f^* is fully faithful.*

Proof The assertion follows from the definition of the quotient functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ with respect to a class of morphisms S in $\mathcal{C}$; see also Lemma 1.1.1. □

We can be more specific when $f\colon \mathcal{C} \to \mathcal{D}$ is exact.

Lemma 11.1.30. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories. Then $f_!\colon \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ is exact. Moreover, f^* is fully faithful if and only if f induces an equivalence $\mathcal{C}/(\operatorname{Ker} f) \xrightarrow{\sim} \mathcal{D}$.*

Proof We embed $\mathcal{C}$ into $\mathcal{A} = \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ via the Yoneda functor. Any exact sequence in $\mathcal{A}$ can be written as a filtered colimit of exact sequences in $\mathcal{C}$; see Remark 11.1.20. Now use that $f_!$ preserves filtered colimits and that filtered colimits in $\operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ are exact.

We have already seen in Lemma 11.1.29 that f^* is fully faithful when f induces an equivalence $\mathcal{C}/(\operatorname{Ker} f) \xrightarrow{\sim} \mathcal{D}$. For the converse we apply Proposition 2.2.11. Thus $f_!$ induces an equivalence $\mathcal{A}/(\operatorname{Ker} f_!) \xrightarrow{\sim} \operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$. One checks that the subcategory $\mathcal{C} \subseteq \mathcal{A}$ is right cofinal with respect to the morphisms that are inverted by $f_!$, using that each object in $\mathcal{A}$ is a filtered colimit of objects in $\mathcal{C}$. Then it follows from Lemma 1.2.5 that $f_!$ restricts to an equivalence $\mathcal{C}/(\operatorname{Ker} f) \xrightarrow{\sim} \mathcal{D}$. □

Proposition 11.1.31. *Let $\mathcal{C}$ be an essentially small abelian category and $\mathcal{D} = \mathcal{C}/\mathcal{B}$ the quotient with respect to a Serre subcategory $\mathcal{B} \subseteq \mathcal{C}$. Then the diagram*

$$\mathcal{B} \overset{i}{\rightarrowtail} \mathcal{C} \overset{p}{\twoheadrightarrow} \mathcal{D}$$

induces a localisation sequence of abelian categories

$$\operatorname{Lex}(\mathcal{B}^{\mathrm{op}}, \mathrm{Ab}) \underset{i^*}{\overset{i_!}{\rightleftarrows}} \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \underset{p^*}{\overset{p_!}{\rightleftarrows}} \operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}).$$

In particular, the functors $i_!$ and $p_!$ are exact and induce equivalences

$$\operatorname{Lex}(\mathcal{B}^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\sim} \operatorname{Ker} p_!$$

and

$$(\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}))/(\operatorname{Ker} p_!) \xrightarrow{\sim} \operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}).$$

Proof We use the fact that every functor in $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a filtered colimit of representable functors, by Lemma 11.1.8 and Lemma 11.1.14. This is combined with the fact that all functors $i_!$, i^*, $p_!$, p^* preserve colimits.

The functors $i_!$ and $p_!$ are exact by Lemma 11.1.30. The functor p^* is fully

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faithful by Lemma 11.1.29. On the other hand, $\mathrm{id} \cong i^* i_!$ since $i^* i_!$ equals the identity on all representable functors. Thus $i_!$ is fully faithful (Proposition 1.1.3).

It remains to show that $\operatorname{Im} i_! = \operatorname{Ker} p_!$. Then the rest follows from the localisation theory of abelian categories (Proposition 2.2.11).

We have $\operatorname{Im} i_! \subseteq \operatorname{Ker} p_!$ since $pi = 0$. For the other inclusion fix an object X in $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and consider the exact sequence $0 \to X' \to X \to p^* p_!(X)$. We claim that $X' \in \operatorname{Im} i_!$. Then $p_! X = 0$ implies $X \in \operatorname{Im} i_!$. It suffices to show this when $X = h_C$ is representable, given by $C \in \mathcal{C}$. For this we show that every morphism $h_{C_0} \to X'$ with $C_0 \in \mathcal{C}$ factors through h_B for some $B \in \mathcal{B}$; then Lemma 11.1.6 implies that X' is a filtered colimit of representable functors in the image of $i_!$. Now observe that a morphism $h_{C_0} \to h_C$ given by $\phi \colon C_0 \to C$ in $\mathcal{C}$ factors through X' if and only if $p\phi = 0$, by the adjointness of $p_!$ and p^*. This happens if and only if $\operatorname{Im} \phi \in \mathcal{B}$. Thus $h_{C_0} \to X'$ factors through h_B for some $B \in \mathcal{B}$. □

Remark 11.1.32. The injective objects in $\operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ identify via p^* with the injective objects in $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ that vanish on $\mathcal{B}$ when viewed as functors on $\mathcal{C}$ (Corollary 2.2.15).

Corollary 11.1.33. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category such that $\operatorname{fp} \mathcal{A}$ is abelian. If $\mathcal{S} \subseteq \operatorname{fp} \mathcal{A}$ is a Serre subcategory, then $\vec{\mathcal{S}}$ is a localising subcategory of $\mathcal{A}$ satisfying $\vec{\mathcal{S}} \cap \operatorname{fp} \mathcal{A} = \mathcal{C}$. Moreover, the canonical functor $\mathcal{A} \twoheadrightarrow \mathcal{A}/\vec{\mathcal{S}}$ restricts to an equivalence $\mathcal{S}^{\perp} \xrightarrow{\sim} \mathcal{A}/\vec{\mathcal{S}}$.*

Proof This follows from Proposition 11.1.31, using the equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{Lex}((\operatorname{fp} \mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ which identifies the subcategory $\vec{\mathcal{S}}$ with $\operatorname{Lex}(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$. □

11.2 Grothendieck Categories

In this section we study a hierarchy of finiteness conditions for Grothendieck categories. This involves the notion of a generating set of objects.

Given an additive category $\mathcal{A}$, a set of objects $\mathcal{C}$ is *generating* if for any non-zero morphism $\phi \colon X \to Y$ in $\mathcal{A}$ there is $\alpha \colon C \to X$ with $C \in \mathcal{C}$ such that $\phi\alpha \neq 0$. If $\mathcal{A}$ has coproducts, then $\mathcal{C}$ is generating if and only if for every object $X \in \mathcal{A}$ there is an epimorphism $\coprod_{i \in I} C_i \to X$ such that $C_i \in \mathcal{C}$ for all i.

Now fix a Grothendieck category $\mathcal{A}$. We have the following hierarchy of finiteness conditions for an object $X \in \mathcal{A}$:

$$X \text{ of finite length} \implies X \text{ noetherian}$$
$$\implies X \text{ finitely generated} \impliedby X \text{ finitely presented.}$$

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The Grothendieck category $\mathcal{A}$ is called

- *locally finitely generated*, if $\mathcal{A}$ has a generating set of finitely generated objects,
- *locally finitely presented*, if $\mathcal{A}$ has a generating set of finitely presented objects,[2]
- *locally noetherian*, if $\mathcal{A}$ has a generating set of noetherian objects,
- *locally finite*, if $\mathcal{A}$ has a generating set of finite length objects.

Suppose that $\mathcal{A}$ has a set $\mathcal{C}$ of generating objects such that for every pair of objects $C, C' \in \mathcal{C}$ and every subobject $D \subseteq C$ the direct sum $C \oplus C'$ and the quotient C/D are isomorphic to objects in $\mathcal{C}$. Then every object $X \in \mathcal{A}$ can be written as the directed union $X = \sum_i X_i$ of subobjects $X_i \subseteq X$ such that $X_i \in \mathcal{C}$ for all i.

Finitely Generated and Finitely Presented Objects

Let $\mathcal{A}$ be an abelian category, and suppose that filtered colimits in $\mathcal{A}$ are exact. An object X is *finitely generated* whenever $X = \sum_{i \in I} X_i$ for a directed family of subobjects $X_i \subseteq X$ implies $X = X_{i_0}$ for some $i_0 \in I$. We record the following elementary fact.

Lemma 11.2.1. *For an exact sequence* $0 \to X' \to X \to X'' \to 0$ *we have*

$$X', X'' \text{ finitely generated} \implies X \text{ finiteley generated} \\ \implies X'' \text{ finitely generated}. \qquad \square$$

We wish to compare finitely generated and finitely presented objects. Observe that 'finitely generated' is a local property, depending only on the lattice of subobjects. The property of an object to be finitely presented is different; it depends on the ambient category.

We have the following characterisation. In particular, we see that every finitely presented object is finitely generated.

Lemma 11.2.2. *For an object X the following are equivalent.*

(1) *X is finitely generated.*
(2) *The canonical map* $\operatorname{colim}_i \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \operatorname{colim}_i Y_i)$ *is injective for every filtered colimit* $\operatorname{colim}_i Y_i$.

[2] This terminology is consistent: a Grothendieck category $\mathcal{A}$ has a generating set of finitely presented objects if and only if $\operatorname{fp} \mathcal{A}$ is essentially small and every object in $\mathcal{A}$ is a filtered colimit of finitely presented objects.

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(3) *The canonical map* $\sum_i \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \sum_i Y_i)$ *is bijective for every directed family of subobjects* $Y_i \subseteq Y$.

Proof (1) ⇒ (2): A morphism $\phi \in \operatorname{colim}_i \operatorname{Hom}(X, Y_i)$ is given by a morphism $X \to Y_j$ for some index j. For all $j \to i$ consider the composite with $Y_j \to Y_i$ which yields an exact sequence $0 \to X_i \to X \to Y_i$. The colimit $0 \to \operatorname{colim}_i X_i \to X \to \operatorname{colim}_i Y_i$ is exact, and if $X \to \operatorname{colim}_i Y_i$ is zero, then $X = \sum_i X_i$. Thus $X = X_{i_0}$ for some index i_0, and therefore $\phi = 0$.

(2) ⇒ (3): Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{colim}_i \operatorname{Hom}(X, Y_i) & \longrightarrow & \operatorname{Hom}(X, Y) & \longrightarrow & \operatorname{colim}_i \operatorname{Hom}(X, Y/Y_i) \\ & & \downarrow \alpha & & \downarrow \mathrm{id} & & \downarrow \gamma \\ 0 & \longrightarrow & \operatorname{Hom}(X, \operatorname{colim}_i Y_i) & \longrightarrow & \operatorname{Hom}(X, Y) & \longrightarrow & \operatorname{Hom}(X, \operatorname{colim}_i Y/Y_i) \end{array}$$

Then α and γ are injective, and therefore α is bijective.

(3) ⇒ (1): Let $X = \sum_i X_i$. Then the identity $X \to \sum_i X_i$ factors through $X_{i_0} \to \sum_i X_i$ for some index i_0. Thus $X = X_{i_0}$. □

Let $\mathcal{A}$ be a Grothendieck category with a generating set of finitely generated objects. This means that each object is a directed union of its finitely generated subobjects. Also, if $\phi\colon X \to Y$ is an epimorphism such that Y is finitely generated, then there exists a finitely generated subobject $X' \subseteq X$ such that $\phi|_{X'}\colon X' \to Y$ is an epimorphism.

Lemma 11.2.3. *Let* $\mathcal{A}$ *be a Grothendieck category with a generating set of finitely generated objects. For an object* $X \in \mathcal{A}$ *the following are equivalent.*

(1) X *is finitely presented.*

(2) X *is finitely generated and every epimorphism* $X' \to X$ *from a finitely generated object* X' *has a finitely generated kernel.*

Proof (1) ⇒ (2): We have already seen that X is finitely generated. Now fix an epimorphism $\phi\colon X' \to X$ and write $\operatorname{Ker} \phi = \sum_i X_i$ as a directed union of finitely generated subobjects $X_i \subseteq X'$. Then $\operatorname{colim}_i X'/X_i \cong X$, so the identity id_X factors through $X'/X_{i_0} \to X$ for some index i_0. Thus the sequence $0 \to \operatorname{Ker} \phi/X_{i_0} \to X'/X_{i_0} \to X \to 0$ is split exact. It follows that $\operatorname{Ker} \phi/X_{i_0}$ is finitely generated, if X' is finitely generated. Thus $\operatorname{Ker} \phi$ is finitely generated.

(2) ⇒ (1): In view of Lemma 11.2.2, it suffices to show that the canonical map $\operatorname{colim}_i \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \operatorname{colim}_i Y_i)$ is surjective for every filtered colimit $\operatorname{colim}_i Y_i$. Given a morphism $\phi\colon X \to \operatorname{colim}_i Y_i$, we consider the following

pullback.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ \coprod_i Y_i & \xrightarrow{\text{can}} & \operatorname{colim}_i Y_i \end{array}$$

We find a finitely generated subobject $P' \subseteq P$ and an index i_0 such that the pullback restricts to a commutative square

$$\begin{array}{ccc} P' & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ Y_{i_0} & \longrightarrow & \operatorname{colim}_i Y_i \end{array}$$

and π is an epimorphism. Since $\operatorname{Ker}\pi$ is finitely generated, there is an index i_1 such that the composite $\operatorname{Ker}\pi \to P' \to Y_{i_0} \to Y_{i_1}$ is zero, by Lemma 11.2.2. It follows that ϕ factors through $Y_{i_1} \to \operatorname{colim}_i Y_i$, and this yields an element in $\operatorname{colim}_i \operatorname{Hom}(X, Y_i)$ which is mapped to ϕ. □

Locally Noetherian Categories

Let $\mathcal{A}$ be an abelian category. An object in $\mathcal{A}$ is *noetherian* if it satisfies the ascending chain condition on subobjects. We record the following elementary facts.

Lemma 11.2.4. *For an exact sequence* $0 \to X' \to X \to X'' \to 0$ *we have*

$$X', X'' \text{ noetherian} \iff X \text{ noetherian}.$$

If filtered colimits are exact, then an object is noetherian if and only if every subobject is finitely generated. □

A Grothendieck category is called *locally noetherian* if there exists a generating set of noetherian objects. Locally noetherian categories form an important class of locally finitely presented categories.

Proposition 11.2.5. *For a Grothendieck category* $\mathcal{A}$ *the following are equivalent.*

(1) *The category* $\mathcal{A}$ *is locally noetherian.*
(2) *The category* $\mathcal{A}$ *is locally finitely presented and for each* $X \in \mathcal{A}$ *we have*

$$X \text{ finitely presented} \iff X \text{ finitely generated} \iff X \text{ noetherian}.$$

(3) *The category* $\mathcal{A}$ *is locally finitely presented and* $\operatorname{fp}\mathcal{A}$ *is an abelian category consisting of noetherian objects.*

Proof (1) ⇒ (2): Suppose that $\mathcal{A}$ is locally noetherian. Then finitely generated objects and noetherian objects coincide. In particular, finitely generated objects are closed under subobjects. The characterisation of finitely presented objects in Lemma 11.2.3 then implies that finitely generated objects and finitely presented objects coincide. In particular, $\mathcal{A}$ is a locally finitely presented category since every object is a directed union of its finitely generated subobjects, so a filtered colimit of finitely presented objects.

(2) ⇒ (3): Clear.

(3) ⇒ (1): If $\mathcal{A}$ is locally finitely presented, then the finitely presented objects generate $\mathcal{A}$. If an object $X \in \operatorname{fp} \mathcal{A}$ satisfies the ascending chain condition on subobjects in $\operatorname{fp} \mathcal{A}$, then each subobject $U \subseteq X$ in $\mathcal{A}$ is finitely presented since $U = \bigcup_{X' \subseteq U} X'$ where X' runs through all $X' \subseteq X$ in $\operatorname{fp} \mathcal{A}$. Thus X is noetherian in $\mathcal{A}$. □

Corollary 11.2.6. *The assignments* $\mathcal{A} \mapsto \operatorname{fp} \mathcal{A}$ *and* $\mathcal{C} \mapsto \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ *induce, up to equivalence, a bijective correspondence between locally noetherian Grothendieck categories and essentially small abelian categories such that every object is noetherian.*

Proof This correspondence is obtained by restricting the correspondence from Theorem 11.1.15 and Corollary 11.1.19 between locally finitely presented Grothendieck categories and essentially small additive categories. Then apply Proposition 11.2.5 to identify the locally noetherian categories. □

Locally Finite Categories

An object X of an abelian category has *finite length* if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X,$$

that is, each subquotient X_i/X_{i-1} is simple. Note that X has finite length if and only if X satisfies both chain conditions on subobjects.

A Grothendieck category $\mathcal{A}$ is called *locally finite* if there exists a generating set of finite length objects. When $\mathcal{A}$ is a locally finite category, then every noetherian object has finite length, since any object is the directed union of finite length subobjects. Thus for every object $X \in \mathcal{A}$ we have

$$X \text{ finitely presented} \iff X \text{ noetherian} \iff X \text{ of finite length}.$$

Let us discuss some further finiteness properties of locally finite categories. To this end fix an object X of an abelian category. The composition length of X is denoted by $\ell(X)$. The height $\operatorname{ht}(X)$ is the smallest $n \geq 0$ such that

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$\operatorname{soc}^n(X) = X$. When $\ell(X) < \infty$, then $\operatorname{ht}(X) \le \ell(X)$, and $\operatorname{ht}(X)$ equals the smallest $n \ge 0$ such that $\operatorname{rad}^n(X) = 0$.

Lemma 11.2.7. *Let X, E be objects of an abelian category and suppose that E is injective. The assignment*

$$X \supseteq U \longmapsto H(U) := \operatorname{Hom}(X/U, E) \subseteq \operatorname{Hom}(X, E)$$

gives a lattice anti-homomorphism into the lattice of $\operatorname{End}(E)$*-submodules of* $\operatorname{Hom}(X, E)$. *Every finitely generated* $\operatorname{End}(E)$*-submodule is in its image, and the homomorphism is injective when E is a cogenerator.*

Proof Given subobjects U, V of X, the Noether isomorphisms imply that

$$H(U \cap V) = H(U) + H(V) \qquad \text{and} \qquad H(U + V) = H(U) \cap H(V).$$

Clearly, $U \ne V$ implies $H(U) \ne H(V)$ when E is a cogenerator. Now let $\phi\colon X \to E$ be a morphism and set $U = \operatorname{Ker} \phi$. Then a morphism $X \to E$ factors through $X \twoheadrightarrow X/U$ if and only if it factors through ϕ. Thus $\operatorname{End}(E)\phi = H(U)$. It follows that every cyclic $\operatorname{End}(E)$-submodule is in the image of H, and therefore so is every finitely generated submodule by the first part of the proof. □

Proposition 11.2.8. *Let $\mathcal{A}$ be a locally finite Grothendieck category and J the Jacobson radical of the endomorphism ring of an injective object E. Then $\bigcap_{n \ge 0} J^n = 0$. Moreover, for $n \ge 0$ we have*

(1) $\operatorname{ht}(C) \le n$ *for all* $C \in \operatorname{fp} \mathcal{A}$ *implies* $J^n = 0$, *and*
(2) $J^n = 0$ *implies* $\operatorname{ht}(C) \le n$ *for all* $C \in \operatorname{fp} \mathcal{A}$ *when E cogenerates* $\mathcal{A}$.

Proof Let $C \in \operatorname{fp} \mathcal{A}$. A radical morphism $E \to E$ annihilates all simple objects in $\mathcal{A}$, and therefore

$$J^n \operatorname{Hom}(C, E) \subseteq \operatorname{Hom}(C/\operatorname{soc}^n C, E)$$

by induction on n. This implies $\bigcap_{n \ge 0} J^n = 0$ and part (1).

To show (2), assume that E is a cogenerator. An induction on $\ell(C)$ gives

$$\ell_{\operatorname{End}(E)}(\operatorname{Hom}(C, E)) = \ell(C).$$

Thus every submodule of $\operatorname{Hom}(C, E)$ is finitely generated. Then Lemma 11.2.7 implies

$$\operatorname{rad}^n \operatorname{Hom}(C, E) = \operatorname{Hom}(C/\operatorname{soc}^n C, E)$$

for all $n \ge 0$. Observe that $JM = \operatorname{rad} M$ for every $\operatorname{End}(E)$-module M, since $\operatorname{End}(E)/J$ is a product of division rings by Theorem 11.2.12 below. Thus $J^n = 0$ implies $\operatorname{soc}^n C = C$. □

Remark 11.2.9. For $C \in \operatorname{fp}\mathcal{A}$ we have

$$\ell_{\operatorname{End}(E)}(\operatorname{Hom}(C,E)) \le \ell(C) \qquad \text{and} \qquad \operatorname{ht}_{\operatorname{End}(E)}(\operatorname{Hom}(C,E)) \le \operatorname{ht}(C)$$

with equalities when E is a cogenerator.

Injective Objects

In a locally noetherian Grothendieck category we have a very satisfactory decomposition theory for injective objects.

We need some preparations and begin with a version of *Baer's criterion*.

Lemma 11.2.10 (Baer)**.** *Let $\mathcal{A}$ be a Grothendieck category and let $\mathcal{C}$ be a class of objects that is generating and closed under quotients. If $X \in \mathcal{A}$ satisfies* $\operatorname{Ext}^1(C,X) = 0$ *for all $C \in \mathcal{C}$, then X is injective.*

Proof Choose an injective envelope $\alpha\colon X \to E(X)$. If $\operatorname{Coker}\alpha \neq 0$, then there exists a subobject $0 \neq C \subseteq \operatorname{Coker}\alpha$ with $C \in \mathcal{C}$. The pullback of $0 \to X \to E(X) \to \operatorname{Coker}\alpha \to 0$ along the inclusion $C \to \operatorname{Coker}\alpha$ is a split exact sequence. Thus α factors through a monomorphism $X \oplus C \to E(X)$, contradicting the property of an injective envelope. It follows that α is an isomorphism and X is injective. □

We continue with a technical lemma which is crucial for the decomposition of injective objects into indecomposables; it is known as *Chase's lemma*.

For a sequence of morphisms $\gamma = (C_n \to C_{n+1})_{n\in\mathbb{N}}$ we denote by $\gamma_n\colon C_0 \to C_n$ the composite of the first n morphisms. Recall that an object X is *compact* if for any morphism $\phi\colon X \to \coprod_{i\in I} Y_i$ there is a finite set $J \subseteq I$ such that ϕ factors through $\coprod_{i\in J} Y_i$.

Lemma 11.2.11 (Chase)**.** *Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_i)_{i\in I}$ be families of objects in an additive category and let*

$$\phi\colon \prod_{n\in\mathbb{N}} X_n \longrightarrow \coprod_{i\in I} Y_i$$

be a morphism. If $\gamma = (C_n \to C_{n+1})_{n\in\mathbb{N}}$ is a sequence of morphisms and $C = C_0$ is compact, then there exists $m \in \mathbb{N}$ such that for almost all $j \in I$ each composite

$$C \xrightarrow{\gamma_m} C_m \xrightarrow{\theta} \prod_{n\in\mathbb{N}} X_n \xrightarrow{\phi} \coprod_{i\in I} Y_i \twoheadrightarrow Y_j$$

with $\theta_n = 0$ for $n < m$ factors through $\gamma_n\colon C \to C_n$ for all $n \in \mathbb{N}$.

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It is convenient to introduce further notation. For a morphism $\gamma\colon C \to D$ and an object X we denote by X_γ the image of the map

$$\operatorname{Hom}(D,X) \xrightarrow{-\circ\gamma} \operatorname{Hom}(C,X).$$

Then a sequence of morphisms $\gamma = (C_n \to C_{n+1})_{n\in\mathbb{N}}$ yields a descending chain

$$\cdots \subseteq X_{\gamma_2} \subseteq X_{\gamma_1} \subseteq X_{\gamma_0} = \operatorname{Hom}(C_0, X).$$

We can now rephrase the statement of the lemma as follows. Set $X = \prod_{n\in\mathbb{N}} X_n$, $Y = \coprod_{i\in I} Y_i$, and write

$$\phi_i\colon\ \operatorname{Hom}(C,X) \xrightarrow{\phi\circ-} \operatorname{Hom}(C,Y) \longrightarrow \operatorname{Hom}(C,Y_i) \qquad (i\in I).$$

There exists $m \in \mathbb{N}$ such that for almost all $i \in I$ we have

$$\phi_i\Big(\Big(\prod_{n\geq m} X_n\Big)_{\gamma_m}\Big) \subseteq \bigcap_{n\geq 0} (Y_i)_{\gamma_n}.$$

Proof Assume the conclusion to be false. We construct inductively sequences of elements $n_j \in \mathbb{N}$, $i_j \in I$, and $\theta_j \in \operatorname{Hom}(C,X)$ with $j\in\mathbb{N}$ and satisfying

(1) $n_{j+1} > n_j$,
(2) $\theta_j \in (\prod_{n\geq n_j} X_n)_{\gamma_{n_j}}$,
(3) $\phi_{i_j}(\theta_j) \notin (Y_{i_j})_{\gamma_{n_{j+1}}}$,
(4) $\phi_{i_j}(\theta_k) = 0$ for $k < j$.

We proceed as follows. Set $n_0 = 0$. Then there exists $i_0 \in I$ such that

$$\phi_{i_0}(X_{\gamma_0}) \nsubseteq \bigcap_{n\geq 0} (Y_{i_0})_{\gamma_n},$$

and hence we may select $\theta_0 \in X_{\gamma_0}$ and $n_1 > 0$ such that $\phi_{i_0}(\theta_0) \notin (Y_{i_0})_{\gamma_{n_1}}$. Thus conditions (1)–(4) are satisfied for $j = 0$.

Proceeding by induction on j, assume that elements $n_{k+1} \in \mathbb{N}$, $i_k \in I$ and $\theta_k \in \operatorname{Hom}(C,X)$ have been constructed for $k < j$ such that conditions (1)–(4) are satisfied. Using that C is compact, there exists a finite subset $I' \subseteq I$ such that for $i \in I \setminus I'$ we have $\phi_i(\theta_k) = 0$ for $k < j$. We may then select $i_j \in I \setminus I'$ such that

$$\phi_{i_j}\Big(\Big(\prod_{n\geq n_j} X_n\Big)_{\gamma_{n_j}}\Big) \nsubseteq \bigcap_{n\geq 0} (Y_{i_j})_{\gamma_n},$$

because otherwise the lemma would be true. Thus there exists an element $\theta_j \in (\prod_{n\geq n_j} X_n)_{\gamma_{n_j}}$ and $n_{j+1} > n_j$ such that $\phi_{i_j}(\theta_j) \notin (Y_{i_j})_{\gamma_{n_{j+1}}}$. It is then clear that the elements $n_{k+1} \in \mathbb{N}$, $i_k \in I$, and $\theta_k \in \operatorname{Hom}(C,X)$ for $k \leq j$ satisfy the conditions (1)–(4).

Now let $\theta = \sum_{j\in\mathbb{N}} \theta_j \in \operatorname{Hom}(C,X)$, which is well defined since the sum

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for each component $C \to X_n$ is finite. For each $j \in \mathbb{N}$ we have $\phi_{i_j}(\theta) = \phi_{i_j}(\theta_j) + \phi_{i_j}(\sum_{k>j} \theta_k) \neq 0$, since the second summand lies in $(Y_{i_j})_{\gamma_{n_{j+1}}}$, whereas the first does not. On the other hand, the morphism $\phi\theta$ factors through a finite sum $\coprod_{i\in J} Y_i$ for some $J \subseteq I$, since C is compact. This contradiction finishes the proof. □

We have the following characterisation of local noetherianness.

Theorem 11.2.12. *For a locally finitely generated Grothendieck category $\mathcal{A}$ the following are equivalent.*

(1) *The category $\mathcal{A}$ is locally noetherian.*
(2) *The subcategory of injective objects in $\mathcal{A}$ is closed under filtered colimits.*
(3) *The subcategory of injective objects in $\mathcal{A}$ is closed under coproducts.*
(4) *Every injective object decomposes into a coproduct of indecomposable objects with local endomorphism rings.*
(5) *There is an object E such that every object in $\mathcal{A}$ is a subobject of a coproduct of copies of E.*

Proof (1) ⇒ (2): If $\mathcal{A}$ is locally noetherian, then $\operatorname{fp}\mathcal{A}$ is closed under quotients. Thus the equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{Lex}((\operatorname{fp}\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ identifies the injective objects with the exact functors $(\operatorname{fp}\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$, by Lemma 11.1.26 and Lemma 11.2.10. It remains to note that a filtered colimit of exact functors is exact.

(2) ⇒ (3): Clear.

(3) ⇒ (1): Fix an injective cogenerator E and let $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ be an ascending chain of subobjects of a finitely generated object C. Choose morphisms $C/C_i \to E$ for all i such that the restriction to C_{i+1}/C_i is non-zero provided that $C_{i+1}/C_i \neq 0$, and consider for $j \leq n$ the composite $\phi_{nj} \colon C_n \to C \to C/C_j \to E$. For each n these yield a morphism $\phi_n \colon C_n \to \coprod_{i=1}^{n} E$ and we obtain a morphism $\phi\colon \sum_{n\geq 1} C_n \to \coprod_{i\geq 1} E$, since the ϕ_n are compatible. The morphism ϕ extends to a morphism $C \to \coprod_{i\geq 1} E$, since we assume $\coprod_{i\geq 1} E$ to be injective, and this factors through a finite sum $\coprod_{i=1}^{m} E$ for some m, since C is finitely generated. Thus $C_n = C_m$ for $n \geq m$, so C is noetherian.

(2) ⇒ (4): Let $X \neq 0$ be injective and fix a finitely generated subobject $0 \neq C \subseteq X$. Using Zorn's lemma and the fact that injectives are closed under filtered colimits, there exists a maximal injective subobject $X' \subseteq X$ not containing C. Then $X = X' \oplus X''$, and we claim that X'' is indecomposable. For, if $X'' = U \oplus V$, then $(X'+U) \cap (X'+V) = X'$ implies that one of the objects $X'+U$ and $X'+V$ does not contain C. Thus $U = 0$ or $V = 0$ by the maximality of X'.

Using again Zorn's lemma, there exists a maximal family of indecomposable injective subobjects $(X_i)_{i\in I}$ of X such that the sum $X' = \sum_{i\in I} X_i$ is direct. This yields a decomposition $X = X' \oplus X''$, and $X'' = 0$ by the previous observation.

It remains to note that every indecomposable injective object has a local endomorphism ring (Lemma 2.5.7).

(4) ⇒ (5): Let E be the coproduct of indecomposable injective objects, taking one representative from each isomorphism class. Note that there is only a set of such representatives, since each indecomposable injective is the injective envelope of a quotient G/U when G is a generator of $\mathcal{A}$. Then every object in $\mathcal{A}$ is a subobject of a coproduct of copies of E, since every object embeds into an injective object (Corollary 2.5.4).

(5) ⇒ (1): Let $C \in \mathcal{A}$ be a finitely generated object. We wish to show that C is noetherian. To this end fix a chain of finitely generated subobjects $0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ and set $C_n = C/B_n$. This yields a sequence of epimorphisms $\gamma = (C_n \twoheadrightarrow C_{n+1})_{n\in\mathbb{N}}$. For $X \in \mathcal{A}$ we set $X_{\bar{\gamma}_n} = \operatorname{Hom}(B_{n+1}/B_n, X)$ and obtain an exact sequence

$$0 \longrightarrow X_{\gamma_{n+1}} \longrightarrow X_{\gamma_n} \longrightarrow X_{\bar{\gamma}_n} \longrightarrow 0$$

provided that X is injective or a coproduct of injective objects.

Now consider a cogenerator E such that each object of $\mathcal{A}$ embeds into a coproduct of copies of E. We may assume that E is injective by replacing E with its injective envelope. Let $\kappa = \max(\aleph_0, \operatorname{card}\operatorname{Hom}(C, E))$ and choose a monomorphism

$$\phi\colon \prod_{n\in\mathbb{N}} E^{\kappa} \longrightarrow \coprod_{i\in I} E.$$

For each $m \in \mathbb{N}$ we apply $\operatorname{Hom}(C_m, -)$ and obtain a monomorphism

$$\phi_m\colon \prod_{n\in\mathbb{N}} (E_{\gamma_m})^{\kappa} \longrightarrow \coprod_{i\in I} E_{\gamma_m}$$

since $X \mapsto X_{\gamma_m}$ preserves products and coproducts. Then it follows from Lemma 11.2.11 that for some $m \in \mathbb{N}$ the map ϕ_m restricts to an embedding

$$\prod_{n\geq m} (E_{\gamma_m})^{\kappa} \longrightarrow \Big(\coprod_{i\in J} E_{\gamma_\infty}\Big) \amalg \Big(\coprod_{\text{finite}} E_{\gamma_m}\Big)$$

for some cofinite subset $J \subseteq I$, where $E_{\gamma_\infty} = \bigcap_{n\geq 0} E_{\gamma_n}$. Comparing this with ϕ_{m+1} and passing to the quotient yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{n\geq m} (E_{\gamma_{m+1}})^{\kappa} & \longrightarrow & \prod_{n\geq m} (E_{\gamma_m})^{\kappa} & \longrightarrow & \prod_{n\geq m} (E_{\bar{\gamma}_m})^{\kappa} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\coprod_{i\in J} E_{\gamma_\infty}) \amalg (\coprod_{\text{finite}} E_{\gamma_{m+1}}) & \to & (\coprod_{i\in J} E_{\gamma_\infty}) \amalg (\coprod_{\text{finite}} E_{\gamma_m}) & \longrightarrow & \coprod_{\text{finite}} E_{\bar{\gamma}_m} & \longrightarrow & 0
\end{array}$$

where we use the fact that E is injective. The vertical map on the right is

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a monomorphism because it is a restriction of $\mathrm{Hom}(B_{m+1}/B_m, \phi)$. From the choice of κ it follows that $E_{\bar{\gamma}_m} = 0$, cf. Lemma 11.2.13 below. Thus $C_m = C_{m+1}$ since E cogenerates $\mathcal{A}$. We conclude that C is noetherian. □

Lemma 11.2.13. *Let A be an abelian group with $\alpha = \operatorname{card} A$ and let $\kappa \geq \max(\aleph_0, \alpha)$. If there is a monomorphism $A^\kappa \to A^n$ for some $n \in \mathbb{N}$, then $A = 0$.*

Proof Suppose $A \neq 0$. Then we have

$$\operatorname{card}(A^\kappa) = \alpha^\kappa \geq 2^\kappa > \kappa = \kappa^n \geq \alpha^n = \operatorname{card}(A^n).$$

This contradicts the fact that there is an injective map $A^\kappa \to A^n$. □

Remark 11.2.14. The Krull–Remak–Schmidt–Azumaya theorem implies that a decomposition into indecomposable objects with local endomorphism rings is essentially unique (Theorem 2.5.8).

We formulate some consequences of Theorem 11.2.12 and its proof.

Corollary 11.2.15. *Let $\mathcal{C}$ be an essentially small abelian category. Then all exact functors in $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ are injective if and only if all objects in $\mathcal{C}$ are noetherian.* □

A variation of the above theorem will be needed later.

Proposition 11.2.16. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category such that $\operatorname{fp}\mathcal{A}$ is abelian. Suppose that $X \in \mathcal{A}$ is an object satisfying $\operatorname{Ext}^1(C, X) = 0$ for all $C \in \operatorname{fp}\mathcal{A}$ and that $\operatorname{Hom}(C, X) = 0$ implies $C = 0$ for all $C \in \operatorname{fp}\mathcal{A}$. Then the following are equivalent.*

(1) *The category $\mathcal{A}$ is locally noetherian*
(2) *The canonical monomorphism $X^{(\mathbb{N})} \to X^{\mathbb{N}}$ splits.*
(3) *There exists a decomposition $X^{\mathbb{N}} = \coprod_{i \in I} X_i$ such that $\operatorname{End}(X_i)$ is local for all $i \in I$.*
(4) *There exists an object Y such that every product of copies of X is a subobject of a coproduct of copies of Y.*

Moreover, in this case the object X is injective.

Proof (1) ⇒ (2) & (3) & (4): If $\mathcal{A}$ is locally noetherian, then $\operatorname{fp}\mathcal{A}$ is closed under quotients. It follows from Lemma 11.2.10 that X is injective. Now apply Theorem 11.2.12.

(2) ⇒ (1): Choose a splitting $\phi\colon X^{\mathbb{N}} \to X^{(\mathbb{N})}$. Let $C = C_0$ be a finitely presented object and let $\gamma = (C_i \twoheadrightarrow C_{i+1})_{i \in \mathbb{N}}$ be a chain of epimorphisms. We wish to show that C is noetherian and apply Lemma 11.2.11 to ϕ as above.

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Thus $X_{\gamma_m} \subseteq \bigcap_{n\geq 0} X_{\gamma_n}$ for some $m \in \mathbb{N}$, and $C_m = C_{m+1} = \cdots$ follows, since X cogenerates $\operatorname{fp}\mathcal{A}$. We conclude that every object in $\operatorname{fp}\mathcal{A}$ is noetherian.

(3) $\Rightarrow$ (1): First observe that any indecomposable direct summand Y of X occurs, up to isomorphism, an infinite number of times in the family $(X_i)_{i\in I}$, by the Krull–Remak–Schmidt–Azumaya theorem (Theorem 2.5.8). Now let $\gamma = (C_i \twoheadrightarrow C_{i+1})_{i\in\mathbb{N}}$ be a chain of epimorphisms in $\operatorname{fp}\mathcal{A}$. Then it follows from Lemma 11.2.11 that there exists $m \in \mathbb{N}$ such that $Y_{\gamma_m} \subseteq \bigcap_{n\in\mathbb{N}} Y_{\gamma_n}$ for all indecomposable direct summands Y of X. Therefore $X_{\gamma_m} \subseteq \bigcap_{n\in\mathbb{N}} X_{\gamma_n}$, and $C_m = C_{m+1} = \cdots$ follows, since X cogenerates $\operatorname{fp}\mathcal{A}$. Thus every object in $\operatorname{fp}\mathcal{A}$ is noetherian.

(4) $\Rightarrow$ (1): Adapt the proof of Theorem 11.2.12, keeping in mind that X cogenerates $\operatorname{fp}\mathcal{A}$. □

11.3 Gröbner Categories

Given an essentially small category $\mathcal{C}$, we study the problem when for any locally noetherian Grothendieck category $\mathcal{A}$ the functor category $\operatorname{Fun}(\mathcal{C},\mathcal{A})$ is again locally noetherian. This problem is motivated by Hilbert's basis theorem and leads to the notion of a Gröbner category.

Hilbert's Basis Theorem

Let A be a (not necessarily commutative) ring and denote by $A[t]$ the polynomial ring in one variable. We can identify modules over $A[t]$ with pairs (X,ϕ) given by an A-module X and a morphism $\phi\colon X \to X$ that sends $x \in X$ to xt.

We view the set of non-negative integers as a category $\bar{\mathbb{N}}$ with a single object $*$, morphisms given by $\operatorname{Hom}(*,*) = \mathbb{N}$, and composition given by addition. Then there is an obvious equivalence

$$\operatorname{Fun}(\bar{\mathbb{N}}, \operatorname{Mod} A) \xrightarrow{\sim} \operatorname{Mod} A[t]$$

which sends a functor $F\colon \bar{\mathbb{N}} \to \operatorname{Mod} A$ to $F(*)$.

Now consider the partially ordered set of non-negative integers as a category $\vec{\mathbb{N}}$ with set of objects $\mathbb{N}$ and a single morphism $m \to n$ if and only if $m \leq n$. We view $A[t] = \bigoplus_{n\geq 0} A[t]_n$ as an $\mathbb{N}$-graded ring where $A[t]_n$ denotes the set of homogeneous polynomials of degree n. If we denote by $\operatorname{GrMod} A[t]$ the category of $\mathbb{N}$-graded $A[t]$-modules (with degree zero morphisms), then there is an obvious equivalence

$$\operatorname{Fun}(\vec{\mathbb{N}}, \operatorname{Mod} A) \xrightarrow{\sim} \operatorname{GrMod} A[t]$$

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which sends a functor $F\colon \vec{\mathbb{N}} \to \operatorname{Mod} A$ to $\bigoplus_{n\geq 0} F(n)$.

The following is a reformulation of *Hilbert's basis theorem.*

Theorem 11.3.1 (Hilbert)**.** *Let A be a right noetherian ring. Then the polynomial ring $A[t]$ is right noetherian; it is also right noetherian as a graded ring. Therefore* $\operatorname{Mod} A[t]$ *and* $\operatorname{GrMod} A[t]$ *are both locally noetherian Grothendieck categories.* □

Noetherian Posets

Let $\mathcal{C}$ be a poset. A subset $\mathcal{D} \subseteq \mathcal{C}$ is an *ideal* if the conditions $x \leq y$ in $\mathcal{C}$ and $y \in \mathcal{D}$ imply $x \in \mathcal{D}$. The ideals in $\mathcal{C}$ are partially ordered by inclusion.

A poset $\mathcal{C}$ is *noetherian* if every ascending chain of elements in $\mathcal{C}$ stabilises, and $\mathcal{C}$ is *strongly noetherian* if every ascending chain of ideals in $\mathcal{C}$ stabilises.

For a poset $\mathcal{C}$ and $x \in \mathcal{C}$, set $\mathcal{C}(x) = \{t \in \mathcal{C} \mid t \leq x\}$. The assignment $x \mapsto \mathcal{C}(x)$ yields an embedding of $\mathcal{C}$ into the poset of ideals in $\mathcal{C}$.

Lemma 11.3.2. *For a poset $\mathcal{C}$ the following are equivalent.*

(1) *The poset $\mathcal{C}$ is strongly noetherian.*
(2) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there exists $i \in \mathbb{N}$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$.*
(3) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there is a map $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\alpha(i) < \alpha(j)$ and $x_{\alpha(j)} \leq x_{\alpha(i)}$.*
(4) *For every infinite sequence $(x_i)_{i\in\mathbb{N}}$ of elements in $\mathcal{C}$ there are $i < j$ in $\mathbb{N}$ such that $x_j \leq x_i$.*

Proof (1) ⇒ (2): Suppose that $\mathcal{C}$ is strongly noetherian and let $(x_i)_{i\in\mathbb{N}}$ be elements in $\mathcal{C}$. For $n \in \mathbb{N}$ set $\mathcal{C}_n = \bigcup_{i\leq n} \mathcal{C}(x_i)$. The chain $(\mathcal{C}_n)_{n\in\mathbb{N}}$ stabilises, say $\mathcal{C}_n = \mathcal{C}_N$ for all $n \geq N$. Thus there exists $i \leq N$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$.

(2) ⇒ (3): Define $\alpha\colon \mathbb{N} \to \mathbb{N}$ recursively by taking for $\alpha(0)$ the smallest $i \in \mathbb{N}$ such that $x_j \leq x_i$ for infinitely many $j \in \mathbb{N}$. For $n > 0$ set

$$\alpha(n) = \min\{i > \alpha(n-1) \mid x_j \leq x_i \leq x_{\alpha(n-1)} \text{ for infinitely many } j \in \mathbb{N}\}.$$

(3) ⇒ (4): Clear.

(4) ⇒ (1): Suppose there is a properly ascending chain $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of ideals in $\mathcal{C}$. Choose $x_n \in \mathcal{C}_{n+1} \setminus \mathcal{C}_n$ for each $n \in \mathbb{N}$. There are $i < j$ in $\mathbb{N}$ such that $x_j \leq x_i$. This implies $x_j \in \mathcal{C}_{i+1} \subseteq \mathcal{C}_j$ which is a contradiction. □

Functor Categories

Let $\mathcal{C}$ be an essentially small category. We simplify the notation by setting

$$\mathcal{C}(x, y) := \mathrm{Hom}_{\mathcal{C}}(x, y) \qquad \text{for objects } x, y \in \mathcal{C}.$$

For a Grothendieck category $\mathcal{A}$ we denote by $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. The morphisms between two functors are the natural transformations. Note that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ is a Grothendieck category.

Given an object $x \in \mathcal{C}$, the evaluation functor

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}) \longrightarrow \mathcal{A}, \quad F \mapsto F(x)$$

admits a left adjoint

$$\mathcal{A} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}), \quad M \mapsto M[\mathcal{C}(-, x)]$$

where for any set X we denote by $M[X]$ a coproduct of copies of M indexed by the elements of X. Thus we have for objects $M \in \mathcal{A}$ and $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ a natural isomorphism

$$\mathrm{Hom}(M[\mathcal{C}(-, x)], F) \cong \mathrm{Hom}(M, F(x)). \tag{11.3.3}$$

Lemma 11.3.4. *Let $(M_i)_{i \in I}$ be a set of generators of $\mathcal{A}$. Then the functors $M_i[\mathcal{C}(-, x)]$ with $i \in I$ and $x \in \mathcal{C}$ generate* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.

Proof Use the adjointness isomorphism (11.3.3). □

Recall that a Grothendieck category $\mathcal{A}$ is *locally noetherian* if $\mathcal{A}$ has a generating set of noetherian objects. In that case an object $M \in \mathcal{A}$ is noetherian if and only if M is *finitely presented*, that is, the representable functor $\mathrm{Hom}(M, -)$ preserves filtered colimits; see Proposition 11.2.5

Lemma 11.3.5. *Let $\mathcal{A}$ be locally noetherian. Then* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian if and only if $M[\mathcal{C}(-, x)]$ is noetherian for every noetherian $M \in \mathcal{A}$ and $x \in \mathcal{C}$.*

Proof First observe that $M[\mathcal{C}(-, x)]$ is finitely presented if M is finitely presented. This follows from the isomorphism (11.3.3) since evaluation at $x \in \mathcal{C}$ preserves colimits. Now the assertion of the lemma is an immediate consequence of Lemma 11.3.4. □

Noetherian Functors

Let $\mathcal{C}$ be a small category and fix an object $x \in \mathcal{C}$. Set

$$\mathcal{C}(x) := \coprod_{t \in \mathcal{C}} \mathcal{C}(t, x).$$

Given $f, g \in \mathcal{C}(x)$, let $\langle f\rangle$ denote the set of morphisms in $\mathcal{C}(x)$ that factor through f, and set $f \leq_x g$ if $\langle f\rangle \subseteq \langle g\rangle$. We identify f and g when $\langle f\rangle = \langle g\rangle$. This yields a poset which we denote by $\bar{\mathcal{C}}(x)$.

A functor is *noetherian* if every ascending chain of subfunctors stabilises.

Lemma 11.3.6. *The functor* $\mathcal{C}(-,x)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *is noetherian if and only if the poset* $\bar{\mathcal{C}}(x)$ *is strongly noetherian.*

Proof Sending $F \subseteq \mathcal{C}(-,x)$ to $\bigcup_{t\in\mathcal{C}} F(t)$ induces an inclusion preserving bijection between the subfunctors of $\mathcal{C}(-,x)$ and the ideals in $\bar{\mathcal{C}}(x)$. □

For a poset $\mathcal{T}$ let $\mathrm{Set} \wr \mathcal{T}$ denote the category consisting of pairs (X, ξ) given by a set X and a map $\xi\colon X \to \mathcal{T}$. A morphism $(X, \xi) \to (X', \xi')$ is a map $f\colon X \to X'$ such that $\xi(a) \leq \xi' f(a)$ for all $a \in X$.

A functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$ is given by a pair (F, ϕ) consisting of a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and a map $\phi\colon \coprod_{t\in\mathcal{C}} F(t) \to \mathcal{T}$ such that $\phi(a) \leq \phi(F(f)(a))$ for every $a \in F(t)$ and $f\colon t' \to t$ in $\mathcal{C}$.

Lemma 11.3.7. *Let* $\mathcal{T}$ *be a noetherian poset. If the functor* $\mathcal{C}(-,x)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ *is noetherian, then every functor* $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$ *whose composite with the canonical functor* $\mathrm{Set} \wr \mathcal{T} \to \mathrm{Set}$ *equals* $\mathcal{C}(-,x)$ *is also noetherian.*

Proof Fix a functor $(F, \phi)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \wr \mathcal{T}$, and let $(F_n, \phi_n)_{n\in\mathbb{N}}$ be a strictly ascending chain of subfunctors of (F, ϕ). The chain $(F_n)_{n\in\mathbb{N}}$ stabilises since $\mathcal{C}(-,x)$ is noetherian. Thus we may assume that $F_n = F$ for all $n \in \mathbb{N}$, and we find $f_n \in \coprod_{t\in\mathcal{C}} F(t)$ such that $\phi_n(f_n) < \phi_{n+1}(f_n)$. The poset $\bar{\mathcal{C}}(x)$ is strongly noetherian by Lemma 11.3.6. It follows from Lemma 11.3.2 that there is a map $\alpha\colon \mathbb{N} \to \mathbb{N}$ such that $i < j$ implies $\alpha(i) < \alpha(j)$ and $f_{\alpha(j)} \leq_x f_{\alpha(i)}$. Thus

$$\phi_{\alpha(n)}(f_{\alpha(n)}) < \phi_{\alpha(n)+1}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n)}) \leq \phi_{\alpha(n+1)}(f_{\alpha(n+1)}).$$

This yields a strictly ascending chain in $\mathcal{T}$, contradicting the assumption on $\mathcal{T}$ to be noetherian. □

A partial order $\leq$ on $\mathcal{C}(x)$ is *admissible* if the following holds.

(Ad1) The order $\leq$ restricted to $\mathcal{C}(t,x)$ is total and noetherian for every $t \in \mathcal{C}$.

(Ad2) For $f, f' \in \mathcal{C}(t,x)$ and $e \in \mathcal{C}(s,t)$, the condition $f < f'$ implies $fe < f'e$.

Assume there is given an admissible partial order $\leq$ on $\mathcal{C}(x)$ and an object M in a Grothendieck category $\mathcal{A}$. Let $\mathrm{Sub}(M)$ denote the poset of subobjects of M and consider the functor

$$\mathcal{C}(-,x) \wr M\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \mathrm{Sub}(M), \quad t \mapsto \big(\mathcal{C}(t,x), (M)_{f\in\mathcal{C}(t,x)}\big).$$

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For a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ define a subfunctor $\tilde{F} \subseteq \mathcal{C}(-,x) \wr M$ as follows:

$$\tilde{F}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \wr \mathrm{Sub}(M), \quad t \mapsto \Big(\mathcal{C}(t,x), \big(\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t))\big)_{f \in \mathcal{C}(t,x)}\Big)$$

where $\mathcal{C}(t,x)_f = \{g \in \mathcal{C}(t,x) \mid f \leq g\}$ and $\pi_f\colon M[\mathcal{C}(t,x)_f] \to M$ is the projection onto the factor corresponding to f. For a morphism $e\colon t' \to t$ in $\mathcal{C}$, the morphism $\tilde{F}(e)$ is induced by precomposition with e. Note that

$$\pi_f(M[\mathcal{C}(t,x)_f] \cap F(t)) \subseteq \pi_{fe}(M[\mathcal{C}(t',x)_{fe}] \cap F(t'))$$

since $\leq$ is compatible with the composition in $\mathcal{C}$.

Lemma 11.3.8. *Suppose there is an admissible partial order on $\mathcal{C}(x)$. Then the assignment which sends a subfunctor $F \subseteq M[\mathcal{C}(-,x)]$ to $\tilde{F}$ preserves proper inclusions. Therefore $M[\mathcal{C}(-,x)]$ is noetherian provided that $\mathcal{C}(-,x) \wr M$ is noetherian.*

Proof Let $F \subseteq G \subseteq M[\mathcal{C}(-,x)]$. Then $\tilde{F} \subseteq \tilde{G}$. Now suppose that $F \neq G$. Thus there exists $t \in \mathcal{C}$ such that $F(t) \neq G(t)$. We have $\mathcal{C}(t,x) = \bigcup_{f \in \mathcal{C}(t,x)} \mathcal{C}(t,x)_f$, and this union is directed since $\leq$ is total. Thus

$$F(t) = \sum_{f \in \mathcal{C}(t,x)} \big(M[\mathcal{C}(t,x)_f] \cap F(t)\big)$$

since filtered colimits in $\mathcal{A}$ are exact. This yields f such that

$$M[\mathcal{C}(t,x)_f] \cap F(t) \neq M[\mathcal{C}(t,x)_f] \cap G(t).$$

Choose $f \in \mathcal{C}(t,x)$ maximal with respect to this property, using that $\leq$ is noetherian. Now observe that the projection π_f induces an exact sequence

$$0 \longrightarrow \sum_{f<g} \big(M[\mathcal{C}(t,x)_g] \cap F(t)\big) \longrightarrow F(t) \longrightarrow \pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \longrightarrow 0$$

since the kernel of π_f equals the directed union $\sum_{f<g} M[\mathcal{C}(t,x)_g]$. For the directedness one uses again that $\leq$ is total. Thus

$$\pi_f\big(M[\mathcal{C}(t,x)_f] \cap F(t)\big) \neq \pi_f\big(M[\mathcal{C}(t,x)_f] \cap G(t)\big)$$

and therefore $\tilde{F} \neq \tilde{G}$. □

Proposition 11.3.9. *Let $x \in \mathcal{C}$. Suppose that $\mathcal{C}(-,x)$ is noetherian and that $\mathcal{C}(x)$ has an admissible partial order. If $M \in \mathcal{A}$ is noetherian, then $M[\mathcal{C}(-,x)]$ is noetherian.*

Proof Combine Lemma 11.3.7 and Lemma 11.3.8. □

Gröbner Categories

A small category $\mathcal{C}$ is a *Gröbner category* if the following holds.

(Gr1) The functor $\mathcal{C}(-,x)$ is noetherian for every $x \in \mathcal{C}$.
(Gr2) There is an admissible partial order on $\mathcal{C}(x)$ for every $x \in \mathcal{C}$.

Theorem 11.3.10. *Let $\mathcal{C}$ be a Gröbner category and $\mathcal{A}$ a Grothendieck category. If $\mathcal{A}$ is locally noetherian, then* $\operatorname{Fun}(\mathcal{C}^{\text{op}}, \mathcal{A})$ *is locally noetherian.*

Proof Combine Lemma 11.3.4 and Proposition 11.3.9. □

Example 11.3.11. A strongly noetherian poset (viewed as a category) is a Gröbner category.

Example 11.3.12. Consider the additive monoid $\bar{\mathbb{N}}$ of non-negative integers, viewed as a category with a single object, and the poset $\vec{\mathbb{N}}$ of non-negative integers, again viewed as a category. Then $\bar{\mathbb{N}}^{\text{op}}$ and $\vec{\mathbb{N}}^{\text{op}}$ are Gröbner categories. Let $\mathcal{A}$ be the module category of a right noetherian ring A. Then $\operatorname{Fun}(\bar{\mathbb{N}}, \mathcal{A})$ and $\operatorname{Fun}(\vec{\mathbb{N}}, \mathcal{A})$ identify with categories of modules over the polynomial ring in one variable over A (ungraded and graded). Thus Theorem 11.3.10 generalises Hilbert's basis theorem (Theorem 11.3.1).

Base Change

Given functors $F, G \colon \mathcal{C}^{\text{op}} \to \text{Set}$, we write $F \rightsquigarrow G$ if there is a finite chain

$$F = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = G$$

of epimorphisms and monomorphisms of functors $\mathcal{C}^{\text{op}} \to \text{Set}$.

A functor $\phi \colon \mathcal{C} \to \mathcal{D}$ is *contravariantly finite* if the following holds.

(Con1) Every object $y \in \mathcal{D}$ is isomorphic to $\phi(x)$ for some $x \in \mathcal{C}$.
(Con2) For every object $y \in \mathcal{D}$ there are objects $x_1, \dots, x_n$ in $\mathcal{C}$ such that

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) \rightsquigarrow \mathcal{D}(\phi -, y).$$

The functor ϕ is *covariantly finite* if $\phi^{\text{op}} \colon \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is contravariantly finite.

Note that a composite of contravariantly finite functors is contravariantly finite.

Lemma 11.3.13. *Let $f \colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a Grothendieck category. Fix $M \in \mathcal{A}$ and suppose that $M[\mathcal{C}(-,x)]$ is noetherian for all $x \in \mathcal{C}$. Then $M[\mathcal{D}(-,y)]$ is noetherian for all $y \in \mathcal{D}$.*

Proof A finite chain

$$\coprod_{i=1}^{n} \mathcal{C}(-, x_i) = F_0 \twoheadrightarrow F_1 \leftarrowtail F_2 \twoheadrightarrow \cdots \twoheadrightarrow F_{n-1} \leftarrowtail F_n = \mathcal{D}(\phi-, y)$$

of epimorphisms and monomorphisms induces a chain

$$\coprod_{i=1}^{n} M[\mathcal{C}(-, x_i)] = \bar{F}_0 \twoheadrightarrow \bar{F}_1 \leftarrowtail \bar{F}_2 \twoheadrightarrow \cdots \twoheadrightarrow \bar{F}_{n-1} \leftarrowtail \bar{F}_n = M[\mathcal{D}(\phi-, y)]$$

of epimorphisms and monomorphisms in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. Thus $M[\mathcal{D}(\phi-, y)]$ is noetherian. It follows that $M[\mathcal{D}(-, y)]$ is noetherian, since precomposition with ϕ yields a faithful and exact functor $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$. □

Proposition 11.3.14. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a contravariantly finite functor and $\mathcal{A}$ a locally noetherian Grothendieck category. If the category* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian, then* $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof Combine Lemma 11.3.5 and Lemma 11.3.13. □

Categories of Finite Sets

Let Γ denote the category of finite sets; a skeleton is given by the sets $\mathbf{n} = \{1, 2, \dots, n\}$. The subcategory of finite sets with surjective morphisms is denoted by Γ_{sur}. A surjection $f\colon \mathbf{m} \to \mathbf{n}$ is *ordered* if $i < j$ implies $\min f^{-1}(i) < \min f^{-1}(j)$. We write Γ_{os} for the subcategory of finite sets whose morphisms are ordered surjections. Given a surjection $f\colon \mathbf{m} \to \mathbf{n}$, let $f^!\colon \mathbf{n} \to \mathbf{m}$ denote the map given by $f^!(i) = \min f^{-1}(i)$. Note that $ff^! = \mathrm{id}$, and $gf = f^! g^!$ provided that f and g are ordered surjections.

Lemma 11.3.15. *The inclusions $\Gamma_{\mathrm{os}} \to \Gamma_{\mathrm{sur}}$ and $\Gamma_{\mathrm{sur}} \to \Gamma$ are both contravariantly finite.*

Proof For each integer $n \geq 0$ there is an isomorphism

$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \xrightarrow{\sim} \Gamma_{\mathrm{sur}}(-, \mathbf{n})$$

which sends a pair (f, σ) to σf. The inverse sends a surjective map $g\colon \mathbf{m} \to \mathbf{n}$ to $(\tau^{-1} g, \tau)$ where $\tau \in \mathfrak{S}_n$ is the unique permutation such that $g^! \tau$ is increasing.

For each integer $n \geq 0$ there is an isomorphism

$$\coprod_{\mathbf{m} \hookrightarrow \mathbf{n}} \Gamma_{\mathrm{sur}}(-, \mathbf{m}) \xrightarrow{\sim} \Gamma(-, \mathbf{n})$$

which is induced by the injective maps $\mathbf{m} \to \mathbf{n}$. □

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Fix an integer $n \geq 0$. Given $f, g \in \Gamma(\mathbf{n})$ we set $f \leq g$ if there exists an ordered surjection h such that $f = gh$.

Lemma 11.3.16. *The poset* $(\Gamma(\mathbf{n}), \leq)$ *is strongly noetherian.*

Proof We fix some notation for each $f \in \Gamma(\mathbf{m}, \mathbf{n})$. Set $\lambda(f) = m$. If f is not injective, set

$$\mu(f) = m - \max\{i \in \mathbf{m} \mid \text{there exists } j < i \text{ such that } f(i) = f(j)\}$$

and $\pi(f) = f(m - \mu(f))$. Define $\tilde{f} \in \Gamma(\mathbf{m} - \mathbf{1}, \mathbf{n})$ by setting $\tilde{f}(i) = f(i)$ for $i < m - \mu(f)$ and $\tilde{f}(i) = f(i + 1)$ otherwise.

Note that $f \leq \tilde{f}$. Moreover, $\mu(f) = \mu(g)$, $\pi(f) = \pi(g)$, and $\tilde{f} \leq \tilde{g}$ imply $f \leq g$.

Suppose that $(\Gamma(\mathbf{n}), \leq)$ is not strongly noetherian. Then there exists an infinite sequence $(f_r)_{r \in \mathbb{N}}$ in $\Gamma(\mathbf{n})$ such that $i < j$ implies $f_j \not\leq f_i$; see Lemma 11.3.2. Call such a sequence *bad*. Choose the sequence *minimal* in the sense that $\lambda(f_i)$ is minimal for all bad sequences $(g_r)_{r \in \mathbb{N}}$ with $g_j = f_j$ for all $j < i$. There is an infinite subsequence $(f_{\alpha(r)})_{r \in \mathbb{N}}$ (given by some increasing map $\alpha \colon \mathbb{N} \to \mathbb{N}$) such that μ and π agree on all $f_{\alpha(r)}$, since the values of μ and π are bounded by n. Now consider the sequence $f_0, f_1, \ldots, f_{\alpha(0)-1}, \tilde{f}_{\alpha(0)}, \tilde{f}_{\alpha(1)}, \ldots$ and denote this by $(g_r)_{r \in \mathbb{N}}$. This sequence is not bad, since $(f_r)_{r \in \mathbb{N}}$ is minimal. Thus there are $i < j$ in $\mathbb{N}$ with $g_j \leq g_i$. Clearly, $j < \alpha(0)$ is impossible. If $i < \alpha(0)$, then

$$f_{\alpha(j-\alpha(0))} \leq \tilde{f}_{\alpha(j-\alpha(0))} = g_j \leq g_i = f_i,$$

which is a contradiction, since $i < \alpha(0) \leq \alpha(j - \alpha(0))$. If $i \geq \alpha(0)$, then $f_{\alpha(j-\alpha(0))} \leq f_{\alpha(i-\alpha(0))}$; this is a contradiction again. Thus $(\Gamma(\mathbf{n}), \leq)$ is strongly noetherian. □

Proposition 11.3.17. *The category* Γ_{os} *is a Gröbner category.*

Proof Fix an integer $n \geq 0$. The poset $\bar{\Gamma}_{\mathrm{os}}(\mathbf{n})$ is strongly noetherian by Lemma 11.3.16, and it follows from Lemma 11.3.6 that the functor $\Gamma_{\mathrm{os}}(-, \mathbf{n})$ is noetherian.

The admissible partial order on $\Gamma_{\mathrm{os}}(\mathbf{n})$ is given by the lexicographic order. Thus for $f, g \in \Gamma_{\mathrm{os}}(\mathbf{m}, \mathbf{n})$, we have $f < g$ if there exists $j \in \mathbf{m}$ with $f(j) < g(j)$ and $f(i) = g(i)$ for all $i < j$. □

Theorem 11.3.18. *Let* $\mathcal{A}$ *be a locally noetherian Grothendieck category. Then the category* $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ *is locally noetherian.*

Proof The category Γ_{os} is a Gröbner category by Proposition 11.3.17. It follows from Theorem 11.3.10 that $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian. The inclusion $\Gamma_{\mathrm{os}} \to \Gamma$ is contravariantly finite by Lemma 11.3.15. Thus $\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 11.3.14. □

FI-Modules

Let Γ_{inj} denote the category whose objects are finite sets and whose morphisms are injective maps. When $\mathcal{A}$ is the category of modules over a ring, then a functor $\Gamma_{\mathrm{inj}} \to \mathcal{A}$ is called an *FI-module* (F = finite sets, I = injective maps).

Theorem 11.3.19. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then the category* $\mathrm{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ *is locally noetherian.*

Proof Consider the functor $\phi\colon \Gamma_{\mathrm{os}} \to (\Gamma_{\mathrm{inj}})^{\mathrm{op}}$ which is the identity on objects and takes a map $f\colon \mathbf{m} \to \mathbf{n}$ to $f^!\colon \mathbf{n} \to \mathbf{m}$ given by $f^!(i) = \min f^{-1}(i)$. This functor is contravariantly finite, since for each integer $n \geq 0$ the morphism

$$\Gamma_{\mathrm{os}}(-, \mathbf{n}) \times \mathfrak{S}_n \longrightarrow \Gamma_{\mathrm{inj}}(\mathbf{n}, \phi-)$$

which sends a pair (f, σ) to $f^!\sigma$ is an epimorphism.

It follows from Proposition 11.3.14 that the category $\mathrm{Fun}(\Gamma_{\mathrm{inj}}, \mathcal{A})$ is locally noetherian, since $\mathrm{Fun}((\Gamma_{\mathrm{os}})^{\mathrm{op}}, \mathcal{A})$ is locally noetherian by Proposition 11.3.17 and Theorem 11.3.10. □

Generic Representations

Let A be a ring. We denote by $\mathcal{F}(A)$ the category of finitely generated free A-modules. Note that $\mathcal{F}(A)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{F}(A^{\mathrm{op}})$. Now fix the module category $\mathcal{A} = \mathrm{Mod}\, k$ of a commutative ring k. Then a functor $F\colon \mathcal{F}(A) \to \mathcal{A}$ yields a family $F(A^n)$ of k-linear representations of $\mathrm{GL}_n(A)$ for $n \geq 0$ via evaluation; so one calls F a *generic representation* of A. In fact, F is equivalent to a compatible family of k-linear representations of $M_n(A)$, where $M_n(A)$ denotes the semigroup of all $n \times n$ matrices over A.

Suppose that A is *finite*, that is, the underlying set has finite cardinality. Then the functor $\Gamma \to \mathcal{F}(A)$ sending X to $A[X]$ is a left adjoint of the forgetful functor $\mathcal{F}(A) \to \Gamma$.

Lemma 11.3.20. *Let A be finite. Then the functor $\Gamma \to \mathcal{F}(A)$ is contravariantly finite.*

Proof The assertion follows from the adjointness isomorphism

$$\mathcal{F}(A)(A[X], P) \cong \Gamma(X, P). \qquad \square$$

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Theorem 11.3.21. *Let A be a finite ring and $\mathcal{A}$ a locally noetherian Grothendieck category. Then the category* $\operatorname{Fun}(\mathcal{F}(A), \mathcal{A})$ *is locally noetherian.*

Proof We reduce the assertion about $\mathcal{F}(A)$ to the category of finite sets, using Lemma 11.3.20 and Proposition 11.3.14. Then one applies Theorem 11.3.18. □

There is the following immediate consequence, also known as the *artinian conjecture*, because it amounts to the fact that the standard injective objects are artinian.

Corollary 11.3.22. *For a finite field $\mathbb{F}$ the category of generic representations* $\operatorname{Fun}(\operatorname{mod}\mathbb{F}, \operatorname{Mod}\mathbb{F})$ *is locally noetherian.* □

Notes

Locally finitely presented categories were introduced by Gabriel and Ulmer [84]. For the special case of abelian categories and the properties of injective objects, see Gabriel's thesis [79]. In particular, that work contains the idea of using categories of left exact functors. The decomposition theory of injective objects in locally noetherian categories goes back to results for modules by Matlis [143] and Papp [155]; see also the exposition of Roos [177, 178].

Chase's lemma appears as an argument in [49] and is formulated explicitly in [50].

In a seminal paper Mitchell pointed out the parallel between modules and additive functors, introducing the term *ring with several objects* for a preadditive category [145].

The concept of a Gröbner category and the corresponding generalisation of Hilbert's basis theorem [112] is due to Richter [169] and was rediscovered by Sam and Snowden [180]. In particular, [180] contains a proof of the artinian conjecture. Lannes and Schwartz formulated this conjecture and were motivated by their study of unstable modules over the Steeenrod algebra [109]. The fact that FI-modules over a noetherian ring form a locally noetherian category is due to Church, Ellenberg, Farb and Nagpal [51]. Our exposition follows notes of Djament [63] which are motivated by applications to generic representation theory; see also the expository articles by Kuhn, Powell and Schwartz in [133].