J. Aust. Math. Soc. **97** (2014), 127–143
doi:10.1017/S1446788714000196

SOME REMARKS ON THE Q CURVATURE TYPE PROBLEM ON $\mathbb{S}^N$

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(Received 5 April 2013; accepted 22 April 2014; first published online 16 June 2014)

Communicated by A. Hassell

Abstract

In this paper, we prove the existence, uniqueness and multiplicity of positive solutions of a nonlinear perturbed fourth-order problem related to the Q curvature.

2010 *Mathematics subject classification*: primary 35J10; secondary 35J35, 35J65.

Keywords and phrases: fourth-order problem, critical exponent, uniqueness, multiplicity.

1. Introduction

In recent years, there has been an intensive study of the relationship between conformally covariant operators and partial differential equations. See some recent survey papers by Chang [8] and Chang and Yang [10]. Given a smooth four-dimensional compact Riemannian manifold (M, g), let R_g and Ric_g be the scalar curvature and the Ricci curvature of g, respectively, div_g the divergence operator and d the de Rham differential; then the Paneitz operator is defined in the following way:

$$P_g\psi = \Delta_g^2\psi - div_g(\tfrac{2}{3}R_g - 2Ric_g)\, d\psi;$$

see Paneitz [22]. For the case $N \geq 5$, the Paneitz operator P_g is defined by

$$P_g = \Delta_g^2 - div_g[a_N R_g g + b_N Ric_g] + \frac{N-4}{2}Q_g.$$

Here

$$Q_g = \frac{1}{2(N-1)}\Delta R_g + \frac{N^3 - 4N^2 + 16N - 16}{8(N-1)^2(N-2)^2}R_g^2 - \frac{2}{(N-2)^2}|Ric|^2$$

and

$$a_N = \frac{(N-2)^2 + 4}{2(N-1)(N-2)},$$

$$b_N = -\frac{4}{N-2}.$$

The author was supported by the Australian Research Council.

When $N \geq 5$, the operator P_g has the following property: if $\overline{g} = u^{4/(N-4)}g$ is a conformal metric of g, then for all $\varphi \in C^\infty(M)$

$$P_g(\varphi u) = \varphi^{(N+4)/(N-4)} P_{\overline{g}}(u).$$

In particular,

$$P_g(\varphi) = \frac{N-4}{2} Q_{\overline{g}} \varphi^{(N+4)/(N-4)}.$$

Many interesting results on the Paneitz operator and related topics have been recently studied by Branson [5], Branson *et al.* [6], Chang and Yang [10], Gursky [18], Ben Ayed and El Mehdi [4], Chtioui and Rigane [11], Esposito and Robert [15], Sandeep [24] and many others. In particular, when $N \geq 5$, Djadli *et al.* [12] studied the coercivity of the Paneitz operator and the positivity of solutions. Moreover, Djadli *et al.* [13] and Hebey and Robert [19] studied the blow-up analysis of the Q curvature equation.

Let us now consider the question: *given a smooth function Q on $\mathbb{S}^N$ ($N \geq 5$), does there exist a metric g conformal to the standard metric g_0 such that $Q = Q_g$?*

If we assume a conformal transformation of the form $g = w^{4/(N-4)} g_0$, the answer to the above question is 'yes' if and only if we can solve for w in the equation

$$\begin{cases} P_{g_0} w = \dfrac{N-4}{2} Q(x) w^{(N+4)/(N-4)} & \text{in } \mathbb{S}^N, \\ w > 0 & \text{in } \mathbb{S}^N. \end{cases} \tag{1.1}$$

The problem of finding Q such that (1.1) possesses a solution can be seen as the generalization to the Paneitz operator of the so-called 'Nirenberg problem' Q; namely: which functions on $\mathbb{S}^N$ are the scalar curvature of a metric conformal to the standard one? The Nirenberg problem has been studied by several authors; we mention Ambrosetti *et al.* [2], Chang and Yang [10], Chang *et al.* [9] and Kazdan and Warner [20]. A detailed bibliography on the Nirenberg problem can be found in Ambrosetti and Malchiodi [3].

It can be checked that the Paneitz operator on $(\mathbb{S}^N, g_0)$ is given by

$$P_{g_0} w = \Delta^2_{\mathbb{S}^N} w - \frac{1}{2}(N^2 - 2N - 4)\Delta_{\mathbb{S}^N} w + \frac{(N-4)N(N^2-4)}{16} w. \tag{1.2}$$

Consider the inverse of the stereographic projection

$$\Pi : \mathbb{R}^N \to \mathbb{S}^N$$

given by

$$x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right).$$

The spherical metric g_0 is given in terms of the stereographic coordinate system as

$$g_0 = \frac{4\, dx^2}{(1+|x|^2)^2}.$$

Hence, by a direct computation,

$$P_{g_0}\Phi(u) = \Big(\frac{1+|x|^2}{2}\Big)^{(N+4)/2}\Delta^2 u \quad \text{for all } u \in C^\infty(\mathbb{R}^N),$$

where

$$\Phi(u)(y) = u(\Pi(x))\Big(\frac{1+|\Pi(x)|^2}{2}\Big)^{(N-4)/2}, \quad y = \Pi(x).$$

Then (1.2) reduces to

$$\Delta^2 u = \tilde{Q}(x)u^{(N+4)/(N-4)} \quad \text{in } \mathbb{R}^4, \quad \text{where } \tilde{Q} = Q \circ \Pi. \tag{1.3}$$

Let us consider the problem (1.1) by taking Q to be a perturbation of a constant function. More precisely, we let $Q = (1+\varepsilon h)$, where h is a smooth function on $\mathbb{S}^N$ and $\varepsilon > 0$ is a small parameter. Using the stereographic projection from $\mathbb{S}^N$ to $\mathbb{R}^N$, we transform (1.3) (with f denoting the transformed function h) to the following problem:

$$\begin{cases} \Delta^2 u = (1+\varepsilon f(x))u^{(N+4)/(N-4)} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

But, in this paper, we consider the nonlinear perturbed problem

$$\begin{cases} \Delta^2 u = u^{(N+4)/(N-4)} + \varepsilon f(x)u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

with $f(\not\equiv 0) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, ε being a positive parameter and $1 < q \le (N+4)/(N-4)$. Note that when $q = (N+4)/(N-4)$, then (1.5) reduces to (1.4). When $q = (N+4)/(N-4)$, it is enough to have $f \in L^\infty(\mathbb{R}^N)$.

Note that (1.5) is related to the entire space problem

$$\begin{cases} \Delta^2 U = U^{(N+4)/(N-4)} & \text{in } \mathbb{R}^N, \\ U \in \mathcal{D}^{2,2}(\mathbb{R}^N), \end{cases}$$

where $\mathcal{D}^{2,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-4)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\Delta u|^2\, dx < +\infty\}$, and the solutions are given by Lin [21] as

$$U_{1,0}(x) = C_N\Big(\frac{1}{1+|x|^2}\Big)^{(N-4)/2},$$

$$U_{\lambda,\xi}(x) = \lambda^{-(N-4)/2}U_{1,0}\Big(\frac{x-\xi}{\lambda}\Big) \tag{1.6}$$

and

$$\langle (x-\xi), \nabla U_{\lambda,\xi}\rangle = -\Big(\lambda\frac{\partial U_{\lambda,\xi}}{\partial\lambda} + \frac{N-4}{2}U_{\lambda,\xi}\Big), \tag{1.7}$$

where $C_N = [N^2(N^2-4)(N-4)]^{(N-4)/8}$. Here

$$\|u\|^2_{\mathcal{D}^{2,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\Delta u|^2\, dx.$$

Note that when $1 < q < (N+4)/(N-4)$, we have interaction with the critical dimension as $U_{1,0}^{q+1}$ is integrable provided $q > 4/(N-4)$, that is, the cases $N = 5, 6, 7$ are the worst case scenario and that is the reason why we require $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let us define a finite-dimensional functional $\mathcal{J}$, where

$$\mathcal{J}(\lambda, \xi) = \frac{1}{q+1}\int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}{}^{q+1}(x)\,dx = \frac{\lambda^{N-\theta}}{q+1}\int_{\mathbb{R}^N} f(\xi + \lambda x) U_{1,0}^{q+1}(x)\,dx, \tag{1.8}$$

where $\theta = ((N-4)(q+1))/2$. Using the Hölder inequality in (1.8) and choosing $N/(N-4) < s < 2N/(N-4)$,

$$\begin{aligned}|\mathcal{J}(\lambda,\xi)| &\le C\Big(\int_{\mathbb{R}^N} |f(x)|^{s/(s-1)}\,dx\Big)^{(s-1)/s}\Big(\int_{\mathbb{R}^N} U_{\lambda,\xi}^{s}(x)\,dx\Big)^{(q+1)/s}\\ &\le c\lambda^{(N(q+1)/s)-\theta}\|f\|_{L^{(s-1)/s}}\|U_{1,0}\|_{L^s}^{q+1}.\end{aligned}$$

Hence,

$$|\mathcal{J}(\lambda,\xi)| \to 0 \quad \text{as } \lambda \to 0. \tag{1.9}$$

As a result, we can extend $\mathcal{J}(\lambda,\xi)$ on $\mathbb{R}\times\mathbb{R}^N$ in an odd way as

$$\tilde{\mathcal{J}}(\lambda,\xi) = -\mathcal{J}(-\lambda,\xi) \quad \text{for } \lambda < 0.$$

Without loss of generality, we consider $\tilde{\mathcal{J}}(\lambda,\xi) = \mathcal{J}(\lambda,\xi)$. Moreover, from (1.8) and the fact that $U_{1,0}$ is bounded,

$$\begin{aligned}\mathcal{J}(\lambda,\xi) &= \frac{\lambda^{N-\theta}}{q+1}\int_{\mathbb{R}^N} f(\xi+\lambda x) U_{1,0}^{q+1}(x)\\ &\le c\lambda^{N-\theta}\|f\|_{L^1}.\end{aligned}$$

Noting the fact that $N-\theta$ is negative, we conclude the fact that $\mathcal{J}(\lambda,\xi) \to 0$ as $|\lambda| \to \infty$. Furthermore, if $\lambda \to \lambda_\star > 0$ and $|\xi| \to \infty$, by the dominated convergence theorem,

$$\mathcal{J}(\lambda,\xi) = \frac{\lambda^{-\theta}}{q+1}\int_{\mathbb{R}^N} f(x) U^{q+1}\Big(\frac{x-\xi}{\lambda}\Big) \to 0.$$

Hence,

$$\lim_{|\lambda|+|\xi|\to\infty} \mathcal{J}(\lambda,\xi) = 0. \tag{1.10}$$

Hence, from (1.9) and (1.10), there exists (λ,ξ) with $\lambda > 0$ such that $\mathcal{J}$ has a critical point (a global maximum or a global minimum) at (λ,ξ). Let

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx - \frac{\varepsilon}{q+1}\int_{\mathbb{R}^N} f(x)|u|^{q+1}\,dx.$$

Hence, by Felli [16] as well as Lemma 2.2, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, $J_\varepsilon \in C^2(\mathcal{D}^{2,2}(\mathbb{R}^N),\mathbb{R})$ admits a critical point $u_\varepsilon \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ near $\mathcal{M}$ and hence u_ε is a solution of (1.5), where $p+1 = 2N/(N-4)$ and

$$\mathcal{M} = \{U_{\lambda,\xi} : (\lambda,\xi) \in \mathbb{R}^+ \times \mathbb{R}^N\}$$

is an $(N+1)$-dimensional manifold of solutions. Note that the existence of a solution is dependent on some sort of 'nondegeneracy' condition of the critical point of $\mathcal{J}$.

Let $K \subset \mathbb{R}^+ \times \mathbb{R}^N$ be a compact set and define

$$d(u, \mathcal{M}_K) = \inf_{(\lambda,\xi)\in K} \|u - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)}.$$

In this paper we discuss the existence, uniqueness and multiplicity of positive solutions of (1.5) under the assumption that $f \in L^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Now we state the following theorems motivated by [23].

THEOREM 1.1. *Let (λ, ξ) be a nondegenerate critical point of $\mathcal{J}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, (1.5) admits a positive solution u_ε. Moreover, $\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} = O(\varepsilon)$.*

COROLLARY 1.2. *Let u_ε be a sequence of solutions of (1.5) such that*

$$\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Then $\nabla\mathcal{J}(\lambda, \xi) = 0$.

THEOREM 1.3 (Uniqueness). *Let (λ, ξ) be a nondegenerate critical point of $\mathcal{J}$. Furthermore, suppose $|\nabla f(x)| \le C$ and there exists two sequences of solutions $\{u_{\varepsilon,i}\}$ $(i = 1, 2)$ of (1.5) such that*

$$\|u_{\varepsilon,i} - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.11}$$

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $u_{\varepsilon,1} \equiv u_{\varepsilon,2}$.

REMARK 1.4. Note that if $q = 1$ and $N > 8$, positive solutions of (1.5) are nonunique for ε sufficiently small. See Felli [16]. In fact, Esposito [14] proved existence of two positive solutions of the Paneitz operator on $\mathbb{S}^N$ (see (1.2))

$$Pu = \frac{N^2(N-4)(N^2-4)}{16}|u|^{8/(N-4)}u + (\varepsilon f + o(\varepsilon))|u|^{q-1}u$$

and $1 \le q \le (N+4)/(N-4)$ when f changes sign and $q \ge 4/(N-4)$ or $q < 4/(N-4)$ and $\int_{\mathbb{S}^N} f = 0$. Note that our uniqueness is different in this context.

THEOREM 1.5 (Multiplicity). *Assume that there is a compact set $K \subset \mathbb{R}^+ \times \mathbb{R}^N$ with nonempty interior such that the critical points of $\mathcal{J}$ in K are finite and nondegenerate. Furthermore, suppose $|\nabla f(x)| \le C$. Then there exists $\rho_0 = \rho_0(K) > 0$ and $\varepsilon_0 = \varepsilon_0(\rho_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the number of solutions to the problem (1.5) with $d(u, \mathcal{M}_K) < \rho_0$ is the same as the number of nondegenerate critical points of $\mathcal{J}$.*

COROLLARY 1.6. *Furthermore, the conclusions of Theorems 1.1–1.5 hold for the equation*

$$(-\Delta)^m u = (1 + \varepsilon f(x))u^{(N+2m)/(N-2m)} \quad \text{in } \mathbb{R}^N$$

whenever $\|f\|_\infty + \|\nabla f\|_\infty \le C$, $N > 2m$ and $m \in \mathbb{N}$. The construction of positive solutions follows from Wei and Xu [25].

Remark 1.7. Note that the conclusions of Theorems 1.1–1.5 are not only applicable to the powers of Laplacians, but also applicable for the coercive Hardy equation $-\Delta u - (\mu/|x|^2)u = (1+\varepsilon f(x))u^{(N+2)/(N-2)}$ with $N \geq 3$ and $\mu > 0$. Here proving the results becomes much easier as $Ker\{-\Delta - (\mu/|x|^2) - ((N+2)/(N-2))u^{4/(N-2)}\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is one dimensional due to the scaling invariance of the operator.

2. Preliminaries

Lemma 2.1 (Nondegeneracy). *The kernel of the linearized operator*

$$\mathcal{L} = \Delta^2 - \frac{N+4}{N-4} {U_{\lambda,\xi}}^{4/(N-4)}$$

in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ *is* $N+1$ *dimensional and*

$$Ker(\mathcal{L}) = \left\{ \frac{\partial U_{\lambda,\xi}}{\partial \lambda}, \frac{\partial U_{\lambda,\xi}}{\partial \xi_1}, \frac{\partial U_{\lambda,\xi}}{\partial \xi_2}, \dots, \frac{\partial U_{\lambda,\xi}}{\partial \xi_N} \right\}.$$

Proof. This follows from Djadli *et al.* [13]. □

Let H be a Hilbert space and $J_\varepsilon(u) = J_0(u) - \varepsilon G(u)$ be a perturbed functional, where $J_0, G \in C^2(H, \mathbb{R})$. Moreover, assume that J_0 satisfies:

(f1) J_0 has a finite-dimensional manifold of critical points $\mathcal{M}$; let $c = J_0(z)$ for all $z \in \mathcal{M}$;
(f2) for all $z \in \mathcal{M}$, $J_0''(z)$ is a Fredholm operator of index zero;
(f3) for all $z \in \mathcal{M}$, $T_z\mathcal{M} = Ker\ J_0''(z)$. We denote $\mathcal{J} = G|_{\mathcal{M}}$.

Lemma 2.2. *Let* J_0 *satisfy (f1)–(f3) and suppose there exists* $z \in \mathcal{M}$ *which is a critical point of* $\mathcal{J}$ *such that one of the following conditions holds:*

(1) *z is nondegenerate;*
(2) *z is a global maximum or global minimum;*
(3) *z is isolated and the local degree of $\nabla\mathcal{J}$ at z is different from zero.*

Then there exists $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *the functional* J_ε *has a critical point* u_ε *such that* $u_\varepsilon \to z$ *as* $\varepsilon \to 0$.

Proof. The proof of this lemma follows from Ambrosetti and Badiale [1]. Also, see Ambrosetti *et al.* [2, page 122] and the book by Ambrosetti and Malchiodi [3]. Note that Lemma 2.2 is a very general theorem; it is not restricted to Laplacian operators only. Note that in Felli's proof [16], condition (2) of the lemma holds. □

Lemma 2.3 (Caristi and Mitidieri [7]). *Let* Ω *be an open subset of* $\mathbb{R}^N$ $(N \geq 5)$ *and* $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ *be a weak solution of*

$$\Delta^2 u = a(x)u \quad \textit{in } \Omega,$$

where $a \in L^{\alpha}_{\text{loc}}(\Omega)$ *with* $\alpha > N/4$. *Then, for any* $0 < \beta < +\infty$, *there exist* $C > 0$ *and* $R > 0$ *such that*

$$\sup_{B(y,r)\cap\Omega} |u| \le C\Big[\frac{1}{r^N}\int_{B(y,2r)\cap\Omega} |u|^{\beta+1}\Big]^{1/(\beta+1)}$$

for any $y \in \mathbb{R}^N$ *and* $0 < r < R$.

Lemma 2.4. *Let* u_ε *be a sequence of solutions of (1.5) with* $\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ *as* $\varepsilon \to 0$ *for some* $(\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N$. *Then the asymptotic behavior for derivatives of* u_ε *at infinity is given by*

$$|\nabla^{(\beta)} u_\varepsilon(x)| = O(1)|x|^{4-N-|\beta|} \tag{2.1}$$

for $0 \le |\beta| \le 3$ *whenever* $|x| \gg 1$.

Proof. First note that if $u_\varepsilon \to U_{\lambda,\xi}$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} u_\varepsilon^{2N/(N-4)}(x)\,dx \to \int_{\mathbb{R}^N} U_{\lambda,\xi}^{2N/(N-4)}(x)\,dx$$

as $\varepsilon \to 0$. Moreover, as $f \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, by the Hölder inequality,

$$\Big|\int_{\mathbb{R}^N} f(x) u_\varepsilon^{q+1}(x)\,dx\Big| \le C,$$

$$\int_{\mathbb{R}^N} f(x) u_\varepsilon^{q+1}(x)\,dx \to \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^{q+1}(x)\,dx.$$

Also, by elliptic regularity, $u_\varepsilon \to U_{\lambda,\xi}$ in $C^4_{\text{loc}}(\mathbb{R}^N)$. Hence, u_ε is locally uniformly bounded. So, we need to study the decay of u_ε at infinity. Define the Kelvin transform of u_ε as

$$\hat{u}_\varepsilon(x) := |x|^{4-N} u_\varepsilon\Big(\frac{x}{|x|^2}\Big).$$

By the application of the Kelvin transform on (1.5),

$$\Delta^2 \hat{u}_\varepsilon = [\hat{u}_\varepsilon^{8/(N-4)} + \varepsilon \hat{f}(x)|x|^{-\tau}\hat{u}_\varepsilon^{q-1}]\hat{u}_\varepsilon \quad \text{in } \mathbb{R}^N\backslash\{0\},$$

where $\tau = N + 4 - q(N-4)$ and $\hat{f}(x) = f(x/|x|^2)$. Let $a_\varepsilon(x) = \hat{u}_\varepsilon^{8/(N-4)} + \varepsilon\hat{f}(x)|x|^{-\tau}\hat{u}_\varepsilon^{q-1}$. But $\hat{f}(x)|x|^{-\tau}$ is bounded near 0. Hence, by Lemma 2.3, there exist $R > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{B_R(0)} |\hat{u}_\varepsilon(x)| \le C\Big[\frac{1}{R^N}\int_{B_{2R}} |\hat{u}_\varepsilon(z)|^{2N/(N-4)}\,dz\Big]^{(N-4)/2N} \le C.$$

This implies that, for $|x| \gg 1$,

$$u_\varepsilon(x) = O(|x|^{4-N}).$$

And, hence, by the Schauder estimates,

$$|\nabla^{(\beta)} u_\varepsilon| \le C|x|^{4-N-|\beta|}.$$

Note that in the above estimate $C > 0$ is independent of $\varepsilon > 0$. □

Lemma 2.5. *Let w_ε be a sequence of solutions of*

$$\begin{cases} \Delta^2 w = c_\varepsilon(x)w + \varepsilon f(x)d_\varepsilon(x)w & \text{in } \mathbb{R}^N \\ w \in \mathcal{D}^{2,2}(\mathbb{R}^N) \end{cases} \tag{2.2}$$

with $\|w_\varepsilon\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \le C$, where $u_{\varepsilon,i}$ $(i = 1, 2)$ are solutions of (1.5)

$$c_\varepsilon(x) = \int_0^1 [tu_{\varepsilon,1}(x) + (1-t)u_{\varepsilon,2}(x)]^{8/(N-4)}\, dt$$

and

$$d_\varepsilon(x) = \int_0^1 [tu_{\varepsilon,1}(x) + (1-t)u_{\varepsilon,2}(x)]^{q-1}\, dt.$$

Then, for $|x| \gg 1$, we have a uniform estimate

$$|\nabla^{(\beta)} w_\varepsilon(x)| = O(1)|x|^{4-N-|\beta|} \tag{2.3}$$

for $0 \le |\beta| \le 3$.

Proof. By the standard regularity, w_ε is locally uniformly bounded. Let us consider the Kelvin transform of w_ε

$$\hat{w}_\varepsilon(x) := |x|^{4-N} w_\varepsilon\Big(\frac{x}{|x|^2}\Big),$$

$$\hat{u}_\varepsilon(x) = |x|^{4-N} u_\varepsilon\Big(\frac{x}{|x|^2}\Big), \quad \hat{w}_\varepsilon(x) = |x|^{4-N} w_\varepsilon\Big(\frac{x}{|x|^2}\Big), \quad x \in \mathbb{R}^N\backslash\{0\}.$$

Furthermore, define

$$\hat{c}_\varepsilon(x) = \int_0^1 [t\hat{u}_{\varepsilon,1} + (1-t)\hat{u}_{\varepsilon,2}]^{8/(N-4)}\, dt,$$

$$\hat{d}_n(x) = \int_0^1 [t\hat{u}_{\varepsilon,1} + (1-t)\hat{u}_{\varepsilon,2}]^{q-1}\, dt.$$

Then, by (2.2), $\hat{w}_\varepsilon$ satisfies

$$\Delta^2 \hat{w}_\varepsilon = \hat{c}_\varepsilon \hat{w}_\varepsilon + \varepsilon |x|^{-\tau} f\left(\frac{x}{|x|^2}\right)\hat{d}_\varepsilon \hat{w}_\varepsilon \quad \text{in } \mathbb{R}^N\backslash\{0\}. \tag{2.4}$$

So, we are going to study boundedness of (2.4) near a neighborhood of the origin. From Lemma 2.4, $\hat{c}_\varepsilon$, $|x|^{-\tau}\hat{d}_\varepsilon f(x/|x|^2)$ is uniformly bounded near the origin. Hence, by Lemma 2.3, there exist $C, R > 0$ such that

$$\sup_{B(y,R)\cap\Omega} |\hat{w}_\varepsilon| \le C\bigg[\frac{1}{R^N}\int_{B(y,2R)\cap\Omega} |\hat{w}_\varepsilon(z)|^{2N/(N-4)}\, dz\bigg]^{(N-4)/(2N)} \le C.$$

Hence, $\hat{w}_\varepsilon$ is uniformly bounded near the origin and hence $|w_\varepsilon(x)| \le C|x|^{4-N}$ when $|x| \gg 1$. The decay of higher derivatives follows from the standard elliptic estimates. □

Lemma 2.6 (Kazdan–Warner-type identities). *Let u_ε be a solution of (1.5) such that $\|u_\varepsilon - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$ for some $(\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^N$. Then, we have the following two types of Pohozaev identities:*

$$\int_{\mathbb{R}^N} f(x) u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i} = 0, \quad i = 1, 2 \tag{2.5}$$

and

$$\int_{\mathbb{R}^N} f(x) u_\varepsilon^q \Big[(x - \xi) \cdot \nabla u_\varepsilon + \Big(\frac{N-4}{2} \Big) u_\varepsilon \Big] = 0. \tag{2.6}$$

Proof. In order to prove (2.5), we multiply (1.5) by $\partial u_\varepsilon(x)/\partial x_i$, $i = 1, 2, \ldots, N$, and integrate by parts on the ball $B_R(0)$ to get

$$\int_{B_R(0)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x) u_\varepsilon^q) \frac{\partial u_\varepsilon}{\partial x_i} = \int_{\partial B_R(0)} \frac{\partial \Delta u_\varepsilon}{\partial \nu} \frac{\partial u_\varepsilon}{\partial x_i} d\sigma - \int_{B_R(0)} \nabla \Delta u_\varepsilon \cdot \frac{\partial}{\partial x_i} (\nabla u_\varepsilon). \tag{2.7}$$

By (2.1), we obtain

$$\int_{\partial B_R(0)} \Big| \frac{\partial \Delta u_\varepsilon}{\partial \nu} \frac{\partial u_\varepsilon}{\partial x_i} \Big| d\sigma = O\Big(\frac{1}{R^{2(N-2)}} \Big) \quad \text{as } R \to \infty.$$

Again, by a suitable integration by parts and using (2.1) and Lemma 2.4, we get, as $R \to \infty$,

$$\int_{B_R(0)} \nabla \Delta u_\varepsilon \cdot \frac{\partial}{\partial x_i} (\nabla u_\varepsilon) = \int_{\partial B_R(0)} \Big(\Delta u_\varepsilon \frac{\partial}{\partial \nu} \Big(\frac{\partial u_\varepsilon}{\partial x_i} \Big) - \frac{1}{2R} x_i |\Delta u_\varepsilon|^2 \Big) d\sigma = O\Big(\frac{1}{R^{2(N-2)}} \Big).$$

Hence, from the last two relations,

$$\lim_{R \to \infty} \{\text{Right-hand side of (2.7)}\} = 0. \tag{2.8}$$

We note that, again integrating by parts,

$$\int_{B_R(0)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x) u_\varepsilon^q) \frac{\partial u_\varepsilon}{\partial x_i} = \frac{1}{R} \int_{\partial B_R(0)} x_i u_\varepsilon^{2N/(N-4)} \, d\sigma + \varepsilon \int_{B_R(0)} f(x) u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i}.$$

Using (2.1) and letting $R \to \infty$ in the above equation,

$$\lim_{R \to \infty} \int_{B_R(0)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x) u_\varepsilon^q) \frac{\partial u_\varepsilon}{\partial x_i} = \varepsilon \int_{\mathbb{R}^N} f(x) u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i}. \tag{2.9}$$

Therefore, we obtain, using (2.9) and (2.8),

$$\varepsilon \int_{\mathbb{R}^N} f(x) u_\varepsilon^q \frac{\partial u_\varepsilon}{\partial x_i} = \lim_{R \to \infty} \{\text{Left-hand side of (2.7)}\} = 0,$$

which proves (2.5).

For (2.6), we multiply (1.5) by $(x-\xi)\cdot\nabla u_\varepsilon + ((N-4)/2)u_\varepsilon$ on either side and integrate on the ball $B_R(y)$ as before to obtain

$$\begin{aligned}\int_{B_R(y)} (u_\varepsilon^{(N+4)/(N-4)} + \varepsilon f(x)u_\varepsilon^q)\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big)\\ = \int_{B_R(y)} \Delta^2 u_\varepsilon\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big).\end{aligned} \tag{2.10}$$

Integrating by parts,

$$\begin{aligned}\text{Left-hand side of (2.10)} = R\int_{\partial B_R(y)} u_\varepsilon^{(N+4)/(N-4)}\,d\sigma\\ + \varepsilon\int_{B_R(y)} f(x)u_\varepsilon^q\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big).\end{aligned}$$

Again integrating by parts suitably,

$$\begin{aligned}\text{Right-hand side of (2.10)} = \int_{\partial B_R(y)} \Big(|x-\xi|\Big[\frac{1}{2}|\Delta u_\varepsilon|^2 + \frac{\partial u_\varepsilon}{\partial r}\frac{\partial}{\partial r}(\Delta u_\varepsilon)\Big]\\ - \Delta u_\varepsilon\frac{\partial}{\partial r}\Big(r\frac{\partial u_\varepsilon}{\partial r}\Big)\Big)\,d\sigma.\end{aligned}$$

Using the decay estimate (2.1),

$$\lim_{R\to\infty}\{\text{Left-hand side of (2.10)}\} = \varepsilon\int_{\mathbb{R}^N} f(x){u_\varepsilon}^q\Big((x-\xi)\cdot\nabla u_\varepsilon + \Big(\frac{N-4}{2}\Big)u_\varepsilon\Big)$$

and

$$\lim_{R\to\infty}\{\text{Right-hand side of (2.10)}\} = 0.$$

Hence, (2.6) follows. □

REMARK 2.7. Note that when $q=(N+4)/(N-4)$ one can derive the Kazdan and Warner [20] kind of identities using the concept of an integral equation in $\mathcal{D}^{2,2}(\mathbb{R}^N)$;

$$u_\varepsilon(x) = \int_{\mathbb{R}^N} (1+\varepsilon f(y))F(x,y)u_\varepsilon^{(N+4)/(N-4)}(y)\,dy, \tag{2.11}$$

where $F(x,y) = 1/(4-N)\sigma_N|x-y|^{N-4}$ is the fundamental solution of Δ^2 and σ_N is the area of the unit sphere in $\mathbb{R}^N$. The main idea is the fact that

$$\Delta^2 u = f \quad \text{in } \mathbb{R}^N$$

can be written as $u=u_1+u_2$, where $u_i\in\mathcal{D}^{2,2}(\mathbb{R}^N)$; $i=1,2$, $u_1(x)=\int_{\mathbb{R}^N}F(x,y)g(y)\,dy$ and $\Delta^2 u_2 = 0$. But this implies $u_2=0$. As a result, we end up getting (2.11).

Proof of Corollary 1.2. By the Schauder estimates, $u_\varepsilon \to U_{\lambda,\xi}$ in $C^4_{\text{loc}}(\mathbb{R}^N)$, and by Lemma 2.6 and the dominated convergence theorem we can pass to the limit in (2.5) and (2.6). Using (1.7),

$$\int_{\mathbb{R}^3} f(x) U^q_{\lambda,\xi} \frac{\partial U_{\lambda,\xi}}{\partial x_i} = 0, \quad i = 1, 2, \ldots, N \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} f(x) U^q_{\lambda,\xi} \frac{\partial U_{\lambda,\xi}}{\partial \lambda} = 0. \tag{2.13}$$

Hence, we obtain $\nabla \mathcal{J}(\lambda, \xi) = 0$. □

Lemma 2.8. *If (λ_0, ξ_0) is a critical point of $\mathcal{J}$, then*

$$\begin{aligned}
\lambda_0 \frac{\partial^2 \mathcal{J}}{\partial \lambda^2}(\lambda_0, \xi_0) = &-\theta \int_{\mathbb{R}^N} f(z) U^q_{\lambda_0,\xi_0}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z)\, dz \\
&- N \int_{\mathbb{R}^N} f(z) U^q_{\lambda_0,\xi_0}(z) \Big\langle z - \xi_0, \nabla \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z) \Big\rangle\, dz \\
&- Nq \int_{\mathbb{R}^N} f(z) U^{q-1}_{\lambda_0,\xi_0}(z) \langle z - \xi_0, \nabla U_{\lambda_0,\xi_0} \rangle \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z)\, dz.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\partial^2 \mathcal{J}}{\partial \lambda \partial \xi_i}(\lambda_0, \xi_0) = &- \int_{\mathbb{R}^N} f(z) U^q_{\lambda_0,\xi_0}(z) \frac{\partial}{\partial z_i} \Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z) \Big)\, dz \\
&- q \int_{\mathbb{R}^N} f(z) U^{q-1}_{\lambda_0,\xi_0}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial \lambda}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\, dz.
\end{aligned}$$

Moreover, for $1 \le i, j \le N$,

$$\begin{aligned}
\frac{\partial^2 \mathcal{J}}{\partial \xi_i \partial \xi_j}(\lambda_0, \xi_0) = &- \int_{\mathbb{R}^N} f(z) U^q_{\lambda_0,\xi_0}(z) \frac{\partial}{\partial z_i} \Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z) \Big)\, dz \\
&- q \int_{\mathbb{R}^N} f(z) U^{q-1}_{\lambda_0,\xi_0}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z) \frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\, dz,
\end{aligned}$$

where $z = \xi + \lambda x$.

Proof. As $U_{\lambda,\xi}$ satisfies (1.6) and (1.7),

$$\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \lambda}(\lambda, \xi) = &\frac{\lambda^{N-\theta}}{q+1} \int_{\mathbb{R}^N} \langle x, \nabla f(\lambda x + \xi) \rangle U^{q+1}_{1,0}(x)\, dx \\
&+ \frac{N-\theta}{q+1} \lambda^{N-\theta-1} \int_{\mathbb{R}^N} f(\lambda x + \xi) U^{q+1}_{1,0}(x)\, dx, \\
\frac{\partial \mathcal{J}}{\partial \xi_i}(\lambda, \xi) = &\frac{\lambda^{N-\theta}}{(q+1)\lambda} \int_{\mathbb{R}^N} \frac{\partial f(\lambda x + \xi)}{\partial x_i} U^{q+1}_{1,0}(x)\, dx.
\end{aligned}$$

Also, note that $\theta = (N-4)(q+1)/2$. Integrating by parts,

$$\begin{aligned}\lambda\frac{\partial\mathcal{J}}{\partial\lambda}(\lambda,\xi) = &- \frac{N}{q+1}\lambda^{N-\theta}\int_{\mathbb{R}^N} f(\lambda x+\xi)U_{1,0}^{q+1}(x)\,dx\\ &- N\lambda^{N-\theta}\int_{\mathbb{R}^N} f(\lambda x+\xi)U_{1,0}^{q}\langle x,\nabla U_{1,0}(x)\rangle\,dx\\ &+ \frac{N-\theta}{q+1}\lambda^{N-\theta}\int_{\mathbb{R}^N} f(\lambda x+\xi)U_{1,0}^{q+1}(x)\,dx\\ = &- \frac{\theta}{q+1}\lambda^{N-\theta}\int_{\mathbb{R}^N} f(\lambda x+\xi)U_{1,0}^{q+1}(x)\,dx\\ &- N\lambda^{N-\theta}\int_{\mathbb{R}^N} f(\lambda x+\xi)U_{1,0}^{q}\langle x,\nabla U_{1,0}(x)\rangle\,dx\end{aligned}$$

and

$$\frac{\partial\mathcal{J}}{\partial\xi_i}(\lambda,\xi) = -\lambda^{N-\theta-1}\int_{\mathbb{R}^N} f(\lambda x+\xi)U_{1,0}^{q}(x)\frac{\partial U_{1,0}}{\partial x_i}\,dx.$$

Since (λ_0,ξ_0) is a critical point of $\mathcal{J}$, we must have $(\partial\mathcal{J}/\partial\lambda)(\lambda_0,\xi_0) = 0$ and $(\partial\mathcal{J}/\partial\xi_i)(\lambda_0,\xi_0) = 0$. Hence, letting $z = \xi + \lambda x$,

$$\begin{aligned}\lambda_0\frac{\partial^2\mathcal{J}}{\partial\lambda^2}(\lambda_0,\xi_0) = &-\theta\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\,dz\\ &- N\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q}(z)\Big\langle z-\xi_0,\nabla\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\Big\rangle\,dz\\ &- Nq\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q-1}(z)\langle z-\xi_0,\nabla U_{\lambda_0,\xi_0}\rangle\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\,dz.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{\partial^2\mathcal{J}}{\partial\lambda\partial\xi_i}(\lambda_0,\xi_0) = &-\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q}(z)\frac{\partial}{\partial z_i}\Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\Big)\,dz\\ &- q\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q-1}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial\lambda}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\,dz.\end{aligned}$$

Moreover, for $1 \le i,j \le N$,

$$\begin{aligned}\frac{\partial^2\mathcal{J}}{\partial\xi_i\partial\xi_j}(\lambda_0,\xi_0) = &-\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q}(z)\frac{\partial}{\partial z_i}\Big(\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z)\Big)\,dz\\ &- q\int_{\mathbb{R}^N} f(z)U_{\lambda_0,\xi_0}^{q-1}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_j}(z)\frac{\partial U_{\lambda_0,\xi_0}}{\partial z_i}(z)\,dz.\end{aligned}$$ □

3. Proof of the main theorems

Proof of Theorem 1.1. Let (λ,ξ) be a nondegenerate critical point of $\mathcal{J}$. Then $\nabla\mathcal{J}(\lambda,\xi) = 0$ and $det(\nabla^2\mathcal{J}(\lambda,\xi)) \neq 0$. Hence, $\nabla^2\,\mathcal{J}(\lambda,\xi)$ is an invertible matrix of order $N+1$. Our aim is to obtain a solution of (1.5) which is of the form $u_\varepsilon = U_{\lambda,\xi} + \phi_\varepsilon$. Note that

$$J_\varepsilon(u) = J_0(u) - \frac{\varepsilon}{q+1}\int_{\mathbb{R}^N} f(x)|u|^{q+1}\,dx,$$

where

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx$$

and $Ker(\mathcal{L})$ is $N+1$-dimensional; see Lemma 2.1. Moreover, it is easy to check that J_0 satisfies (f1)–(f3). Hence, by Lemma 2.2, (1) holds and we obtain a solution of (1.5) for sufficiently small $\varepsilon > 0$.

Proof of Theorem 1.3. If possible, let there exist a sequence $\varepsilon_n \to 0$ and two distinct functions $u_{1,\varepsilon_n} \equiv u_{1,n}$, $u_{2,\varepsilon_n} \equiv u_{2,n}$ which solve (1.5) with $\varepsilon = \varepsilon_n$ and $\|u_{i,n} - U_{\lambda,\xi}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for $i = 1, 2$. Set $\tilde{w}_n = u_{1,n} - u_{2,n}$. Then $\|\tilde{w}_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Hence, by Lemma 2.4, $\|\tilde{w}_n\|_{L^\infty(\mathbb{R}^N)} \le C$.

Define $w_n = \tilde{w}_n / \|\tilde{w}_n\|_{L^\infty(\mathbb{R}^N)}$. Then there exists $x_n \in \mathbb{R}^N$ such that $|w_n(x_n)| \ge \frac{1}{2}$. Then w_n satisfies

$$\Delta^2 w_n = c_n(x) w_n + \varepsilon f(x) d_n(x) w_n \quad \text{with } c_n(x) = \int_0^1 [t u_{1,n} + (1-t) u_{2,n}]^{8/(N-4)}\,dt$$

and

$$d_n(x) = \int_0^1 [t u_{1,n} + (1-t) u_{2,n}]^{q-1}\,dt.$$

Using Schauder estimates, we obtain $w_n \to w$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$, where w satisfies the entire problem

$$\Delta^2 w = \frac{N+4}{N-4} U_{\lambda,\xi}^{8/(N-4)} w \quad \text{in } \mathbb{R}^N.$$

By the nondegeneracy result in Lemma 2.1,

$$w = c_0 \frac{\partial U_{\lambda,\xi}}{\partial \lambda} + \sum_{j=1}^N c_j \frac{\partial U_{\lambda,\xi}}{\partial x_j}$$

for some $c_j \in \mathbb{R}$, $j = 1, \ldots, N$. We claim that $c_j = 0$ for all $j = 0, 1, \ldots, N$. By the identity (2.5),

$$\int_{\mathbb{R}^N} f(x) {u_{i,n}}^q \frac{\partial u_{i,n}}{\partial x_j} = 0, \quad j = 1, 2, \ldots, N. \tag{3.1}$$

We derive from (3.1) and (2.1)

$$\int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j} u_{\varepsilon,i}^{q+1} = 0, \quad i = 1, 2 \text{ and } j = 1, 2, \ldots, N.$$

Therefore,

$$\int_{\mathbb{R}^N} \left(\frac{\partial f}{\partial x_j} {u_{1,n}}^{q+1} - \frac{\partial f}{\partial x_j} {u_{2,n}}^{q+1} \right) = 0 \quad \text{for } j = 1, 2, \ldots, N$$

and, using the fundamental theorem of integral calculus,

$$\int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j}\Big(\int_0^1 [tu_{1,n} + (1-t)u_{2,n}]^q\, dt\Big)\tilde{w}_n\, dx = 0 \quad \text{for } j = 1, 2, \ldots, N. \tag{3.2}$$

Letting $\varepsilon \to 0$ in (3.2),

$$\int_{\mathbb{R}^N} \frac{\partial f}{\partial x_j} {U_{\lambda,\xi}}^q \left(c_0 \frac{\partial U_{\lambda,\xi}}{\partial \lambda} + \sum_{i=1}^{N} c_i \frac{\partial U_{\lambda,\xi}}{\partial x_i}\right) = 0, \quad j = 1, 2, \ldots, N.$$

That is, integrating by parts again,

$$\int_{\mathbb{R}^N} f \frac{\partial}{\partial x_j}({U_{\lambda,\xi}}^q w) = 0, \quad j = 1, 2, \ldots, N.$$

This implies that

$$q \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^{q-1} \frac{\partial U_{\lambda,\xi}}{\partial x_j} w + \int_{\mathbb{R}^N} f(x) U_{\lambda,\xi}^{q} \frac{\partial w}{\partial x_j} = 0. \tag{3.3}$$

Furthermore, we obtain by integrating on $B_R(y)$

$$\int_{B_R(y)} (x-\xi)\cdot \nabla(f u_{i,n}^{q+1}) = R \int_{\partial B_R(y)} f(x) u_{i,n}^{q+1} - N \int_{B_R(y)} f(x) u_{i,n}^{q+1} \quad \text{for } i = 1, 2.$$

This implies that as $R \to +\infty$

$$\int_{\mathbb{R}^N} (x-\xi)\cdot \nabla(f u_{i,n}^{q+1}) = -N \int_{\mathbb{R}^N} f(x) u_{i,n}^{q+1} \quad \text{for } i = 1, 2.$$

And, as a result,

$$\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle u_{i,n}^{q+1} + (q+1)\int_{\mathbb{R}^N} f(x)\langle (x-\xi), \nabla u_{i,n}\rangle u_{i,n}^{q} = -N \int_{\mathbb{R}^N} f(x) u_{i,n}^{q+1}.$$

Hence, by the Pohozaev identity (2.6), we have for $i = 1, 2$

$$\begin{aligned} \int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle u_{i,n}^{q+1} &= \left[\frac{(N-4)(q+1) - 2N}{2}\right] \int_{\mathbb{R}^N} f(x) u_{i,n}^{q+1} \\ &= \gamma \int_{\mathbb{R}^N} f(x) u_{i,n}^{q+1}, \end{aligned}$$

where $\gamma = (N-4)(q+1) - 2N/2$. This implies that

$$\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle u_{1,n}^{q+1} - \int_{\mathbb{R}^N} \langle (x-\xi)\cdot \nabla f(x)\rangle u_{2,n}^{q+1} = \gamma \int_{\mathbb{R}^N} f(x)[u_{1,n}^{q+1} - u_{2,n}^{q+1}]$$

and, by the application of the mean value theorem,

$$\begin{aligned} &\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x)\rangle \Big(\int_0^1 (tu_{1,n} + (1-t)u_{1,n})^q\, dt\Big) w_n \\ &\quad = \gamma \int_{\mathbb{R}^N} f(x) \Big(\int_0^1 (tu_{1,n} + (1-t)u_{1,n})^q\, dt\Big) w_n. \end{aligned}$$

And, letting $n \to \infty$,

$$\int_{\mathbb{R}^N} \langle (x-\xi), \nabla f(x) \rangle U^q_{\lambda,\xi} w = \gamma \int_{\mathbb{R}^N} f(x) U^q_{\lambda,\xi} w = 0 \tag{3.4}$$

because of (2.5) and (2.6) and passing to the limit as $\varepsilon \to 0$. Again, integrating by parts (3.4),

$$\int_{\mathbb{R}^N} f(x) U^q_{\lambda,\xi}[Nw + \langle (x-\xi), \nabla w \rangle] + q \int_{\mathbb{R}^N} f(x) U^{q-1}_{\lambda,\xi} w \langle (x-\xi), \nabla U_{\lambda,\xi} \rangle = 0. \tag{3.5}$$

From (3.3), (3.5), Corollary 1.2 and Lemma 2.8, $\nabla \mathcal{J}(\lambda, \xi) = 0$ and

$$\nabla^2 \mathcal{J}(\lambda, \xi)(c_0, c_1, \ldots, c_N)^T = 0$$

with $\nabla^2 \mathcal{J}(\lambda, \xi)$ an invertible matrix, which implies $c_0 = c_1 = c_2 \cdots = c_N = 0$. Also, note that there will be some cancelation in Lemma 2.8 due to (2.12) and (2.13). This proves that $w \equiv 0$ in $\mathbb{R}^N$ and hence $w_n \to 0$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$. Hence, we must have $|x_n| \to \infty$. As usual, we define the Kelvin transform of the functions $u_{i,n}(x)$ and $w_n(x)$ as

$$\hat{u}_{i,n}(x) = |x|^{4-N} u_{i,n}\Big(\frac{x}{|x|^2}\Big), \quad i = 1, 2, \quad \hat{w}_n(x) = |x|^{4-N} w_n\Big(\frac{x}{|x|^2}\Big), \quad x \in \mathbb{R}^N \backslash \{0\}.$$

Furthermore, define

$$\hat{c}_n(x) = \int_0^1 [t\hat{u}_{1,n} + (1-t)\hat{u}_{2,n}]^{8/(N-4)}\, dt,$$
$$\hat{d}_n(x) = \int_0^1 [t\hat{u}_{1,n} + (1-t)\hat{u}_{2,n}]^{q-1}\, dt.$$

Clearly, we have $|\hat{w}_n(x_n/|x_n|^2)| \geq \frac{1}{2}$ for all large n. It is easily seen that $\hat{w}_n$ satisfies the following equation:

$$\Delta^2 \hat{w}_n = \hat{c}_n \hat{w}_n + \varepsilon f\left(\frac{x}{|x|^2}\right) |x|^{-(N+4)+q(N-4)} \hat{d}_n \hat{w}_n.$$

By the decay estimate, we obtain $|\hat{w}_n(x)| \leq 1$ for all n and all $x \in B_1(0) \backslash \{0\}$. Since $\hat{w}_n \to 0$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N \backslash \{0\})$, by the dominated convergence theorem, we obtain $\hat{w}_n \to 0$ in $L^p(B_1(0))$ for all $p \geq 1$. Using the assumption $f \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and the estimate (2.3),

$$\hat{c}_n(x), f\Big(\frac{x}{|x|^2}\Big)|x|^{-\tau} \hat{d}_n(x)$$

are bounded sequences in $L^2(B_1(0))$. Using L^p theory on $\hat{w}_n$ [17, Corollary 2.23, page 45],

$$\|\hat{w}_n\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C \|\hat{w}_n\|_{L^p(B_1(0))} \to 0.$$

This gives a contradiction, since

$$\|\hat{w}_n\|_{L^\infty(B_{\frac{1}{2}}(0))} \geq \left|\hat{w}_n\Big(\frac{x_n}{|x_n|^2}\Big)\right| \geq \frac{1}{2}$$

for all large n. This proves the theorem.

PROOF OF THEOREM 1.5. By the assumptions, the nondegenerate critical points of $\mathcal{J}$ are contained in the interior of a ball $K = \overline{B}_R(0) \subset \mathbb{R}^+ \times \mathbb{R}^N$ for some $R > 0$. Let (λ_i, ξ_i) be the nondegenerate critical points of $\mathcal{J}$ $(i = 1, 2, \ldots, s)$ contained in K. Then, by Theorem 1.1 and Corollary 1.2, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the problem (1.5) has at least s solutions $u_{\varepsilon,i}$ and s points (λ_i, ξ_i) such that $u_{\varepsilon,i} - U_{\lambda_i,\xi_i} \to 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. For any $\mu > 0$, define

$$\mathcal{S}_\mu = \{u \text{ solves (1.5) for } \varepsilon \in (0, \mu)\} \backslash \{u_{\varepsilon,i}\}_{0<\varepsilon<\mu, 1\le i\le s}.$$

Let

$$\theta_\mu = \inf_{u\in\mathcal{S}_\mu} d(u, \mathcal{M}_K).$$

We now claim that

$$\theta_0 = \liminf_{\mu\to 0} \theta_\mu > 0.$$

If possible, let $\theta_0 = 0$; then there exist sequences $\{u_n\} \subset \mathcal{S}_\mu$ and $\{(\lambda_n, \xi_n)\} \subset K$ such that $\|u_n - U_{\lambda_n,\xi_n}\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Let $(\lambda_n, \xi_n) \to (\lambda, \xi)$. Then $(\lambda, \xi) \in K$ and $\nabla\mathcal{J}(\lambda, \xi) = 0$ and hence $\{u_n\}$ is a sequence of solutions bifurcating from (λ, ξ). But, by the uniqueness theorem (Theorem 1.3) and $\{u_n\} \subset \mathcal{S}_\mu$, we obtain a contradiction. This proves the claim.

As a result, we can choose $\mu_0 > 0$ small such that $\theta_\mu \ge \theta_0/2$ for all $\mu < \mu_0$. By Theorem 1.1, there exist some $C > 0$ and $\varepsilon' > 0$ such that

$$d(u_{\varepsilon,i}, \mathcal{M}_K) \le C\varepsilon, \quad i = 1, \ldots, s, \quad \varepsilon \in (0, \varepsilon').$$

Choosing $\rho_0 = \theta_0/2$ and $\varepsilon_1 = \min\{\theta_0/2C, \mu_0, \varepsilon'\}$, we obtain the required result.

References

[1] A. Ambrosetti and M. Badiale, 'Variational perturbative methods and bifurcation of bound states from the essential spectrum', *Proc. Roy. Soc. Edinburgh Sect. A* **128**(6) (1998), 1131–1161.

[2] A. Ambrosetti, A. Garcia and I. Peral, 'Perturbation of $\Delta u + u^{N+2N-2} = 0$, the scalar curvature problem in $\mathbb{R}^N$ and related topics', *J. Funct. Anal.* **165** (1999), 117–149.

[3] A. Ambrosetti and A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^N$*, Progress in Mathematics, 240 (Birkhäuser, Basel, 2006).

[4] M. Ben Ayed and K. El Mehdi, 'The Paneitz curvature problem on lower-dimensional spheres', *Ann. Global Anal. Geom.* **31**(1) (2007), 1–36.

[5] T. Branson, 'Differential operators canonically associated to a conformal structure', *Math. Scand.* **57** (1985), 293–345.

[6] T. Branson, A. Chang and P. Yang, 'Estimates and extremals for zeta function determinants on four-manifolds', *Comm. Math. Phys.* **149**(2) (1992), 241–262.

[7] G. Caristi and E. Mitidieri, 'Harnack inequality and applications to solutions of biharmonic equations', in: *Partial Differential Equations and Functional Analysis*, Operator Theory Advances and Applications, 168 (Birkhäuser, Basel, 2006), 1–26.

[8] A. Chang, 'On Paneitz operator—a fourth-order partial differential equation in conformal geometry', in: *Harmonic Analysis and Partial Differential Equations; Essays in honor of Alberto P. Calderon*, Chicago Lectures in Mathematics, 1999 (eds. M. Christ, C. Kenig and C. Sadorsky) (University of Chicago Press, 1996), Ch. 8, 127–150.

[9] A. Chang, M. Gursky and P. Yang, 'The scalar curvature equation on 2- and 3-spheres', *Calc. Var. Partial Differential Equations* **1** (1993), 205–229.
[10] A. Chang and P. Yang, 'On a fourth order curvature invariant', in: *Spectral Problems in Geometry and Arithmetic*, Contemporary Mathematics, 237 (American Mathematical Society, Providence, RI), 9–28.
[11] H. Chtioui and A. Rigane, 'On the prescribed Q-curvature problem on $\mathbb{S}^N$', *J. Funct. Anal.* **261**(10) (2011), 2999–3043.
[12] Z. Djadli, E. Hebey and M. Ledoux, 'Paneitz-type operators and applications', *Duke Math. J.* **104**(1) (2000), 129–169.
[13] Z. Djadli, A. Malchiodi and M. O. Ahmedou, 'Prescribing a fourth order conformal invariant on the standard sphere, Part I: a perturbation result', *Commun. Contemp. Math.* **4** (2002), 1–34.
[14] P. Esposito, 'Perturbations of Paneitz–Branson operators on $\mathbb{S}^N$', *Rend. Semin. Mat. Univ. Padova* **107** (2002), 165–184.
[15] P. Esposito and F. Robert, 'Mountain pass critical points for Paneitz–Branson operators', *Calc. Var. Partial Differential Equations* **15**(4) (2002), 493–517.
[16] V. Felli, 'Existence of conformal metrics on $\mathbb{S}^N$ with prescribed fourth-order invariant', *Adv. Differential Equations* **7**(1) (2002), 47–76.
[17] F. Gazzola, H. C. Grunau and G. Sweers, 'Polyharmonic boundary value problems', in: *Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains*, Lecture Notes in Mathematics, 1991 (Springer, Berlin, 2010).
[18] M. Gursky, 'The Weyl functional, de Rham cohomology, and Kahler–Einstein metrics', *Ann. of Math.* (2) **148** (1998), 315–337.
[19] E. Hebey and F. Robert, 'Asymptotic analysis for fourth order Paneitz equations with critical growth', *Adv. Calc. Var.* **4**(3) (2011), 229–275.
[20] J. L. Kazdan and F. W. Warner, 'Curvature functions for compact 2-manifolds', *Ann. of Math.* (2) **99**(1) (1974), 14–47.
[21] C. S. Lin, 'A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^N$', *Comment. Math. Helv.* **73**(2) (1998), 206–231.
[22] S. Paneitz, 'A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds', *SIGMA Symmetry Integrability Geom. Methods Appl.* **4** (2008), 1–3.
[23] S. Prashanth, S. Santra and A. Sarkar, 'On the perturbed Q-curvature problem on $\mathbb{S}^4$', *J. Differential Equations* **255**(8) (2013), 2363–2391.
[24] K. Sandeep, 'A compactness type result for Paneitz–Branson operators with critical nonlinearity', *Differential Integral Equations* **18**(5) (2005), 495–508.
[25] J. Wei and X. Xu, 'Classification of solutions of higher order conformally invariant equations', *Math. Ann.* **313**(2) (1999), 207–228.

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