A COMBINATION OF SURPLUS AND EXCESS REINSURANCE OF A FIRE PORTFOLIO

GUNNAR BENKTANDER AND JAN OHLIN
University of Stockholm

Reinsurance forms can roughly be classified into proportional and non-proportional. The authors of this paper had planned to investigate the "efficiency" of two different reinsurance forms, one from each of these categories. Efficiency is here understood as reduction in the variance of the annual results of the risk business achieved per unit of ceded reinsurance risk premium. This investigation may be carried out in full later.

This note will only deal with the interplay between surplus and excess of loss reinsurance; more specifically the effect of changes in the volume of surplus cessions on the excess of loss risk premium.

The study came out of a practical Fire Reinsurance rating problem and will be carried through under very simplified assumptions. Thus we will ignore the conflagration hazard and the possibility of a wrongly taxed PML. This means that if amounts above a PML of M are ceded on a surplus basis the highest loss per event will be M, and an excess cover above a priority m will never pay more than M-m per event.

The following notations will be used

$R(M)$ ceded risk premium volume on surplus basis, the PML retention being M.

$\pi_M(m)$ excess of loss risk premium if priority is m and surplus cessions are made above a PML of M.

Obviously $R(\infty) = 0$ and $\pi_m(m) = 0$ and

$$\frac{dR}{dM} < 0; \quad \frac{d\pi_M}{dM} > 0; \quad \frac{d\pi_M}{dm} < 0.$$

The volume of risk premiums ceded on surplus *and* excess basis is

$$\pi_M(m) + R(M)$$

This quantity obviously decreases when M increases, which gives

$$\frac{d\pi_M}{dM} + \frac{dR}{dM} < 0 \qquad \text{or} \qquad 0 < -\frac{d\pi_M}{dR} < 1.$$

This means that an increase in the volume of surplus cessions by a certain amount will lead to a decrease in the excess risk premium by a smaller amount. It is easily seen that $-d\pi_M/dR$ decreases when R increases. Starting from $M = \infty$ $(R = 0)$ the first small volume of surplus cessions will have the relatively highest reducing effect on π.

To investigate the behaviour of π_M and the interplay between π_M and R we introduce the following functions and notations

$g(s)$ — the frequency function of the PML-size of claimed risks.

$S(M) = \int\limits_{M}^{\infty} g(s)ds.$

$\varphi_s(x)$ — the probability of a claimed amount exceeding x, given that the PML-size of the claimed risk is s.

$\gamma(s) = -\frac{1}{s}\int\limits_{0}^{s} x\, d\,\varphi_s(x) = \frac{1}{s}\int\limits_{0}^{s} \varphi_s(x)dx$ — the expected damage degree, given s.

$H_M(x)$ — the probability of a claimed amount for own account exceeding x, after surplus sessions above PML M.

$H(x) = H_\infty(x).$

To express $H_M(x)$ in terms of $g(s)$ and $\varphi_s(x)$ we have to integrate the simultaneous density of s and x, $-g(s)ds\, d\varphi_s(x)$, over the shaded area in figure 1.

This gives

$$H_M(x) = \int\limits_{x}^{M} g(s)\,\varphi_s(x)ds + \int\limits_{M}^{\infty} g(s)\,\varphi_s\left(\frac{x}{M}s\right) =$$

$$= \int\limits_{x}^{\infty} g(s)\,\varphi_s(x)ds - \int\limits_{M}^{\infty} g(s)\left(\varphi_s(x) - \varphi_s\left(\frac{x}{M}s\right)\right)ds =$$

$$= H(x) - \int\limits_{M}^{\infty} g(s)\left(\varphi_s(x) - \varphi_s\left(\frac{x}{M}s\right)\right)ds \qquad (1)$$
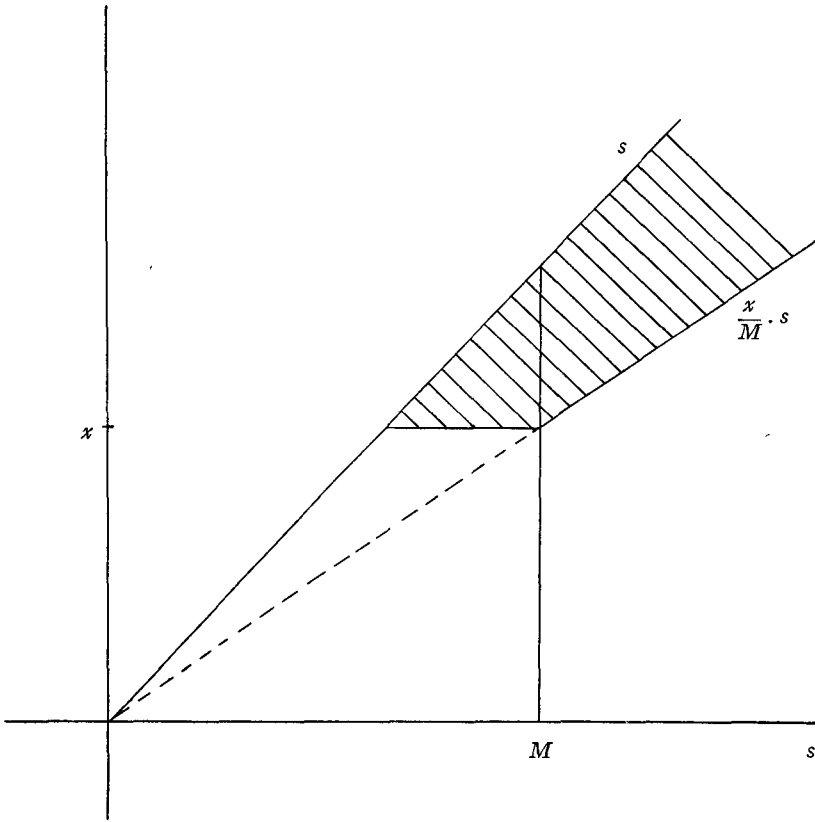
Fig. 1.

The excess of loss risk premium $\pi_M(m)$ is obtained by integrating $H_M(x)$ (see [1])

$$\pi_M(m) = \int_m^M H_M(x)dx = \pi(m) - \pi(M) - \int_m^M \int_M^\infty g(s)\left(\varphi_s(x) - \right.$$

$$\left. - \varphi_s\left(\frac{x}{M}s\right)\right)\,dsdx \qquad (2)$$

where $\pi(x) = \pi_\infty(x) = \int_x^\infty H(t)dt$.

The ceded surplus risk premium volume $R(M)$ is easily found to be

$$R(M) = -\int_M^\infty \int_0^s x\left(1 - \frac{M}{s}\right) d\varphi_s(x) \; g(s)ds = \int_M^\infty \gamma(s)(s\text{-}M)g(s)ds \quad (3)$$

Differentiating with regard to M we get

$$\frac{dR(M)}{dM} = -\int_M^\infty \gamma(s)g(s)ds \quad (4)$$

We will now consider two particular cases with regard to $\varphi_s(x)$, the *uniform case* and *the Pareto case*. In both cases we start by finding general expressions for $\pi_M(m)$ and $d\pi_M/dR$, i.e. expressions valid for any choice of $g(s)$. In order to arrive at explicit formulas that make numerical computations possible, we then consider the following particular choice of $g(s)$.

The Pareto law

$$g(s) = \frac{\alpha}{a}\left(\frac{s}{a}\right)^{-\alpha-1}, \; s \geq a$$

will be assumed to describe the distribution of the PML size s of claimed risks, for that part of the portfolio for which $s > a$. This leaves us some freedom to assume various combinations of claims frequencies and distribution of portfolio according to size for $s > a$, and complete freedom in this respect for $s < a$.

1. *The Uniform Case*

In this case the damage degree is assumed to be uniformly distributed in the interval [0,1], i.e. $\varphi_s(x) = 1 - \dfrac{x}{s}$.

The expected damage degree is constant, $\gamma(s) \equiv \dfrac{1}{2}$, and hence

$$R(M) = \frac{1}{2}\int_M^\infty (s - M)\,g(s)ds$$

and

$$\frac{dR(M)}{dM} = -\frac{1}{2}\int_M^\infty g(s)ds = -\frac{1}{2}S(M).$$

Introducing $\varphi_s(x) = 1 - \dfrac{x}{s}$ in (1) we get

$$H_M(x) = H(x) - \int\limits_M^\infty g\,(s)\left(\frac{x}{M} - \frac{x}{s}\right) ds = H(x) - \frac{x}{M}\,H(M) \qquad (5)$$

and by integrating

$$\pi_M(m) = \pi(m) - \pi(M) - \frac{M^2 - m^2}{2M}\,H(M). \qquad (6)$$

Writing this as

$$\pi(m) - \pi_M(m) = \pi(M) + \frac{M^2 - m^2}{2M}\,H(M)$$

we see that the reduction of the excess risk premium due to surplus cessions above M equals the excess risk premium above M, plus the expected number of claims above M, multiplied by a factor $\dfrac{M^2 - m^2}{2M}$.

We now differentiate $\pi_M(m)$ with regard to M using

$$\frac{d\pi(M)}{dM} = -\,H(M)$$

and

$$\frac{dH\,(M)}{dM} = -\int\limits_M^\infty \frac{g\,(s)}{s}\,ds$$

We get

$$\frac{d\pi_M(m)}{dM} = H(M) - \frac{1}{2}\left(1 + \frac{m^2}{M^2}\right) H(M) + \frac{M^2 - m^2}{2M}\int\limits_M^\infty \frac{g\,(s)}{s}\,ds =$$

$$= \frac{1}{2}\left(1 - \frac{m^2}{M^2}\right)\left(H(M) + M\int\limits_M^\infty \frac{g\,(s)}{s}\,ds\right)$$

But since

$$H(M) = \int\limits_M^\infty g\,(s)\left(1 - \frac{M}{s}\right) ds = S(M) - M\int\limits_M^\infty \frac{g\,(s)}{s}\,ds,$$

the last factor reduces to $S(M)$, and we get

$$\frac{d\pi_M(m)}{dM} = \frac{1}{2} S(M) \left(1 - \frac{m^2}{M^2}\right) \tag{7}$$

We thus obtain

$$-\frac{d\pi_M(m)}{dR(M)} = \left(1 - \frac{m^2}{M^2}\right). \tag{8}$$

We see that in this case $d\pi_M/dR$ does not depend on the function $g(s)$ but only on the priority m and the retention M — in fact only on the ratio m/M. It can be shown that $d\pi_M/dR$ will have this property as soon as the distribution of the damage degree does not depend on s, i.e. when $\varphi_s(x)$ can be written as a function only of x/s. The proof of this will be published later.

We now introduce $g(s) = \dfrac{\alpha}{a} \left(\dfrac{s}{a}\right)^{-\alpha-1}$ $(s > a)$. To calculate $\pi_M(m)$ we need the functions $H(x)$ and $\pi(x)$.

We get

$$H(x) = \int_x^\infty g(s) \left[1 - \frac{x}{s}\right] ds = \frac{1}{\alpha+1} \left(\frac{x}{a}\right)^{-\alpha}$$

$$\pi(x) = \int_x^\infty H(t)dt = \frac{a}{(\alpha+1)(\alpha-1)} \left(\frac{x}{a}\right)^{-\alpha+1}$$

Inserting this in (5) gives

$$\pi_M(m) = \pi(m) - \frac{a}{2(\alpha+1)} \left(\frac{M}{a}\right)^{-\alpha+1} \left(\frac{\alpha+1}{\alpha-1} - \frac{m^2}{M^2}\right) \tag{9}$$

We also get

$$R(M) = \frac{a}{2(\alpha-1)} \left(\frac{M}{a}\right)^{-\alpha+1}$$

Adding $R(M)$ and $\pi_M(m)$ gives the following expression for the total volume of risk premium ceded on surplus *and* excess basis

$$\pi_M(m) + R(M) = \pi(m) + \frac{a}{2(\alpha+1)} \left(\frac{M}{a}\right)^{-\alpha+1} \frac{m^2}{M^2}. \tag{10}$$

2. *The Pareto Case*

In [2] Benckert and Sternberg investigate whether the distribution of the damage degree can be described by a Pareto distribution, modified by concentrating all mass above s discreetly in the point $x = s$. They come to the conclusion that this model gives a reasonably good fit to empirical data for some classes of fire insurance, provided that claims below a certain limit are excluded.

Since most fire insurance policies in Sweden nowadays are written with a deductible, the exclusion of the very smallest claims is not a serious limitation. In theory the introduction of a deductible should be taken into account by reducing all claims by the deductible amount and working with a Pareto distribution with a density of the type

$f(x) = \beta(x + b)^{-\beta-1}$ over the interval $(0, S - b)$. But, since we are mainly interested in the large claims where the influence of the deductible is negligible, we have decided to avoid unnecessary complications in the formulas by simply excluding claims below a certain limit. Following Benckert and Sternberg in [2] we take this limit as the unit of value. We thus arrive at the following expression for $\varphi_s(x)$

$$\varphi_s(x) = \begin{cases} x^{-\beta}, & 1 \le x < s \quad (\beta > 0) \\ 0, & x \ge s \end{cases}$$

The expected damage degree is then

$$\gamma(s) = \frac{1}{s} \left(\int_1^s x\, \beta x^{-\beta-1}\, dx + s \cdot s^{-\beta} \right) = \frac{s^{1-\beta} - \beta}{s(1 - \beta)}$$

(We assume here and in the following that $\beta \ne 1$. The modifications when $\beta = 1$ are self-evident.)

Inserting the expression for $\varphi_s(x)$ in (1) leads to

$$H_M(x) = H(x) - \int_M^\infty g(s) \left(x^{-\beta} - \left(\frac{xs}{M}\right)^{-\beta} \right) ds =$$

$$= H(x) - \left(\frac{x}{M}\right)^{-\beta} (H(M) - C(M)) \tag{11}$$

where

$$C(M) = \int_M^\infty g(s)\, s^{-\beta}\, ds$$

Integrating (11) and denoting $\int_m^M \left(\dfrac{x}{M}\right)^{-\beta} dx$ by $I(M)$

we get

$$\pi_M(m) = \pi(m) - \pi(M) - I(M)\,(H(M) - C(M)) \qquad (12)$$

We now differentiate $\pi_M(m)$ and by noting that

$$\frac{dI(M)}{dM} = 1 + \frac{\beta}{M}\, I(M)$$

and

$$\frac{d(H(M) - C(M))}{dM} = -\frac{\beta}{M}\,(H(M),$$

we find that

$$\frac{d\pi_M(m)}{dM} = H(M) - \left(1 + \frac{\beta}{M} I(M)\right)(H(M) - C(M)) + I(M)\frac{\beta}{M} H(M) =$$

$$= C(M)\left(1 + \frac{\beta}{M} I(M)\right). \qquad (13)$$

Inserting the expression for $\gamma(s)$ in $\dfrac{dR}{dM}$ we get

$$\frac{dR(M)}{dM} = -\frac{1}{1-\beta} \int_M^\infty \left(s^{-\beta} - \frac{\beta}{s}\right) g(s)ds = -$$

$$-\frac{1}{1-\beta}\left(C(M) - \beta \int_M^\infty \frac{g(s)}{s}\, ds\right) \qquad (14)$$

The resulting expression for $\dfrac{d\pi_M}{dR}$ will hence be

$$-\frac{d\pi_M(m)}{dR(M)} = \frac{(1-\beta)\,C(M)}{C(M) - \beta \displaystyle\int_M^\infty \frac{g(s)}{s}\, ds}\left(1 + \frac{\beta}{M} \int_m^M \left(\frac{x}{M}\right)^{-\beta} dx\right) =$$

$$= \frac{1}{1 - \dfrac{\beta}{C(M)} \displaystyle\int_{M}^{\infty} \dfrac{g(s)}{s}\, ds} \left(1 - \beta \left(\frac{m}{M}\right)^{1-\beta}\right) \qquad (15)$$

We see that the first factor does not depend on m whereas the second depends only on m/M. The first factor is obviously always greater than one. If $\beta < 1$ (and according to [2] it seems reasonable to work with values of β in the vicinity of 0,5) the ratio $\dfrac{1}{C(M)} \displaystyle\int_{M}^{\infty} \dfrac{g(s)}{s}\, ds$

will be a decreasing function of M and, since

$$g(s) \left(\frac{s}{M}\right)^{-1} < g(s) \left(\frac{s}{M}\right)^{-\beta}, \text{ for } s \geq M,$$

we see that $\quad \dfrac{1}{C(M)} \displaystyle\int_{M}^{\infty} \dfrac{g(s)}{s}\, ds < M^{\beta-1}.$

This means that the first factor in $-\dfrac{d\pi_M}{dR}$ is bounded above by

$$\frac{1}{(1 - \beta\, M^{\beta-1})}$$

and for large values of M this quantity is close to one, the more so the smaller β is. We have thus shown that $-d\pi_M/dR$ will, for large M, be approximately independent of $g(s)$ and depend only on m/M.

We now introduce $g(s) = \dfrac{\alpha}{a} \left(\dfrac{s}{a}\right)^{-\alpha-1} \quad (s > a).$

We get

$$H(x) = a^{\alpha}\, x^{-\alpha-\beta}$$

$$\pi(x) = \int_{x}^{\infty} H(t)dt = \frac{a^{\alpha}\, x^{-\alpha-\beta+1}}{\alpha + \beta - 1} = \frac{x}{\alpha + \beta - 1}\, H(x)$$

$$C(x) = \int_{x}^{\infty} g(s)\, s^{-\beta}\, ds = \frac{\alpha a^{\alpha}\, x^{-\alpha-\beta}}{\alpha + \beta} = \frac{\alpha}{\alpha + \beta}\, H(x)$$

$$\int\limits_{M}^{\infty} \frac{g(s)}{s}\, ds = \frac{\alpha a^{\alpha}\, M^{-\alpha-1}}{\alpha + 1}$$

Inserting in (12) gives

$$\pi_M(m) = \pi(m) - \frac{M}{\alpha+\beta-1}\, H(M) - I(M)\left(H(M) - \frac{\alpha}{\alpha+\beta}H(M)\right) =$$

$$= \pi(m) - MH(M)\left(\frac{1}{\alpha+\beta-1} + \frac{\beta}{\alpha+\beta}\frac{I(M)}{M}\right) =$$

$$= \pi(m) - \left(\frac{M}{a}\right)^{-\alpha}\left(\frac{M^{1-\beta}}{\alpha+\beta-1} + \frac{\beta(M^{1-\beta}-m^{1-\beta})}{(\alpha+\beta)(1-\beta)}\right) \qquad (16)$$

Inserting in (15) gives

$$-\frac{d\pi_M(m)}{dR(M)} = \frac{1}{1 - \dfrac{\beta(\alpha+\beta)}{\alpha+1}M^{\beta-1}}\left(1 - \beta\left(\frac{m}{M}\right)^{1-\beta}\right) \qquad (17)$$

Tedious but elementary calculations yield

$$R(M) = \frac{\alpha}{1-\beta}\left(\frac{M}{a}\right)^{-\alpha}\left(\frac{M^{1-\beta}}{(\alpha+\beta-1)(\alpha+\beta)} - \frac{\beta}{\alpha(\alpha+1)}\right) \qquad (18)$$

If we add this to $\pi_M(m)$ the terms containing $M^{1-\beta}$ cancel out, and we get the following expression for the total ceded risk premium volume

$$\pi_M(m) + R(M) = \pi(m) + \frac{\beta}{1-\beta}\left(\frac{M}{a}\right)^{-\alpha}\left(\frac{m^{1-\beta}}{\alpha+\beta} - \frac{1}{\alpha+1}\right) \qquad (19)$$

The apparent lack of dimensional consistency in (18) and (19) is a result of our particular choice of $\varphi_s(x)$. All terms in $\pi_M(m)$ have the dimension $1 - \beta$ in m or M, but due to the form of $\gamma(s)$, $R(M)$ will also contain a term which appears to be dimensionless. If we had called the lower limit of x b instead of chosing it as our unit, both π_M and R would have been of dimension one (in a or b).

Numerical examples and conclusions

In order to illustrate the behaviour of $\pi_M(m)$ we have computed $\pi_M(m)$ numerically under the assumption that $g(s)$ is of the Pareto type with $\alpha = 2$. The results are given in table 1 for the uniform case and in table 2a for the Pareto case with $\beta = 0.5$.

When m and M are chosen as multiples of a the computational work involved is very slight. After computing $R(M)$ and $\pi(m)$ it only remains to compute the product of two factors, the first depending on M and the second on m (see (10) and (19)). It is further seen from (10) that in the uniform case the parameter a simply plays the role of norming constant. In the Pareto case, however, this is not so, since the damage degree distribution depends on s. It is therefore necessary to fix the value of a and in the tables 1 and 2a the value $a = 400$ has been used.

Table 1

$\pi_M(m)$. Uniform case.
($\alpha = 2. a = 400$)

M	a	2a	3a	4a	5a	10a	20a	50a	100a	∞
R(M)	200	100	67	50	40	20	10	4	2	0
m										
a	0	42	69	84	94	113	123	129	131	133
2a		0	10	21	29	47	57	62	65	67
3a			0	4	9	25	35	40	42	44
4a				0	2	14	23	29	31	33
5a					0	8	17	23	25	27

One sees immediately that the values in table 1 are much larger than the corresponding values in tables 2a. This is a natural consequence of the difference in expected damage degree. In the uniform case $\gamma(s)$ is 50% whereas in the Pareto case, with $\beta = 0.5$, $\gamma(s)$ is less than 10% for $s \geq 400$. To facilitate comparison of

Table 2

$\pi_m(m)$. Pareto Case.
($\alpha = 2. a = 400. \beta = 0,5$)
a. Absolute values.

M	a	2a	3a	4a	5a	10a	20a	50a	100a	∞
R(M)	21.0	7.46	4.07	2.65	1.90	0.671	0.238	0.060	0.002	0
m										
a	0	7.8	10.1	11.1	11.7	12.7	13.1	13.3	13.3	13.3
2a		0	1.87	2.75	3.26	4.15	4.50	4.66	4.71	4.71
3a			0	0.77	1.21	2.03	2.36	2.51	2.57	2.57
4a				0	0.40	1.15	1.47	1.61	1.67	1.67
5a					0	0.70	1.00	1.14	1.19	1.19

b. Normed values.

M $\diagdown$	a	$2a$	$3a$	$4a$	$5a$	$10a$	$20a$	$50a$	$100a$	∞
$R(M)$	200	71	39	25	18	6	2	0.6	0.02	0
m $\diagdown$										
a	0	74	96	106	112	121	125	126	127	127
$2a$		0	18	26	31	40	43	44	45	45
$3a$			0	7	12	19	22	24	24	24
$4a$				0	4	11	14	15	16	16
$5a$					0	7	10	11	11	11

relative sizes we have therefore normed the values for the Pareto case, by putting $R(a) = 200$ and changing all other values in table 2a in proportion. The results are given in table 2b.

A comparison of table 1 with 2b can now be said to show the effect of the "decreasing damage degree" in the Pareto case. $R(M)$ and $\pi(m)$ decrease more rapidly than in the uniform case and $\pi_M(m)$ approaches its limit $\pi(m)$ quicker.

Table 3

$$-\frac{d\pi_M}{dR}$$

$\dfrac{m}{M}$	Uniform Case	Pareto Case $(\beta = 0.5; M = \infty)$
0.005	1.000	0.965
0.01	1.000	0.950
0.05	0.998	0.888
0.1	0.990	0.842
0.2	0.960	0.776
0.3	0.910	0.726
0.4	0.840	0.684
0.5	0.750	0.646
0.6	0.640	0.613
0.7	0.510	0.582
0.8	0.360	0.553
0.9	0.190	0.526
1.0	0.000	0.500

The behaviour of $- d\pi_M/dR$ is illustrated in table 3 and figure 2. Table 3 gives values of $- d\pi_M/dR$ in the uniform case and asymptotic values of $- d\pi_M/dR$ (i.e. the second factor in (15)) in the Pareto case

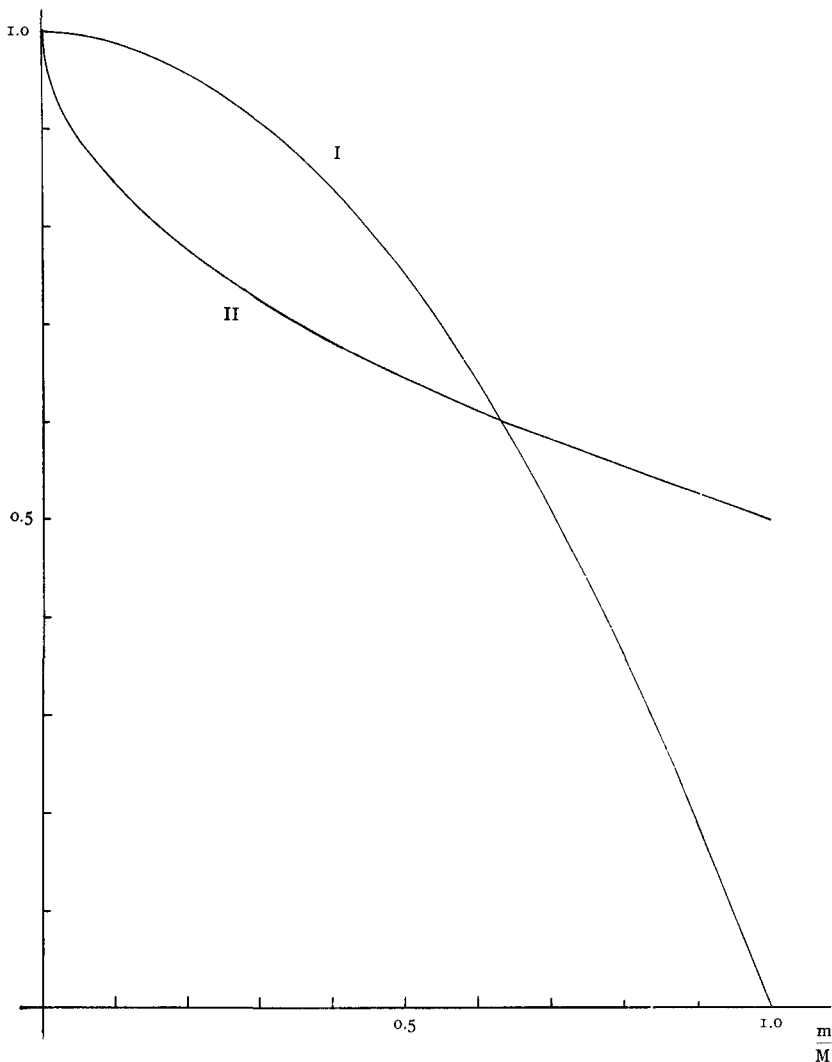with $\beta = 0.5$. The same functions are shown in graphic form in figure 2.

Figure 2.
I. Uniform Case   II. Pareto Case $(\beta = 0.5; M = \infty)$

Table 4, finally, gives values of the "correction factor for finite M", i.e. the first factor in (17) computed for $\alpha = 2$ and $\beta = 0.5$.

Table 4

Pareto Case. Correction factor for finite M.

($\alpha = 2$, $\beta = 0.5$)

M		M		M	
100	1.043	800	1.015	4000	1.007
200	1.030	1200	1.012	8000	1.005
300	1.025	1600	1.010	20000	1.003
400	1.021	2000	1.009	40000	1.002

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