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DIFFERENTIAL EQUATION FOR CLASSICAL-TYPE ORTHOGONAL POLYNOMIALS

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ABSTRACT. The second order differential equation of Littlejohn-Shore for Laguerre type orthogonal polynomials is generalized in two ways. First the positive Dirac mass can be situated at any point and secondly the weight can be any classical weight modified by an arbitrary number of Dirac distributions.

1. **Introduction.** Modification of a given weight $\rho(x)$ $(\rho > 0, a < x < b)$ connects nontrivially the family of polynomials $p_n(x)$, orthogonal with respect to ρ in (a, b), to the new family $\bar{p}_n(x)$ of polynomial orthogonal with respect to the modified weight $\bar{\rho}$ (in the same interval).

In the two following situations, the links between p_n and $\bar{p}_n$ are particularly simple:

1. *Rational Case.* $\bar{\rho} = \pi\rho$, when $\pi \equiv \pi(x)$ is a rational function $(\pi = N/D)$ with poles and zeros outside the support of ρ.

2. δ *Dirac distribution.*

$$\bar{\rho} = \rho + \sum_{k=1}^{K} \lambda_k \delta(x - x_k),$$

where the positive mass λ_k is located at x_k, x_k outside or *inside* the support of ρ.

In the first case, the relation between the family $\bar{p}_n$ and p_n was given by Christoffel [3] when π is a polynomial and the full case $\pi = N/D$ is credited to Uvarov [15] (see Gautschi) [3].

The development of the form

$$\pi N \bar{p}_n = \sum_{i=n-k}^{n+q} h_{i,n} p_i \qquad (n \geqq k) \tag{1}$$

where q and k are the degree respectively of N and D and $h_{i,n}$ are constants given by the minors of $p_i(x)$ in the Christoffel determinant.

In the second case, relation between $\bar{p}_n$ and p_n was given by Uvarov [15], and Nevai [9] in the real case, and by Cachafeiro-Marcellan [1], in a more general situation and

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reads as:

$$\bar{p}_n = H_n p_n + p_n \sum_{k=1}^{K} (H_{k,n})/(x - x_k) + p_{n-1} \sum_{r=1}^{K} (G_{r,n})/(x - x_r) \tag{2}$$

where the constants H_n, $H_{k,n}$ and $G_{r,n}$ are given explicitly in [1]. In fact these two modifications are closely related as shown in the Cachafeiro-Marcellan paper.

In both cases, the new polynomials $\bar{p}_n$ are *semi-classical* [4, 8] (the appellation "Semi-Classical" was coined for the first time by Hendriksen and van Rossum [4] and extended by Maroni [8]) and therefore are solutions of a second order differential equation of the Laguerre type [6].

$$\bar{\sigma}\Theta_n \bar{p}_n'' + (\bar{\tau}\Theta_n - \bar{\sigma}\Theta_n')\bar{p}_n' + K_n \bar{p}_n = 0, \tag{3}$$

where $\bar{\sigma}$ and $\bar{\tau}$ are polynomials defining the new weight $\bar{\rho}$, via the weight differential equation:

$$(\bar{\sigma}\bar{\rho})' = \bar{\tau}\bar{\rho}, \tag{4}$$

and Θ_n and K_n are polynomials in x of *fixed* degree related to the degree of $\bar{\sigma}$ and $\bar{\tau}$.

In the Laguerre differential equation the polynomial coefficient Θ_n and K_n are difficult to obtain in explicit form except in some very peculiar situations. For instance the classical case: $\Theta_n = 1, K_n =$ constant (depending on n) and the super classical case [12, 13] ("Super Classical" is used in [12, 13] and refer to polynomials orthogonal with respect to a classical weight time rational functions and for which one explicit differential equation can be written [12]): $\bar{\rho} = \pi\rho$, π rational function and where ρ is a classical weight.

The aim of this paper is to use the Cachafeiro-Marcellan representation in order to build explicitly the differential equation satisfied by $\bar{p}_n$ when p_n is any classical orthogonal polynomial and x_k is located at any point.

The building algorithm is similar to the algorithm described before [12] and adapted from Shohat [14].

The basic ingredients are, for each classical orthogonal polynomial p_n (Jacobi, Laguerre, Hermite) the three relations:

$$xp_n = \alpha_n p_{n+1} + \beta_n p_n + \gamma_n p_{n-1} \tag{5}$$

$$\sigma p_n' = \alpha_n^* p_{n+1} + (\beta_n^* + x\gamma_n^*) p_n \tag{6}$$

$$\sigma p_n'' + \tau p_n' + K_n p_n = 0, \tag{7}$$

where the constants $\alpha_n, \beta_n, \gamma_n, \alpha_n^*, \beta_n^*, \gamma_n^*, K_n$, are given for instance in [10, 11], $\sigma = \sigma_J, \sigma_L, \sigma_H$, with

(8) $$\sigma_J = 1 - x^2, \sigma_L = x, \sigma_H = 1,$$

and τ is defined by:

(9) $$\tau = (\sigma\rho)'/\rho,$$

where

(10) $$\rho_J = (1-x)^\alpha (1+x)^\beta, \rho_L = x^\alpha e^{-x}, \rho_H = e^{-x^2} (\alpha, \beta > -1).$$

In order to simplify the constants appearing in (2), we prefer to work with *Monic* classical orthogonal polynomials $\hat{p}_n$, in which case the "hat" constants are given, or related, to the previous ones by the obvious relation:

(11) $$\hat{\alpha}_n = 1, \hat{\beta}_n = \beta_n \quad \hat{\gamma}_n = \gamma_n \alpha_{n-1}$$

(12) $$\hat{\alpha}_n^* = \tau' + \sigma''(n - 1/2) = (\alpha_n^*)/(\alpha_n), \hat{\beta}_n^* = \beta_n^*, \hat{\gamma}_n^* = \gamma_n^*.$$

In the following however, in order to simplify the notations, p_n (without hat) will denote Monic classical orthogonal polynomials, and the constants in relations (5) and (6), for these Monic polynomials will be rewrite using (11) and (12).

2. **Algorithm.** By multiplication by

$$\pi(x) = \prod_{k=1}^{K} (x - x_k),$$

relation (2) becomes:

(13) $$\pi \bar{p}_n = q_1 p_n + q_2 p_{n-1}$$

where $q_1 \equiv q_1(x, n)$ and $q_2 \equiv q_2(x, n)$ are polynomials in x which can be computed explicitly for each K. We reach here the starting point of the three steps algorithm described in [13], with a shift of indices. The second step is therefore identical: derivation of relation (13) with respect to x, multiplication by σ and use of the relation (5), (11) and (12) to eliminate p_{n+1}.

The result is written in the following form:

(14) $$\sigma \pi' \bar{p}_n + \pi \sigma \bar{p}_n' = q_3 p_n + q_4 p_{n-1}$$

where the polynomial q_3 and q_4 are (See [13]):

$$q_3 = \sigma q_1' + q_1[(x - \beta_n)\alpha_n^*/\alpha_n + \beta_n^* + x\gamma_n^*] + q_2\alpha_{n-1}^*/\alpha_{n-1} \tag{15}$$

$$q_4 = \sigma q_2' - q_1\gamma_n\alpha_{n-1} + q_2(\beta_{n-1}^* + x\gamma_{n-1}^*]. \tag{16}$$

The final step constructs, by summation as in [12], the differential equation satisfied by p_n starting with relation (13).

The quantity q_1p_n satisfies the differential relation obtained by multiplication of relation (7) by q_1:

$$\sigma(q_1p_n)'' + \tau(q_1p_n)' - 2\sigma q_1'p_n' - \tau q_1'p_n - \sigma q_1''p_n + q_1K_np_n = 0 \tag{17}$$

and a similar relation holds for q_2p_{n-1}.

By summation of these two relations, elimination of p_n, p_{n-1} by inversion of the system (13) and (14), and uses of relation (6) and (5), we obtain the differential equation satisfied by $\bar{p}_n$:

$$\sigma\Delta(\pi\bar{p}_n)'' + \tau\Delta(\pi\bar{p}_n)' + \pi\sigma H\bar{p}_n' + G\bar{p}_n = 0 \tag{18}$$

where

$$\Delta = q_1q_4 - q_2q_3 = \Theta_n \tag{19}$$

and $H = H(x, n)$, and $G = G(x, n)$ are polynomials in x which can be computed explicitely from (17), (13) and (14).

3. **Classical-type orthogonal polynomials.** As an example, let us construct in detail the differential equation satisfied by polynomials p_n orthogonal with respect to a weight $\bar{\rho}$ which is a classical weight modified by one δ Dirac distribution (Koornwinder [5] considered also a similar situation but at the level of the 4$^{\text{th}}$ order differential equation) at $x = c, c \in \mathbf{R}$

$$\bar{\rho} = \rho + \lambda\delta(x - c) \tag{20}$$

The Laguerre-type [7] polynomials for instance belong to this class:

$$\bar{\rho} = e^{-x} + (1/R)\delta(x), c = 0 \tag{21}$$

Relation (2) becomes in this case:

$$\bar{p}_n = p_n + p_nh_n(x - c)^{-1} + p_{n-1}g_n(x - c)^{-1} \tag{22}$$

with [1, 2]

$$h_n = H_{1,n} = -\lambda p_n(c)p_{n-1}(c)\{d_{n-1}^2 + \lambda[p_n'(c)p_{n-1}(c) - p_n(c)p_{n-1}'(c)]\}^{-1}$$

$$g_n = G_{1,n} = \lambda p_n(c)p_n(c)\{d_{n-1}^2 + \lambda[p_n'(c)p_{n-1}(c) - p_n(c)p_{n-1}'(c)]\}^{-1}$$

and

$$d_n^2 = \int_a^b [p_n(x)]^2 \rho(x)dx. \tag{23}$$

The Basic polynomials q_1 and q_2 defined before (eq 13) by:

$$\pi\bar{p}_n = q_1 p_n + q_2 p_{n-1} \qquad (\pi = x - c)$$

become

$$\begin{aligned} q_1 &= x - c + h_n \\ q_2 &= g_n. \end{aligned} \tag{24}$$

Polynomials q_3 and q_4 defined in section 2 become:

$$q_3 = \sigma + q_1[\beta_n^* + x\gamma_n^* + (x - \beta_n)\alpha_n^*/\alpha_n] + q_2\alpha_{n-1}^*/\alpha_{n-1}$$

$$q_4 = -q_1\alpha_n^*\gamma_n\alpha_{n-1}/\alpha_n + q_2(\beta_{n-1}^* + x\gamma_{n-1}^*]. \tag{25}$$

In the differential equation (18), the polynomials H and G can be now explicitly computed from relation (17) and from the solution of the system (13) and (14), $(H = -\Delta' = -\Theta_n')$.

4. **Laguerre-type differential equation.** The constants in (5), (6), (7) for the Laguerre polynomials ($\alpha = 0$) are:

$$\begin{aligned} &\alpha_n = -(n+1), \quad \beta_n = 2n+1, \quad \gamma_n = -n \\ &\alpha_n^* = n+1, \quad \beta_n^* = -(n+1), \quad \gamma_n^* = +1 \\ &K_n = n, \end{aligned} \tag{26}$$

The two basic relations for MONIC Laguerre polynomials can be written as:

$$\hat{L}_n(x) = [x - (2n-1)]\hat{L}_{n-1}(x) - (n-1)^2\hat{L}_{n-2}(x) \tag{27}$$

$$x\hat{L}_{n-1}'(x) = (x - n)\hat{L}_{n-1}(x) - \hat{L}_n(x). \tag{28}$$

Relation (22) becomes:

$$(x - c)\bar{p}_n = (x - c + h_n)\hat{L}_n(x) + g_n\hat{L}_{n-1}(x)$$

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with

$$(29)\qquad \begin{aligned} h_n &= -\lambda l_n l_{n-1}\{[(n-1)!]^2 + \lambda c^{-1}[n^2 l_{n-1}^2 + l_n^2 + (2n-c)l_n l_{n-1}]\}^{-1} \\ g_n &= \lambda l_n^2\{[(n-1)!]^2 + \lambda c^{-1}[n l_{n-1}^2 + l_n^2 + (2n-c)l_n l_{n-1}\}^{-1} \end{aligned}$$

using for short $l_n = L_n(c)$.

The first step polynomials q_1 and q_2 are:

$$(30)\qquad q_1 = x - c + h_n,\ q_2 = g_n$$

and the second step polynomials q_3 and q_4:

$$(31)\qquad q_3 = (n+1)x + (nh_n - nc - g_n),\ q_4 = (n^2 + g_n)x + n(nh_n - cn - g_n).$$

The polynomial $\Delta = q_1 q_4 - q_2 q_3 = \Theta_n$ becomes:

$$(32)\qquad \Delta = (n^2 + g_n)x^2 + [2n(nh_n - nc - g_n) + g_n(h_n - c - 1)]x + (nh_n - nc - g_n)^2.$$

The G polynomial is easily computed:

$$(33)\qquad \begin{aligned} G = (x-c)\{&(n^2+g_n)x^2 + [n(nh_n - nc - g_n) - (n^2 + g_n + ng_n)]x \\ &+ (n+g_n)(nh_n - nc - g_n)\} \\ &- x[g_n x + 2n^2 h_n + g_n h_n - (2n+1)g_n] + n(x-c)\Delta \end{aligned}$$

The differential equation satisfied by $\bar{p}_n = \bar{L}_n$ reads now:

$$(34)\quad x(x-c)\Delta\bar{L}_n'' + [(2x + (1-x)(x-c))\Delta - x(x-c)\Delta']\bar{L}_n' + [(1-x)\Delta + G]\bar{L}_n = 0.$$

REMARKS 1. The Littlejohn-Shore differential equation can easily be recovered with $c = 0$, $\lambda = 1/R$ and

$$(35)\qquad l_n = (-1)^n n!,\ l_n' = \hat{L}_n'(0) = (-1)^{n+1} n! n.$$

The constants h_n and g_n reduce to (eq 23)

$$(36)\qquad h_n = \frac{n}{n+R},\ g_n = \frac{n^2}{n+R}$$

and Δ and G to:

$$(37)\qquad \Delta = n^2\frac{(n+R+1)}{n+R}x^2 - \frac{n^2 R}{(n+R)^2}x$$

$$G = x\left\{\frac{n^2(n+R+1)}{n+R}x^2 - \left[n^2 + \frac{(n+2)n^2}{n+R}\right]x + \frac{n^2 R}{(n+R)^2}\right\} + n + \Delta.$$

After division by n^2x^2 the differential equation becomes:

$$(38)\quad [(n+R+1)(n+R)x^2 - Rx][\bar{L}_n'' + (n+R+1)(n+R)(x-x^2) + R(x-2)]\bar{L}_n' + [(n+R)(n+R+1)x - n(n+1+2R)]\bar{L}_n = 0.$$

This equation coincides with the Littlejohn-Shore one written in the following way:

$$(39)\quad [(R^2+R+\lambda_n)x^2 - Rx]\bar{L}_n'' + [-(R^2+R+\lambda_n)x^2 + (R^2+2R+\lambda_n)x - 2R]\bar{L}_n' + [(2R\lambda_n + 22\lambda_n - \mu_n)x - \lambda_n]\bar{L}_n = 0$$

with the notation [5]

$$(40)\quad \lambda_n = (2R+2)n + n(n-1)$$

$$\mu_n = (3R^2 + 45R + 42)n + 18n(n-1) - n(n-1)(n-2)$$

2. If we multiply the equation (34) by $x - c(c \neq 0)$ we obtain:

$$\bar{\sigma}\Delta\bar{L}_n'' + [\bar{\tau}\Delta - \bar{\sigma}\Delta']\bar{L}_n' + K_n\bar{L}_n = 0$$

with

$$(41)\quad \bar{\sigma} = x(x-c)^2,\ \bar{\tau} = (x-c)^2(1-x) + 2x(x-c)$$

which is the Laguerre equation (3) with the explicit polynomials $\bar{\sigma}$ and $\bar{\tau}$ in (4).

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