# 39

# Correlators in $x$-space

In previous chapters we have discussed correlators in the momentum space. In some applications, some authors prefer to work in the $x$-space. From the pure theoretical point of view, the use of the $x$-space is no better than the use of the momentum space, which is the traditional tool of QSSR [1,3]. However, each representation has its own advantages and inconveniences. The $x$-space approach is described in detail, for example in [386]. In particular, the current correlators are measured in the most direct way on the lattice [393]. In the coordinate space, the two-point functions obey a dispersion representation:

$$\Pi(x) = \frac{1}{4\pi^2}\int_0^\infty dt\ \frac{\sqrt{t}}{x} K_1(x\sqrt{t})\ \mathrm{Im}\Pi(t)\ , \tag{39.1}$$

where $K_1(z)$ is the modified Bessel function, which behaves for small $z$ as:

$$K(z \to 0) \simeq \frac{1}{z} + \frac{z}{2}\ln z\ . \tag{39.2}$$

In the limit $x \to 0$, $\Pi(x)$ coincides with the free-field correlator. For the sake of completeness, we begin with a summary of theoretical expressions for the current correlators, both in the $Q-$ and $x-$spaces. We will focus on the $(V \pm A)$ and $(S \pm P)$ channels since the recent lattice data [393] refer to these channels.

### 39.1 (Axial-)vector correlators

In case of $(V \pm A)$ currents the correlator is defined as:

$$\Pi_{\mu\nu}(q) = i\int d^4x\ e^{iqx}\langle T J_\mu(x) J_\nu(0)^\dagger\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)\ , \tag{39.3}$$

where $-q^2 \equiv Q^2 > 0$ in the Euclidean space–time. For the sake of definiteness we fix the flavour structure of the light-quark current $J_\mu$ as:

$$J_\mu^{V\pm A} = \bar{u}\gamma_\mu(1 \pm \gamma_5)d\ . \tag{39.4}$$



https://doi.org/10.1017/9781009290296.051 Published online by Cambridge University Press

In the chiral limit one has in the $(V+A)$ case (see, e.g., [1,3] and previous chapters):

$$\Pi^{V+A}(Q^2) = \frac{1}{2\pi^2}\left\{ -\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\nu^2} - \frac{\alpha_s}{\pi}\frac{\lambda^2}{Q^2} + \frac{\pi}{3}\frac{\left\langle \alpha_s\left(G^a_{\mu\nu}\right)^2\right\rangle}{Q^4} + \frac{256\pi^3}{81}\frac{\alpha_s\langle\bar{q}q\rangle^2}{Q^6}\right\}. \tag{39.5}$$

The corresponding relation for the $(V-A)$ case reads as:

$$\Pi^{V-A}(Q^2) = \frac{4m_q < \bar{q}q >}{Q^4} - \frac{64\pi}{9}\frac{\alpha_s\langle\bar{q}q\rangle^2}{Q^6} + 8\pi\frac{\alpha_s M_0^2\langle\bar{q}q\rangle^2}{Q^8}, \tag{39.6}$$

where $M_0^2 \approx 0.8\,\mathrm{GeV}^2$ parametrizes the mixed condensate as discussed in previous chapters. In the $x$-space the same correlators, upon dividing by $\Pi^{V+A}_{\mathrm{pert}}$ where $\Pi^{V+A}_{\mathrm{pert}}$ stands for the perturbative correlator, are obtained by applying the equations collected for convenience in the Table G.1 from [394] given in Appendix G. Therefore, one obtains [394]:

$$\frac{\Pi^{V+A}}{\Pi^{V+A}_{\mathrm{pert}}} \to 1 - \frac{\alpha_s}{4\pi}\lambda^2\cdot x^2 - \frac{\pi}{48}\left\langle\alpha_s\left(G^a_{\mu\nu}\right)^2\right\rangle x^4\ln x^2 + \frac{2\pi^3}{81}\alpha_s\langle\bar{q}q\rangle^2 x^6\ln x^2. \tag{39.7}$$

Note that $\ln x^2$ is negative since we start from small $x$. In the $(V-A)$ case:

$$\frac{\Pi^{V-A}}{\Pi^{V+A}_{\mathrm{pert}}} \to \frac{\pi^2}{2}m_q\langle\bar{q}q\rangle x^4\ln x^2 - \frac{\pi^3}{9}\alpha_s\langle\bar{q}q\rangle^2 x^6\ln x^2. \tag{39.8}$$

The $x$-transform of the $Q^2\cdot\Pi(Q^2)$ is given by:

$$\frac{Q^2\cdot\Pi^{V+A}}{Q^2\cdot\Pi^{V+A}_{\mathrm{pert}}} \to 1 - \frac{\pi}{96}\left\langle\alpha_s\left(G^a_{\mu\nu}\right)^2\right\rangle x^4 + \frac{2\pi^3}{81}\alpha_s\langle\bar{q}q\rangle^2 x^6\ln x^2. \tag{39.9}$$

Similarly:

$$\frac{Q^2\cdot\Pi^{V-A}}{Q^2\cdot\Pi^{V+A}_{\mathrm{pert}}} \to -\frac{\pi^3}{9}\alpha_s\langle\bar{q}q\rangle^2 x^6\ln x^2. \tag{39.10}$$

## 39.2 (Pseudo)scalar correlators

Next, we will concentrate on the currents having the quantum numbers of the pion and of $a_0(980)$-meson. The correlator of two pseudoscalar currents is defined as

$$\Pi^P(Q^2) \equiv i\int d^4x\, e^{iqx}\langle T\{J^\pi(x)J^\pi(0)\}\rangle, \tag{39.11}$$

where

$$J^P = i(m_u+m_d)\bar{u}\gamma_5 d, \tag{39.12}$$

https://doi.org/10.1017/9781009290296.051 Published online by Cambridge University Press

In the momentum space, it reads in terms of the renormalized coupling, masses and condensates:

$$\Pi^P(Q^2) \equiv i \int d^4x \, e^{iqx} \langle T\{J^\pi(x) J^\pi(0)\}\rangle$$
$$= \frac{3}{8\pi^2}(m_u+m_d)^2 \left\{ \left[1 + \left(\frac{17}{3} - \ln\frac{Q^2}{\nu^2}\right)\frac{\alpha_s}{\pi}\right] Q^2 \ln\frac{Q^2}{\nu^2} + \frac{4\alpha_s}{\pi}\lambda^2 \ln\frac{Q^2}{\nu^2} \right.$$
$$\left. + \frac{\pi}{3}\frac{\langle \alpha_s (G^a_{\mu\nu})^2\rangle}{Q^2} + \frac{896\pi^3}{81}\frac{\alpha_s\langle \bar{q}q\rangle^2}{Q^4} \right\} . \tag{39.13}$$

Here, the standard OPE terms can be found in [1,3,167] as compiled in previous chapters, while the gluon-mass correction was introduced first in [161]. It is more convenient to introduce the running QCD coupling $\bar{\alpha}_s(Q^2)$, the quark running mass $\bar{m}_i(Q^2)$ and condensate $\langle\overline{\bar{q}q}\rangle(Q^2)$,[1] into the second derivative in $Q^2$ of $\Pi^P(Q^2)$ defined in Eq. (39.13), which obeys an homogeneous RGE:

$$\frac{\partial^2 \Pi^P}{(\partial Q^2)^2} = \frac{3}{8\pi^2}\frac{(\bar{m}_u+\bar{m}_d)^2}{Q^2}\left\{1 + \frac{11}{3}\frac{\bar{\alpha}_s}{\pi} - \frac{4\alpha_s}{\pi}\frac{\lambda^2}{Q^2} \right.$$
$$\left. + 2\frac{\pi}{3}\frac{\langle \alpha_s (G^a_{\mu\nu})^2\rangle}{Q^4} + 2\cdot 3\frac{896\pi^3}{81}\frac{\bar{\alpha}_s\langle\overline{\bar{q}q}\rangle^2}{Q^6}\right\} . \tag{39.14}$$

In what follows, we shall work with the appropriate ratio where the pure perturbative corrections are absorbed into the overall normalization and concentrate on the power corrections assuming that these corrections are responsible for the observed rather sharp variations of the correlation functions. Thus, in the $x$-space we have for the pion channel [394]:

$$\frac{\Pi^P}{\Pi^P_{\rm pert}} \to 1 - \frac{\alpha_s}{2\pi}\lambda^2 x^2 + \frac{\pi}{96}\langle \alpha_s (G^a_{\mu\nu})^2\rangle x^4 - \frac{7\pi^3}{81}\alpha_s\langle\bar{q}q\rangle^2 x^6 \ln x^2 . \tag{39.15}$$

Note that the coefficient in front of the last term in Eq. (39.15) differs both in the absolute value and sign from the corresponding expression in [386].

Similarly, in the $S$-channel, the correlator associated with the scalar current having the quantum number of the $a_0$:

$$J^S = i(m_u - m_d)\bar{u}d \tag{39.16}$$

is obtained from Eq. (39.13) by changing $m_i$ into $-m_i$ and by taking the coefficient in front of the $1/Q^6$ correction to be $-1408\pi^3/81$ instead of $896\pi^3/81$ in Eq. (39.13). This term was found first in [666].

Therefore, we have in the $x-$space:

$$\frac{\Pi^S}{\Pi^S_{\rm pert}} \to 1 - \frac{\alpha_s}{2\pi}\lambda^2 x^2 + \frac{\pi}{96}\langle \alpha_s (G^a_{\mu\nu})^2\rangle x^4 + \frac{11\pi^3}{81}\alpha_s\langle\bar{q}q\rangle^2 x^6 \ln x^2 . \tag{39.17}$$

[1] We assume that $\alpha_s\lambda^2$ does not run like $\langle\alpha_s(G^a_{\mu\nu})^2\rangle$.

https://doi.org/10.1017/9781009290296.051 Published online by Cambridge University Press

The channel which is crucial for the analysis in [393,394] is the $(S + P)$, which is less affected by some eventual direct instanton contributions than the individual $S$ and $P$ correlators. In this channel:

$$R_{P+S} \equiv \frac{1}{2}\left(\frac{\Pi^P}{\Pi^P_{\text{pert}}} + \frac{\Pi^S}{\Pi^S_{\text{pert}}}\right) \to 1 - \frac{\alpha_s}{2\pi}\lambda^2 x^2 + \frac{\pi}{96}\left\langle \alpha_s \left(G^a_{\mu\nu}\right)^2\right\rangle x^4 + \frac{4\pi^3}{81}\alpha_s \langle \bar{q}q\rangle^2 x^6 \ln x^2 \,. \tag{39.18}$$

This expression concludes the summary of the power corrections to the current correlators.

We shall see later on that the QCD expressions of the two-point functions given in this part of the book are crucial inputs in the discussions of QCD spectral sum rules analysis and in various high-energy processes ($e^+e^- \to$ hadrons total cross-section, Higgs decays, ...).