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Tensor Products and Transferability of Semilattices

G. Grätzer and F. Wehrung

Abstract. In general, the tensor product, $A \otimes B$, of the lattices A and B with zero is not a lattice (it is only a join-semilattice with zero). If $A \otimes B$ is a *capped* tensor product, then $A \otimes B$ is a lattice (the converse is not known). In this paper, we investigate lattices A with zero enjoying the property that $A \otimes B$ is a capped tensor product, for *every* lattice B with zero; we shall call such lattices *amenable*.

The first author introduced in 1966 the concept of a *sharply transferable lattice*. In 1972, H. Gaskill defined, similarly, sharply transferable semilattices, and characterized them by a very effective condition (T).

We prove that *a finite lattice A is* amenable *iff it is* sharply transferable *as a join-semilattice*.

For a general lattice A with zero, we obtain the result: *A is amenable iff A is locally finite and every finite sublattice of A is transferable as a join-semilattice*.

This yields, for example, that a finite lattice A is amenable iff $A \otimes \mathrm{F}(3)$ is a lattice iff A satisfies (T), with respect to join. In particular, $M_3 \otimes \mathrm{F}(3)$ is not a lattice. This solves a problem raised by R. W. Quackenbush in 1985 whether the tensor product of lattices with zero is always a lattice.

1 Introduction

The tensor product, $A \otimes B$, of the $\{\vee, 0\}$-semilattices A and B is defined in a very classical fashion, as a free (universal) object with respect to a natural notion of *bimorphism*, see G. Fraser [3], G. Grätzer, H. Lakser, and R. W. Quackenbush [9], and G. Grätzer and F. Wehrung [10]. Unfortunately, the tensor product of two lattices with zero is, in general, not a lattice.

Let A and B be lattices with zero. If the tensor product, $A \otimes B$, satisfies the very natural condition of being *capped*, introduced in [10], then $A \otimes B$ is always a lattice. Capped tensor products have many interesting properties. The most important one is the main result of [10]: If $A \otimes B$ is a capped tensor product, then $\mathrm{Con_c}(A \otimes B)$ and $\mathrm{Con_c}\, A \otimes \mathrm{Con_c}\, B$ are isomorphic. For finite lattices this was proved in G. Grätzer, H. Lakser, and R. W. Quackenbush [9].

In this paper, we study the connections between $A \otimes B$ being a lattice and $A \otimes B$ being capped, for lattices A and B with zero. We do not have the answer to the most obvious question whether these two conditions are equivalent (see Problem 1 in Section 9). We prove, however, that these two conditions are equivalent provided that A is finite (or locally finite), see Theorem 3.

Since we cannot handle the general problem, when is $A \otimes B$ a lattice, we universally quantify one of the variables: Let us call the lattice A with zero *amenable*, if $A \otimes B$ is a capped tensor product, for any lattice B with zero. Then from Theorem 3 we obtain the easy

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corollary that for a (locally) finite lattice A with zero, A is amenable iff $A \otimes B$ is a lattice, for any lattice B with zero. This leads to the central problem of this paper: characterize amenable lattices.

The technique to handle this problem comes from an unexpected source.

The first author introduced in 1966 the concept of *sharp transferability* for lattices (see [7]). A finite lattice A is sharply transferable, if whenever A has an embedding φ into $\mathrm{Id}\, L$, the ideal lattice of a lattice L, then A has an embedding ψ into L satisfying $\psi(x) \in \varphi(y)$ iff $x \leq y$. In 1972, H. Gaskill [5] characterized sharply transferable semilattices by a very effective condition (T). For a lattice A, as usual, we shall denote by $(\mathrm{T}_\vee)$ the condition (T) for $\langle A; \vee\rangle$. Finite lattices satisfying $(\mathrm{T}_\vee)$ are well-understood. In particular, they are exactly the finite, *lower bounded* homomorphic images of finitely generated free lattices, see the survey by R. Freese, J. Ježek, and J. B. Nation [4], from which we shall borrow a number of results.

We characterize finite amenable lattices as follows:

For a finite lattice A, the following conditions are equivalent:

(i) A is amenable.
(ii) $A \otimes B$ is a lattice, for every lattice B with zero.
(iii) A as a join-semilattice is sharply transferable.
(iv) $A \otimes \mathrm{F}(3)$ is a lattice.
(v) A satisfies $(\mathrm{T}_\vee)$.

Our main result (see Corollary 5.4 and Theorem 5) characterizes amenability for general lattices:

For a lattice A with zero, the following conditions are equivalent:

(i) A is amenable.
(ii) A is locally finite and $A \otimes B$ is a lattice, for every lattice B with zero.
(iii) A is locally finite and $A \otimes \mathrm{F}(3)$ is a lattice.
(iv) A is locally finite and every finite sublattice of A satisfies $(\mathrm{T}_\vee)$.

The finite case trivially follows from the general result.

Since M_3 fails $(\mathrm{T}_\vee)$, it follows that $M_3 \otimes \mathrm{F}(3)$ is not a lattice. This answers a problem raised by R. W. Quackenbush in 1985 whether the tensor product of lattices with zero is always a lattice. By a different approach, this was also answered in our paper [11], where we produce two different examples of lattices A and B with zero such that $A \otimes B$ is not a lattice: in the first example, A and B are planar; in the second, A and B are modular.

Sections 2 and 3 are introductory. In Section 2, we review the basic facts about tensor products with special emphasis on the representation by some hereditary subsets of $A \times B$ and the representation by antitone maps, as in J. Anderson and N. Kimura [1]. Section 3 introduces transferability and the technical tools we inherit from transferability: minimal pairs, the condition (T), and the adjustment sequence of a map. Section 3 introduces lower bounded homomorphisms as well, together with some required technical tools, such as the lower limit table.

In Section 4, we recall the definition of a capped tensor product, and we establish a number of technical results; applying a theorem of [10], we prove, for instance, that if $A \otimes B$ is a capped tensor product, then so is $(A/\alpha) \otimes (B/\beta)$, where α is a lattice congruence of A and β is a lattice congruence of B.

Amenable lattices are introduced in Section 5, where we characterize amenable locally finite lattices. In Section 6, we prepare the ground for studying capped tensor products *via* adjustment sequences of maps. Not all maps can be considered, but only those that we call *step functions*, which are "measurable" finite joins of characteristic functions. In Section 7, we utilize the concepts introduced in Sections 3 and 6 to characterize capped tensor products $A \otimes B$, with A and B arbitrary lattices with zero.

In Section 8, we characterize amenable lattices, as locally finite lattices with zero in which every finite sublattice satisfies $(\mathrm{T}_\vee)$. Section 9 concludes the paper discussing some related results and stating some open problems.

2 Tensor Products

2.1 The Basic Concepts

We shall adopt the notation and terminology of our paper [10]. In particular, for a $\{\vee, 0\}$-semilattice A, we use the notation $A^- = A - \{0\}$. Note that A^- is a subsemilattice of A.

Let A and B be $\{\vee, 0\}$-semilattices. We denote by $A \otimes B$ the *tensor product* of A and B, defined as the free $\{\vee, 0\}$-semilattice generated by the set $A^- \times B^-$ subject to the relations $\langle a, b_0\rangle \vee \langle a, b_1\rangle = \langle a, b_0 \vee b_1\rangle$, for $a \in A^-$, $b_0, b_1 \in B^-$; and symmetrically, $\langle a_0, b\rangle \vee \langle a_1, b\rangle = \langle a_0 \vee a_1, b\rangle$, for $a_0, a_1 \in A^-$, $b \in B^-$.

It follows directly from the definition that the sum distributes over tensor product; since the sum for $\{\vee, 0\}$-semilattices is the direct product, we get the formula:

$$(A \times B) \otimes C \cong (A \otimes C) \times (B \otimes C). \tag{2.1}$$

The following two statements on tensor products are taken from Corollary 3.7(iv) and Corollary 3.9 of [10]. For a lattice L, we denote by $\operatorname{Con} L$ the congruence lattice of L.

Lemma 2.1

(i) Let A and B be lattices with zero, let $\alpha \in \operatorname{Con} A$ and $\beta \in \operatorname{Con} B$. If $A \otimes B$ is a lattice, then $(A/\alpha) \otimes (B/\beta)$ is also a lattice.

(ii) Let A, A', B, B' be lattices with zero such that A is a sublattice of A' and B is a sublattice of B'. If $A' \otimes B'$ is a lattice, then $A \otimes B$ is a lattice.

2.2 The Set Representation

In [10], we used the following representation of the tensor product.

First, we introduce the notation:

$$\perp_{A,B} = (A \times \{0\}) \cup (\{0\} \times B).$$

Second, we introduce a partial binary operation on $A \times B$: let $\langle a_0, b_0\rangle, \langle a_1, b_1\rangle \in A \times B$; the *lateral join* of $\langle a_0, b_0\rangle$ and $\langle a_1, b_1\rangle$ is defined if $a_0 = a_1$ or $b_0 = b_1$, in which case, it is the join, $\langle a_0 \vee a_1, b_0 \vee b_1\rangle$.

Third, we define bi-ideals: a nonempty subset I of $A \times B$ is a *bi-ideal* of $A \times B$, if it satisfies the following conditions:

(i) I is hereditary;
(ii) I contains $\perp_{A,B}$;
(iii) I is closed under lateral joins.

The *extended tensor product* of A and B, denoted by $A \mathbin{\bar{\otimes}} B$, is the lattice of all bi-ideals of $A \times B$.

It is easy to see that $A \mathbin{\bar{\otimes}} B$ is an algebraic lattice. For $a \in A$ and $b \in B$, we define $a \otimes b \in A \mathbin{\bar{\otimes}} B$ by

$$a \otimes b = \perp_{A,B} \cup \{\langle x, y\rangle \in A \times B \mid \langle x, y\rangle \leq \langle a, b\rangle\}$$

and call $a \otimes b$ a *pure tensor*. A pure tensor is a principal (that is, one-generated) bi-ideal. Using this notation, the isomorphism (2.1) can be established by the map

$$\langle a, b\rangle \otimes c \mapsto \langle a \otimes c, b \otimes c\rangle. \tag{2.2}$$

Now we can state the representation:

The tensor product $A \otimes B$ can be represented as the $\{\vee, 0\}$-subsemilattice of compact elements of $A \mathbin{\bar{\otimes}} B$.

For every positive integer m, denote by $\mathrm{F}(m)$ the free lattice on m generators $x_0, \dots, x_{m-1}$. One can evaluate any element p of $\mathrm{F}(m)$ at any m-tuple of elements of any lattice, thus giving the notation $p(a_0, \dots, a_{m-1})$. Let p^{d} be the dual of p.

The following purely arithmetic formulas are due to G. A. Fraser [3]. They are easiest to prove using the above representation of tensor products.

Lemma 2.2 *Let A and B be $\{\vee, 0\}$-semilattices. Let $a_0, a_1 \in A$, and $b_0, b_1 \in B$ be such that $a_0 \wedge a_1$ and $b_0 \wedge b_1$ both exist.*

(i) The intersection (meet) of two pure tensors is a pure tensor, in fact,

$$(a_0 \otimes b_0) \cap (a_1 \otimes b_1) = (a_0 \wedge a_1) \otimes (b_0 \wedge b_1).$$

(ii) The join of two pure tensors is the union of four pure tensors, in fact,

$$\begin{aligned}&(a_0 \otimes b_0) \vee (a_1 \otimes b_1)\\ &\quad = (a_0 \otimes b_0) \cup (a_1 \otimes b_1) \cup \big((a_0 \vee a_1) \otimes (b_0 \wedge b_1)\big) \cup \big((a_0 \wedge a_1) \otimes (b_0 \vee b_1)\big).\end{aligned}$$

(iii) Let A and B be lattices with zero, let n be a positive integer, let $a_0, \dots, a_{n-1} \in A$, and let $b_0, \dots, b_{n-1} \in B$. Then

$$\bigvee (a_i \otimes b_i \mid i < n) = \bigcup \big(p(a_0, \dots, a_{n-1}) \otimes p^{\mathrm{d}}(b_0, \dots, b_{n-1}) \mid p \in \mathrm{F}(n)\big).$$

(iv) Let A and B be lattices with zero. Then

$$\begin{aligned}&\bigvee (a_i \otimes b_i \mid i < n) \wedge \bigvee (c_j \otimes d_j \mid j < m)\\ &= \bigcup \big(p(a_0, \dots, a_{n-1}) \wedge q(c_0, \dots, c_{m-1})\big) \otimes \big(p^{\mathrm{d}}(b_0, \dots, b_{n-1}) \wedge q^{\mathrm{d}}(d_0, \dots, d_{m-1})\big),\end{aligned}$$

where p^{d} and q^{d} are the duals of p and q, respectively, and where the union is for all $p \in \mathrm{F}(n)$ and $q \in \mathrm{F}(m)$.

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Note that in (iii) (and similarly, in (iv)) the right side is an infinite union, for $n > 2$.

For any subset X of a lattice L, let $X^\wedge$ (resp., $X^\vee$) denote the meet-subsemilattice (resp., join-subsemilattice) of L generated by X. We also write $X^{\wedge\vee} = (X^\wedge)^\vee$, $X^{\vee\wedge} = (X^\vee)^\wedge$, and so on. We define inductively an increasing sequence of finite *join-subsemilattices* $S_n(m)$ of $\mathrm{F}(m)$ as follows:

(i) $S_0(m) = \{x_0, \dots, x_{m-1}\}^{\wedge\vee}$.
(ii) $S_{n+1}(m) = S_n(m)^{\wedge\vee}$.

In particular, all the $S_n(m)$ are finite and their union equals $\mathrm{F}(m)$.

The following lemma readily follows from Lemma 2.2(iii).

Lemma 2.3 *Let A and B be lattices with zero, let m be a positive integer, let $a_0, \dots, a_{m-1} \in A$, and let $b_0, \dots, b_{m-1} \in B$. Then the following conditions are equivalent:*

(i) The finite join $\bigvee(a_i \otimes b_i \mid i < m)$ is a finite union of pure tensors.
(ii) There exists a positive integer n such that the following finite union:

$$\bigcup\bigl(p(a_0, \dots, a_{m-1}) \otimes p^{\mathrm{d}}(b_0, \dots, b_{m-1}) \mid p \in S_n(m)\bigr)$$

belongs to $A \otimes B$.

2.3 Representation by Homomorphisms

Let A and B be $\{\vee, 0\}$-semilattices. Note that $\mathrm{Id}\, B$, the set of all ideals of $\langle B; \vee\rangle$, is a semilattice under intersection. So we can consider the set of all semilattice homomorphisms from the semilattice $\langle A^-; \vee\rangle$ into the semilattice $\langle \mathrm{Id}\, B; \cap\rangle$,

$$A \mathbin{\vec{\otimes}} B = \mathrm{Hom}(\langle A^-; \vee\rangle, \langle \mathrm{Id}\, B; \cap\rangle),$$

ordered componentwise, that is, $f \leq g$ iff $f(a) \leq g(a)$ (that is, $f(a) \subseteq g(a)$), for all $a \in A^-$. The arrow indicates which way the homomorphisms go. Note that the elements of $A \mathbin{\vec{\otimes}} B$ are *antitone* functions from A^- to $\mathrm{Id}\, B$; indeed, if $a \leq b$, then $a \vee b = b$ and so $f(a) \cap f(b) = f(b)$, that is, $f(a) \supseteq f(b)$.

With any element φ of $A \mathbin{\vec{\otimes}} B$, we associate the subset $\varepsilon(\varphi)$ of $A \times B$:

$$\varepsilon(\varphi) = \bigl\{\langle x, y\rangle \in A \times B \mid y \in \varphi(x)\bigr\} \cup \perp_{A,B}.$$

Proposition 2.4 *The map ε is an isomorphism between $A \mathbin{\vec{\otimes}} B$ and $A \mathbin{\bar{\otimes}} B$. The inverse map, ε^{-1}, sends $H \in A \mathbin{\bar{\otimes}} B$ to $\varepsilon^{-1}(H)\colon A^- \to \mathrm{Id}\, B$, defined by*

$$\varepsilon^{-1}(H)(a) = \{x \in B \mid \langle a, x\rangle \in H\}.$$

Proof We leave the easy computation to the reader. ■

For $a \in A$ and $b \in B$, we can describe $\xi = \varepsilon^{-1}(a \otimes b)\colon A^- \to \mathrm{Id}\, B$ as follows

$$\xi(x) = \begin{cases} (b], & \text{if } x \leq a; \\ \{0\}, & \text{otherwise.} \end{cases}$$

If A is *finite*, then a homomorphism from $\langle A^-; \vee\rangle$ to $\langle \mathrm{Id}\, B; \cap\rangle$ is determined by its restriction to $\mathrm{J}(A)$, the set of all join-irreducible elements of A.

This representation of the tensor product is also utilized in [11] and [12].

3 Sharply Transferable Lattices

In this section, we introduce sharply transferable lattices and semilattices, minimal pairs, the condition (T), the D-relation, and the adjustment sequence of a map. For more information on these topics, the reader is referred to [4] and [6].

3.1 Sharp Transferability

We start with the definition of sharp transferability (see [7]). We get a definition that is easier to utilize by using choice functions ($\mathrm{P}(Y)$ is the power set of Y):

Definition 3.1 Let X and Y be sets and let $\varphi\colon X \to \mathrm{P}(Y)$ be a map. Then a *choice function* for φ is a map $\xi\colon X \to Y$ such that $\xi(x) \in \varphi(x)$, for all $x \in X$.

Definition 3.2 A lattice S is *sharply transferable*, if for every embedding φ of S into $\operatorname{Id} T$, the ideal lattice of a lattice T, there exists an embedding ξ of S into T such that ξ is a choice function for φ satisfying $\xi(x) \in \varphi(y)$ iff $x \leq y$.

Equivalently, $\xi(x) \in \varphi(x)$ and $\xi(x) \notin \varphi(y)$, for any $y < x$.

The motivation for these definitions comes from the fact that the well-known result: *a lattice L is modular iff* $\operatorname{Id} L$ *is modular*, can be recast: *N_5 is a (sharply) transferable lattice.*

The reader should find it easy to verify that N_5 is a sharply transferable lattice. It is somewhat more difficult to see the negative result: M_3 is not a sharply transferable lattice.

3.2 Minimal Pairs

Let P be a poset and let X and Y be subsets of P. Then the relation X *is dominated by* Y, in notation, $X \ll Y$, is defined as follows:

$$X \ll Y \text{ iff for all } x \in X \text{ there exists } y \in Y \text{ such that } x \leq y.$$

The relation $\ll$ defines a *quasi-ordering* on the power set of P, and it is a *partial ordering* on the set of antichains of P.

The following definition is from H. Gaskill [5]:

Definition 3.3 Let A be a finite join-semilattice. A *minimal pair* of A is a pair $\langle p, I\rangle$ such that $p \in \mathrm{J}(A)$, $I \subseteq \mathrm{J}(A)$, and the following three conditions hold:

(i) $p \notin I$.
(ii) $p \leq \bigvee I$.
(iii) For all $J \subseteq \mathrm{J}(A)$, if $J \ll I$ and $p \leq \bigvee J$, then $I \subseteq J$.

If $\langle p, I\rangle$ is a minimal pair, then I is an antichain of $\mathrm{J}(A)$. If I is an antichain of $\mathrm{J}(A)$ and $\langle p, I\rangle$ satisfies conditions 3.3(i), 3.3(ii), then condition 3.3(iii) is equivalent to the following condition:

(iii′) For all $J \subseteq \mathrm{J}(A)$, if $J \ll I$ and $p \leq \bigvee J$, then $I \ll J$.

"Minimal pairs" are minimal in two ways. Firstly, if $\langle p, I\rangle$ is a minimal pair, then $p \leq \bigvee I$ and an element of I cannot be replaced by a set of smaller join-irreducible elements while retaining that their join is over p. Indeed, if $i \in I$ can be replaced by $X \subseteq \mathrm{J}(A)$ with $X \cap I = \varnothing$, that is, $p \leq \bigvee J$, where $J = (I - \{i\}) \cup X$, then $J \ll I$, hence by (iii), $I \subseteq J$, a contradiction. Secondly, the collection of all minimal pairs is the minimal information necessary to describe the join structure of the finite lattice.

If A is a finite join-semilattice and $p \in \mathrm{J}(A)$, then we shall define

$$\mathcal{M}(p) = \{I \subseteq \mathrm{J}(A) \mid \langle p, I\rangle \text{ is a minimal pair}\}.$$

3.3 The Condition (T)

For the following definition, see H. Gaskill [5].

Definition 3.4 A finite join-semilattice A satisfies (T), if $\mathrm{J}(A)$ has a linear order $\trianglelefteq$ such that for every minimal pair $\langle p, J\rangle$ of A, the relation $p \trianglelefteq j$ holds, for any $j \in J$.

A finite lattice A satisfies the condition $(\mathrm{T}_\vee)$, if the semilattice $\langle A; \vee\rangle$ satisfies (T).

One can define sharply transferable semilattices by changing "lattice" to "semilattice" in Definition 3.2. H. Gaskill [5] proved that (T) characterizes finite sharply transferable semilattices.

An interpretation of $(\mathrm{T}_\vee)$ for finite lattices can be found in [4]. Let A be a finite semilattice. For $p \in \mathrm{J}(A)$, let p_* denote the unique element covered by p. The *dependency relation* is the binary relation D defined on $\mathrm{J}(A)$ as follows

$$p \mathrel{D} q \quad \text{iff} \quad p \neq q \text{ and there exists } x \in A \text{ such that } p \leq q \vee x \text{ and } p \nleq q_* \vee x.$$

Lemma 3.5 *Let A be a finite lattice and $p, q \in \mathrm{J}(A)$. Then $p \mathrel{D} q$ iff there exists $I \in \mathcal{M}(p)$ such that $q \in I$.*

Corollary 3.6 *A finite lattice L satisfies $(\mathrm{T}_\vee)$ iff $\mathrm{J}(L)$ has no D-cycle.*

3.4 Lower Bounded Homomorphisms; the Lower Limit Table

We recall in this section some classical concepts about lower bounded homomorphisms, due to A. Day, H. S. Gaskill, B. Jónsson, A. Kostinsky, and R. N. McKenzie, which are presented in Chapter 2 of R. Freese, J. Ježek, and J. B. Nation [4].

Let K and L be lattices. If $h\colon K \to L$ is a lattice homomorphism, we say that h is *lower bounded*, if the set $h^{-1}[a)$ has a least element, for all $a \in L$.

Let us assume that K is finitely generated and let X be a finite generating set of K. We put $H_0 = X^\wedge \cup \{1_K\}$, and $H_n = X^{\wedge(\vee\wedge)^n}$, for all $n > 0$. We note that $\langle H_n \mid n \in \omega\rangle$ is an increasing sequence of *finite* meet-subsemilattices of K, and that their union is K. For $n \in \omega$ and $a \in L$, we denote by $\beta_n(a)$ the least element x of H_n such that $a \leq x$, if it exists. Note that, for $n \in \omega$, $\beta_n(a)$ is defined iff $a \leq h(1_K)$. If this holds, then $\beta_n(a)$ can be computed by the following formulas, see Theorem 2.3 in [4] ($h[X]$ denotes the image of X under h):

$$\beta_0(a) = \bigwedge\bigl(x \in X \mid a \leq h(x)\bigr), \tag{3.1}$$

$$\beta_{n+1}(a) = \beta_n(a) \wedge \bigwedge\Bigl(\bigvee \beta_n[S] \mid S \in \mathcal{C}^*(a), \bigvee S \leq h(1_K)\Bigr), \tag{3.2}$$

where an empty meet in K equals 1_K and

$$\mathcal{C}^*(a) = \Bigl\{S \subseteq K \mid S \text{ is finite nonempty, } x \leq \bigvee S, \text{ and } x \notin (S]\Bigr\},$$

for all $a \in K$.

The sequence $\langle \beta_n \mid n \in \omega\rangle$ is called the *lower limit table* of h. It is uniquely determined by h and X. The homomorphism h is *lower bounded* iff, for all $a \leq h(1_K)$, there exists $n \in \omega$ such that $\beta_n(a) = \beta_{n+1}(a)$. In that case, $\beta_n(a)$ is the least element of $h^{-1}[a)$.

A finitely generated lattice L is *lower bounded*, if there exists a *surjective*, lower bounded lattice homomorphism from a finitely generated free lattice onto L. We recall the following characterization of lower bounded lattices, see Theorem 2.13 in [4]:

Proposition 3.7 *For a finitely generated lattice L, the following are equivalent:*

(i) L is lower bounded.
(ii) For every finitely generated lattice K, every lattice homomorphism from K to L is lower bounded.

Finite lower bounded lattices have a number of characterizations. The following result states one of them. For a proof, we refer to Corollary 2.39 in [4].

Proposition 3.8 *A finite lattice is lower bounded iff it satisfies ($T_\vee$).*

We recall that another characterization of ($\mathrm{T}_\vee$) is given by Corollary 3.6.

We shall also need the following result, see the proof of Theorem 2.6 in [4]:

Proposition 3.9 *Let K be a finitely generated lattice, let L be a lattice, and let $h\colon K \twoheadrightarrow L$ be a surjective lattice homomorphism. Let X be a finite generating subset of K, and let $\langle \beta_n \mid n \in \omega\rangle$ be the associated lower limit table of h. Let $n \in \omega$, and let us assume that $\beta_n = \beta_{n+1}$. Then $L = h[X]^{\wedge(\vee\wedge)^n}$, so that L is finite. Furthermore, h is lower bounded.*

The proof of of Proposition 3.9 is easy to outline. We note first that $\beta_n(a) \in X^{\wedge(\vee\wedge)^n}$, for all $a \in L$. Since $\beta_n = \beta_{n+1}$, it follows that $\beta_n(a)$ is the least element of $h^{-1}[a)$, for all $a \in A^-$. In particular, h is lower bounded. Since h is surjective, the equality $a = h\bigl(\beta_n(a)\bigr)$ holds. It follows that $a \in h[X]^{\wedge(\vee\wedge)^n}$.

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4 Capped Tensor Products of Lattices

In [10] (see Definition 7.1), we introduced *capped tensor products*. We recall the definition here:

Definition 4.1 Let A and B be lattices with zero. We say that $A \otimes B$ is a *capped tensor product*, if every element of $A \otimes B$ is a *finite union* of pure tensors.

It is easy to see by Lemma 2.2 that if $A \otimes B$ is a capped tensor product, then $A \otimes B$ is a lattice (see Lemma 7.2 of [10]).

In this section, we shall establish some basic properties of capped tensor products. Most of the results in this section are technical lemmas with the exception of Theorem 1, stating that $(A \times B) \otimes C$ is capped iff both $A \otimes C$ and $B \otimes C$ are capped.

Lemma 4.2 *Let A and B be lattices with zero. Let n be a positive integer, let $a_0, \dots, a_{n-1} \in A$, and let $b_0, \dots, b_{n-1} \in B$. Then*

$$\bigcup(a_i \otimes b_i \mid i < n) \in A \otimes B$$

iff, for all i, $j < n$, $i \neq j$, the following two conditions hold:

(a) $a_i \wedge a_j = 0$ or there exists $k < n$ such that $a_i \wedge a_j \leq a_k$ and $b_i \vee b_j \leq b_k$;
(b) $b_i \wedge b_j = 0$ or there exists $k < n$ such that $b_i \wedge b_j \leq b_k$ and $a_i \vee a_j \leq a_k$.

Proof It is evident that the hereditary subset $H = \bigcup(a_i \otimes b_i \mid i < n) \subseteq A \times B$ belongs to $A \otimes B$ iff it is closed under lateral joins. The two conditions guarantee this. Now the conclusion easily follows. ■

Corollary 4.3 *Let A, A', B, and B' be lattices with zero, let $f\colon A \to A'$ and $g\colon B \to B'$ be lattice homomorphisms. Let n be a positive integer, let $a_0, \dots, a_{n-1} \in A$, and let $b_0, \dots, b_{n-1} \in B$.*

(a) If f and g are 0-preserving, then $\bigcup(a_i \otimes b_i \mid i < n) \in A \otimes B$ implies that

$$\bigcup\bigl(f(a_i) \otimes g(b_i) \mid i < n\bigr) \in A' \otimes B'.$$

(b) If f and g are lattice embeddings, then $\bigcup\bigl(f(a_i) \otimes g(b_i) \mid i < n\bigr) \in A' \otimes B'$ implies that

$$\bigcup(a_i \otimes b_i \mid i < n) \in A \otimes B.$$

Corollary 4.4 *Let A, A', B, and B' be lattices with zero, let $f\colon A \to A'$ and $g\colon B \to B'$ be lattice homomorphisms. Let n be a positive integer, let $a_0, \dots, a_{n-1} \in A$ and let $b_0, \dots, b_{n-1} \in B$.*

(i) Let f and g be 0-preserving. If $\bigvee(a_i \otimes b_i \mid i < n)$ is a finite union of pure tensors in $A \otimes B$, then $\bigvee\bigl(f(a_i) \otimes g(b_i) \mid i < n\bigr)$ is a finite union of pure tensors in $A' \otimes B'$.

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(ii) Let f and g be lattice embeddings. If $\big(f(a_i) \otimes g(b_i) \mid i < n\big)$ is a finite union of pure tensors in $A' \otimes B'$, then $\bigvee(a_i \otimes b_i \mid i < n)$ is a finite union of pure tensors in $A \otimes B$.

Now we are ready to state our first preservation results about capped tensor products:

Proposition 4.5 *Let $A \otimes B$ be a capped tensor product of the lattices A and B with zero. Then the following holds:*

(i) Let $\alpha \in \operatorname{Con} A$ and $\beta \in \operatorname{Con} B$. Then $(A/\alpha) \otimes (B/\beta)$ is a capped tensor product of A/α and B/β.
(ii) Let $A' \subseteq A$ and $B' \subseteq B$ be lattices with zero. Then $A' \otimes B'$ is a capped tensor product of A' and B'.

Note that we do *not* assume in (ii) that $0_A = 0_{A'}$ or that $0_B = 0_{B'}$.

Proof (i) follows immediately from Corollary 4.4(i), while (ii) follows from Corollary 4.4(ii). ■

Proposition 4.6 *Let A and B be lattices with zero, and let $(A_i \mid i \in I)$ be a directed family of lattices with zero. Suppose that, with appropriate transition maps that are 0-lattice homomorphisms, $A = \varinjlim_i A_i$, and that all $A_i \otimes B$ are capped tensor products. Then $A \otimes B$ is a capped tensor product.*

Now we present the result on direct products:

Theorem 1 *Let A, B, and C be lattices with zero. Then $(A \times B) \otimes C$ is a capped tensor product iff both $A \otimes C$ and $B \otimes C$ are capped tensor products.*

Proof If $(A \times B) \otimes C$ is a capped tensor product, then both $A \otimes C$ and $B \otimes C$ are capped tensor products by Proposition 4.5(i).

Conversely, let us assume that both $A \otimes C$ and $B \otimes C$ are capped tensor products. Let $H \in (A \times B) \otimes C$. We have to prove that H is a finite union of pure tensors. By (2.1), we take instead $H \in (A \otimes C) \times (B \otimes C)$.

Let $\langle u, v\rangle \in (A \otimes C) \times (B \otimes C)$; then u is a finite union of pure tensors $a \otimes c \in A \otimes C$ and v is a finite union of pure tensors $b \otimes c \in B \otimes C$. So $\langle u, v\rangle$ is a finite union of pure tensors of the form $\langle a \otimes c_1, b \otimes c_2\rangle \in (A \otimes C) \times (B \otimes C)$. Since

$$\langle a \otimes c_1, b \otimes c_2\rangle = \langle a \otimes c_1, 0\rangle \vee \langle 0, b \otimes c_2\rangle,$$

by formula (ii) of Lemma 2.2, $\langle a \otimes c_1, b \otimes c_2\rangle$ is a union of (at most) four pure tensors; therefore, $(A \times B) \otimes C$ is a capped tensor product. ■

By using Proposition 4.6, we deduce immediately the following:

Corollary 4.7 *Let $(A_i \mid i \in I)$ be a family of lattices with zero and let A be the discrete direct product of this family. Then, for any lattice B with zero, if $A_i \otimes B$ is a capped tensor product, for all $i \in I$, then $A \otimes B$ is a capped tensor product.*

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5 Amenable Lattices

Now we come to the central concept of this paper:

Definition 5.1 A lattice A with zero is *amenable*, if $A \otimes L$ is a capped tensor product, for every lattice L with zero.

In other words, A is amenable iff $A \otimes L$ is always capped; it follows that then $A \otimes L$ is always a lattice.

Theorem 2 *The class of amenable lattices with zero is preserved under the following operations:*

(i) the formation of sublattices,
(ii) the formation of quotient lattices,
(iii) finite direct products,
(iv) direct limits.

It follows from known results on lower bounded lattices that Theorem 5 is stronger than Theorem 2. However, Theorem 2 is more elementary, and it can easily be generalized to $\mathbf{V}$-amenable lattices, for any variety $\mathbf{V}$ of lattices, see Definition 9.3.

We shall see, in Corollary 8.3, that the class of amenable lattices is *not* closed under arbitrary direct products; in particular, it is not a variety. However, the following result holds:

Corollary 5.2 *Let* $\mathbf{V}$ *be a variety generated by an amenable finite lattice. Then every lattice with zero in* $\mathbf{V}$ *is amenable.*

Proof Let A be an amenable finite lattice generating $\mathbf{V}$. Let $B \in \mathbf{V}$ and let U be a finitely generated, say, n-generated, $\{0\}$-sublattice of B. Let $\mathrm{F}_{\mathbf{V}}(n)$ be the free lattice on n generators in $\mathbf{V}$. Since U belongs to $\mathbf{V}$ and U is n-generated, there exists a surjective lattice homomorphism $\pi\colon \mathrm{F}_{\mathbf{V}}(n) \twoheadrightarrow U$. But $\mathrm{F}_{\mathbf{V}}(n)$ embeds into A^{A^n} (this is a classical result of universal algebra). Since A is amenable and A^n is finite, it follows from Theorem 2 that $\mathrm{F}_{\mathbf{V}}(n)$ is amenable. Since U is a quotient of $\mathrm{F}_{\mathbf{V}}(n)$, U is amenable, again by Theorem 2.

Finally, B is the direct union of all its finitely generated $\{0\}$-sublattices, so the conclusion follows by Proposition 4.6. ■

Again, Corollary 5.2 can easily be generalized to $\mathbf{W}$-amenable lattices, for any variety $\mathbf{W}$ of lattices.

In Example 8.4, we show that "amenable finite lattice" cannot be replaced by "amenable locally finite lattice" in Corollary 5.2.

Corollary 5.3 *Every distributive lattice with zero is amenable.*

If A is an amenable lattice with zero, then $A \otimes L$ is a lattice, for every lattice L with zero. We do not know whether the converse is true in general (see Problem 1 in Section 9), however, we can settle this problem for locally finite lattices:

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Theorem 3 *Let A and B be lattices with zero. Let A be locally finite. Then $A \otimes B$ is a lattice iff $A \otimes B$ is a capped tensor product.*

Proof We prove the nontrivial direction; so we assume that $A \otimes B$ is a lattice.

First Case: A is finite. Let $H \in A \otimes B$; we have to prove that H is a finite union of pure tensors. Let $\xi \colon \langle A^-; \vee\rangle \to \langle \mathrm{Id}\, B; \cap\rangle$ be the antitone map associated with H, as defined in Section 2.3. Then $H = \bigcup\bigl(a \otimes \xi(a) \mid a \in A^-\bigr)$ (where $a \otimes \xi(a)$ is the set of all $a \otimes x$, $x \in \xi(a)$), so it suffices to prove that all $\xi(a)$, $a \in A^-$, are principal ideals of B.

For all $a \in A^-$,

$$\xi(a) = \bigcap\bigl(\xi(p) \mid p \in \mathrm{J}(A) \text{ and } p \leq a\bigr);$$

thus it suffices to prove that $\xi(p)$ is a principal ideal of B, for every $p \in \mathrm{J}(A)$.

Let $u_B \in B$ such that $H \subseteq 1_A \otimes u_B$; this element u_B exists because H is compact.

In Proposition 4.3 of [10], we noted that if a tensor product is a lattice, then the set representation is closed under intersection. Hence

$$U = H \cap (p_* \otimes u_B),$$
$$V = H \cap (p \otimes u_B)$$

belong to $A \otimes B$. Since $U \subseteq V$, there exists a decomposition of V of the form

$$V = U \vee \bigvee(a_i \otimes b_i \mid i < n), \tag{5.1}$$

where n is a nonnegative integer, and $0_A < a_i$ and $0_B < b_i \leq u_B$, for all $i < n$. The inequality $a_i \otimes b_i \leq V$ holds in $A \otimes B$, for all $i < n$. Since a_i and b_i are nonzero, $a_i \leq p$, therefore, $a_i \leq p_*$ or $a_i = p$. But if $a_i < p$, then $a_i \otimes b_i \leq U$, by the definition of U, so that $a_i \otimes b_i$ is absorbed by U in the decomposition (5.1). In other words, in (5.1) we may assume that $a_i = p$, for all $i < n$.

Set $b = \bigvee(b_i \mid i < n)$; then

$$V = U \vee (p \otimes b),$$

and we define $V' = U \cup (p \otimes b)$. We claim that V' is a bi-ideal. It is obvious that V' is a hereditary subset of $A \otimes B$ containing $\perp_{A,B}$. We show that V' is closed under lateral joins.

Let $\langle x_0, y\rangle$ and $\langle x_1, y\rangle$ in V'. Without loss of generality, we can assume that x_0, x_1, and y are nonzero, and $\langle x_0, y\rangle \in U$, $\langle x_1, y\rangle \in p \otimes b$. Then $x_0 \vee x_1 \leq p$ and $y \leq b$, so that $\langle x_0 \vee x_1, y\rangle \in V'$.

Let $\langle x, y_0\rangle$ and $\langle x, y_1\rangle$ in V'. Without loss of generality, x, y_0, and y_1 are nonzero, and $\langle x, y_0\rangle \in U$, $\langle x, y_1\rangle \in p \otimes b$. Since $U \leq p_* \otimes u_B$, it follows that $x \leq p_*$, so

$$\langle x, y_0 \vee y_1\rangle \in H \cap (p_* \otimes u_B) = U \leq V'.$$

This proves that V' belongs to $A \otimes B$, hence $V = V'$. Therefore,

$$V = U \cup (p \otimes b).$$

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It follows that, for all $y \in B$,

$$\begin{aligned} y \in \xi(p) &\text{ iff } p \otimes y \subseteq H, \\ &\text{ iff } p \otimes y \subseteq V, \\ &\text{ iff } p \otimes y \in U \cup (p \otimes b), \\ &\text{ iff } y \leq b. \end{aligned}$$

This proves that $\xi(p) = (b]$, a principal ideal, thus completing the proof of the finite case.

Second Case: A ***is locally finite.*** Then A can be written as the direct union of all of its finite $\{0\}$-sublattices. By Lemma 2.1, the tensor product $U \otimes B$ is a lattice, for each of those lattices U. Thus by the first case, $U \otimes B$ is a capped tensor product. Therefore, by Proposition 4.6, $A \otimes B$ is a capped tensor product. ■

Corollary 5.4 *Let A be a locally finite lattice with zero. Then the following conditions are equivalent:*

(i) *A is amenable.*
(ii) $A \otimes L$ *is a lattice, for every lattice L with zero.*
(iii) $A \otimes \mathrm{F}(3)$ *is a lattice.*

Proof The equivalence of (i) and (ii) follows immediately from Theorem 3. (ii) implies (iii) is trivial.

Finally, assume (iii). For every set X, denote by $\mathrm{F}^{(0)}(X)$ the free $\{0\}$-lattice on the generating set X. By a result of P. M. Whitman [16] (see Theorem VI.2.8 in [8]), $\mathrm{F}^{(0)}(\omega)$ embeds into $\mathrm{F}(3)$ as a $\{0\}$-sublattice (ω denotes the set of all natural numbers). Thus, by Lemma 2.1(ii), $A \otimes \mathrm{F}^{(0)}(\omega)$ is a lattice. For every infinite set X, $\mathrm{F}^{(0)}(X)$ is a direct limit of 0-lattices each one isomorphic to $\mathrm{F}^{(0)}(\omega)$; thus $A \otimes \mathrm{F}^{(0)}(X)$ is a lattice. But every lattice with zero is a quotient of some $\mathrm{F}^{(0)}(X)$, so we conclude the argument by Lemma 2.1(i). ■

We have not yet seen a finite lattice that is not amenable. The smallest example is M_3, as we shall see it in the next few sections.

6 Step Functions and Adjustment Sequences

The referee informed us that some of the arguments in this section date back to A. Kostinsky's unpublished 1971 thesis, which is unavailable to us.

Let A and B be lattices with zero. We shall put $(a]_{\bullet} = (a] \cap A^{-}$, for all $a \in A$. We shall denote by $\operatorname{Int} A$ the Boolean lattice of $\mathrm{P}(A^{-})$ generated by all sets of the form $(a]_{\bullet}$, or $a \in A^{-}$. Furthermore, we shall denote by $\operatorname{Int}_{*} A$ the ideal of $\operatorname{Int} A$ consisting of all $X \in \operatorname{Int} A$ with $X \subseteq (a]_{\bullet}$, for some $a \in A$. For every subset U of A^{-}, we denote by $\operatorname{Max} U$ the set of all *maximal* elements of U.

6.1 Step Functions

We start with the following very simple lemma:

Lemma 6.1 *Let $U \in \operatorname{Int}_* A$. Then the following two properties hold:*

(i) *Every element of U is contained in an element of* $\operatorname{Max} U$.
(ii) $\operatorname{Max} U$ *is finite.*

Proof The elements of $\operatorname{Int}_* A$ are the finite unions of subsets of the form $U = (b]_\bullet - X$, where $b \in A^-$ and X is a finite union of principal ideals of A^-. For such a subset U, the properties (i) and (ii) above are obvious, with $\operatorname{Max} U = \varnothing$, if $X = (b]_\bullet$, $\operatorname{Max} U = \{b\}$, otherwise. Furthermore, the set of all subsets U of A^- satisfying (i) and (ii) above is obviously closed under finite union. Therefore, it contains $\operatorname{Int}_* A$. ∎

Definition 6.2 A map $\xi\colon A^- \to B$ is a *step function*, if the range of ξ is finite, and the inverse image $\xi^{-1}\{b\}$ belongs to $\operatorname{Int}_* A$, for all $b \in B^-$.

If $\xi\colon A^- \to B$ is any map, a *support* of ξ is an element a of A such that $\xi(x) > 0$ implies $x \leq a$, for all $x \in A^-$. By Lemma 6.1, every step function has a support.

The following lemma establishes a useful compactness property of step functions:

Lemma 6.3 **(Compactness of step functions)** *Let $\eta\colon A^- \to B$ be a step function, let $\mathcal{D}$ be an upward directed set of antitone maps from A^- to B. If for all $a \in A^-$, there exists $\xi \in \mathcal{D}$, with $\eta(a) \leq \xi(a)$, then there exists an antitone map $\xi \in \mathcal{D}$ such that $\eta \leq \xi$.*

Proof Put $U = \bigcup(\operatorname{Max} \eta^{-1}\{b\} \mid b \in \eta[A^-] - \{0\})$. Since η is a step function, it follows from Lemma 6.1 that U is finite. For all $a \in U$, there exists, by assumption, an element ξ_a of $\mathcal{D}$ such that $\eta(a) \leq \xi_a(a)$. Since $\mathcal{D}$ is upward directed and U is finite, there exists $\xi \in \mathcal{D}$ such that $\xi_a \leq \xi$, for all $a \in U$. Now let $a \in A^-$; we prove that $\eta(a) \leq \xi(a)$. This is trivial if $\eta(a) = 0$. Now assume that $\eta(a) > 0$. Since $\eta^{-1}\{\eta(a)\}$ belongs to $\operatorname{Int}_* A$, there exists, by Lemma 6.1(i), $x \in \operatorname{Max} \eta^{-1}\{\eta(a)\}$ such that $a \leq x$. Note that $x \in U$. Therefore,

$$\begin{aligned}
\eta(a) &= \eta(x) \quad (\text{since } x \in \eta^{-1}\{\eta(a)\}) \\
&\leq \xi_x(x) \quad (\text{since } x \in U) \\
&\leq \xi(x) \quad (\text{by the definition of } \xi) \\
&\leq \xi(a) \quad (\text{since } a \leq x \text{ and } \xi \text{ is antitone}).
\end{aligned}$$

This holds for all $a \in A^-$, thus $\eta \leq \xi$. ∎

The following lemma provides us with a large supply of step functions:

Lemma 6.4 *Let $n \in \omega$, let $a_0, \dots, a_{n-1} \in A$, let $b_0, \dots, b_{n-1} \in B$. Then the map $\xi\colon A^- \to B$ defined by*

$$\xi(x) = \bigvee(b_i \mid i < n, x \leq a_i), \tag{6.1}$$

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for all $x \in A^-$, is an antitone step function. Furthermore,

$$\langle x, \xi(x) \rangle \in \bigvee (a_i \otimes b_i \mid i < n),$$

for all $x \in A^-$.

Proof Everything in the statement of the lemma is obvious except for the fact that ξ is a step function. Put

$$S(x) = \{ i < n \mid x \leq a_i \},$$

for all $x \in A^-$. If $b \in B^-$, then, for all $x \in A^-$, $\xi(x) = b$ iff there exists a nonempty subset I of n such that $\bigvee(b_i \mid i \in I) = b$ and $I = S(x)$. Therefore, to prove that $\xi^{-1}\{b\}$ belongs to $\mathrm{Int}_* A$, it suffices to prove that the set

$$X_I = \{ x \in A^- \mid I = S(x) \}$$

belongs to $\mathrm{Int}_* A$, for all nonempty $I \subseteq n$. But it is easy to verify that

$$X_I = \Bigl(\bigwedge (a_i \mid i \in I) \Bigr]_\bullet - \bigcup \bigl((a_j]_\bullet \mid j \in n - I \bigr),$$

which belongs to $\mathrm{Int}_* A$ since I is nonempty. ■

6.2 The Adjustment Sequence of a Step Function

The basic technical tool of transferability, see Section 3.1, is the adjustment sequence: we adjust a map to closer reflect the structure. Similar ideas come up in connection with projective lattices (B. Jónsson, first published in [13]) and bounded homomorphisms (R. N. McKenzie [14]); see also the adjustment sequence of the n-modular identity in [11]. The following definition of the adjustment sequence takes into account that the lattices may be *infinite*. Note that we can only adjust maps with finite range.

For all $a \in A^-$, we shall denote by $\mathcal{C}(a)$ the set of all nonempty, finite subsets X of A^- such that $a \leq \bigvee X$.

Definition 6.5 Let $\xi \colon A^- \to B$ be a map with finite range. The *one-step adjustment* of ξ is $\xi^{(1)} \colon A^- \to B$ defined by

$$\xi^{(1)}(x) = \bigvee \Bigl(\bigwedge \xi[S] \mid S \in \mathcal{C}(x) \Bigr), \tag{6.2}$$

for all $x \in A^-$.

Note that since the range of ξ is finite, the right hand side of the equation (6.2) is well-defined.

Remark 6.6 Since $\{x\} \in \mathcal{C}(x)$, for all $x \in A^-$, the inequality $\xi \le \xi^{(1)}$ always holds. Let us assume, in addition, that ξ is *antitone*. Then the expression (6.2) for $\xi^{(1)}(x)$ takes on the following form:

$$\xi^{(1)}(x) = \xi(x) \vee \bigvee\Big(\bigwedge \xi[S] \mid S \in \mathcal{C}^*(x)\Big), \tag{6.3}$$

where $\mathcal{C}^*(x)$ denotes the set of all $S \in \mathcal{C}(x)$ such that $x \notin (S]$, see Section 3.4.

Let us further assume that A is *finite*. Then every map $\xi\colon A^- \to B$ is a step function, and the one-step adjustment of ξ takes, on the join-irreducible elements of A, the following form:

$$\xi^{(1)}(p) = \xi(p) \vee \bigvee\Big(\bigwedge \xi[I] \mid I \in \mathcal{M}(p)\Big),$$

for all $p \in \mathrm{J}(A)$.

Lemma 6.7 *Let $\xi\colon A^- \to B$ be a step function. For every subset X of $\xi[A^-]$, we define a subset $\operatorname{Cov}(\xi;X)$ of A^- as follows:*

$$\operatorname{Cov}(\xi;X) = \{x \in A^- \mid \xi[S] = X, \textit{ for some } S \in \mathcal{C}(x)\}.$$

Then for all $a \in A$, $\operatorname{Cov}(\xi;X) \cap (a]_\bullet$ belongs to $\operatorname{Int}_* A$.

Proof Let $R = \xi[A^-]$ and define $U_b = \xi^{-1}\{b\} \cap (a]_\bullet$, for all $b \in R$. Since ξ is a step function, $\xi^{-1}\{b\}$ belongs to $\operatorname{Int}_* A$, for all $b \in R - \{0\}$, while $\xi^{-1}\{0\}$ belongs to $\operatorname{Int} A$. Thus, U_b belongs to $\operatorname{Int}_* A$, for all $b \in R$.

Furthermore, $U = \bigcup(\operatorname{Max} U_b \mid b \in R)$ is a finite set, by Lemma 6.1(ii). Let $x \in \operatorname{Cov}(\xi;X) \cap (a]_\bullet$, that is, $x \in A^-$, $x \le a$, and there exists $S \in \mathcal{C}(x)$ such that $\xi[S] = X$. For all $s \in S$, $U_{\xi(s)}$ belongs to $\operatorname{Int}_* A$ and it contains s as an element. By Lemma 6.1(i), there exists $s^* \in \operatorname{Max} U_{\xi(s)}$ such that $s \le s^*$. Put $T = \{s^* \mid s \in S\}$. Then $T \in \mathcal{C}(x)$, $\xi[T] = X$, and $T \subseteq U$. Hence, we have proved the equality

$$\operatorname{Cov}(\xi;X) \cap (a]_\bullet = \{x \in (a]_\bullet \mid \xi[T] = X, \text{ for some } T \in \mathcal{C}(x) \text{ with } T \subseteq U\},$$

which can be written, by the definition of $\mathcal{C}(x)$, as

$$\operatorname{Cov}(\xi;X) \cap (a]_\bullet = \bigcup\bigg(\Big(a \wedge \bigvee T\Big]_\bullet \,\Big|\, T \subseteq U, \xi[T] = X\bigg).$$

Since U is finite, $\operatorname{Cov}(\xi;X) \cap (a]_\bullet$ belongs to $\operatorname{Int}_* A$. ■

Lemma 6.8 *Let $\xi\colon A^- \to B$ be a map with finite range. Then $\xi^{(1)}$ is antitone and has finite range. Furthermore, $\xi^{(1)} = \xi$ iff ξ is a semilattice homomorphism from $\langle A^-;\vee\rangle$ to $\langle B;\wedge\rangle$.*

Proof Since $x \le y$ implies that $\mathcal{C}(y) \subseteq \mathcal{C}(x)$, $\xi^{(1)}$ is obviously antitone. If ξ is a semilattice homomorphism from $\langle A^-;\vee\rangle$ to $\langle B;\wedge\rangle$, then $\bigwedge \xi[S] \le \xi(x)$, for all $x \in A^-$ and all $S \in \mathcal{C}(X)$, thus $\xi^{(1)}(x) \le \xi(x)$. Since $\xi \le \xi^{(1)}$, we obtain that $\xi = \xi^{(1)}$. Conversely, suppose that $\xi = \xi^{(1)}$. Let $x, y \in A^-$. Since $\{x, y\} \in \mathcal{C}(x \vee y)$, the inequality $\xi^{(1)}(x \vee y) \ge \xi(x) \wedge \xi(y)$ holds. Since $\xi = \xi^{(1)}$ and $\xi^{(1)}$ is antitone, we obtain that $\xi(x \vee y) = \xi(x) \wedge \xi(y)$. ■

Lemma 6.9 *Let $\xi\colon A^- \to B$ be a map with finite range. Then ξ and $\xi^{(1)}$ have the same supports.*

Proof Since $\xi \leq \xi^{(1)}$, every support of $\xi^{(1)}$ is a support of ξ. Conversely, let a be a support of ξ. Let $x \in A^-$ such that $x \nleq a$. For all $S \in \mathcal{C}(x)$, there exists $s \in S$ such that $s \nleq a$ (otherwise, $x \leq \bigvee S \leq a$), thus $\bigwedge \xi[S] = \xi(s) = 0$; whence $\xi^{(1)}(x) = 0$. Hence a is a support of $\xi^{(1)}$. ■

We now state the main result of this section:

Proposition 6.10 *Let $\xi\colon A^- \to B$ be a step function. Then the one-step adjustment $\xi^{(1)}$ of ξ is an antitone step function.*

Proof The fact that $\xi^{(1)}$ is antitone with finite range has been established in Lemma 6.8. So, to complete the proof that $\xi^{(1)}$ is a step function, it suffices to prove that $(\xi^{(1)})^{-1}\{b\}$ belongs to $\operatorname{Int}_* A$, for all $b \in B^-$. Let a_0 be a support of ξ. Then, by Lemma 6.9, a_0 is also a support of $\xi^{(1)}$. Put $R = \xi[A^-]$, and let I be the set of all nonempty subsets of $\mathrm{P}(R) - \{\varnothing\}$. Note that I is finite. For all $\mathfrak{a} \in I$, define

$$b_{\mathfrak{a}} = \bigvee\Big(\bigwedge X \Bigm| X \in \mathfrak{a}\Big),$$
$$U_{\mathfrak{a}} = \{x \in (a_0]_\bullet \mid \mathfrak{a} = \{\xi[S] \mid S \in \mathcal{C}(x)\}\}.$$

We claim that the following equality holds:

$$(\xi^{(1)})^{-1}\{b\} = \bigcup(U_{\mathfrak{a}} \mid \mathfrak{a} \in I, b_{\mathfrak{a}} = b). \tag{6.4}$$

Indeed, if x belongs to the right hand side of (6.4), then there exists $\mathfrak{a} \in I$ such that $b_{\mathfrak{a}} = b$ and $x \in U_{\mathfrak{a}}$. Hence $\mathfrak{a} = \{\xi[S] \mid S \in \mathcal{C}(x)\}$, so that

$$\xi^{(1)}(x) = \bigvee\Big(\bigwedge \xi[S] \Bigm| S \in \mathcal{C}(x)\Big) = \bigvee\Big(\bigwedge X \Bigm| X \in \mathfrak{a}\Big) = b_{\mathfrak{a}} = b. \tag{6.5}$$

Conversely, let us assume that $\xi^{(1)}(x) = b$. Since $b > 0$ and a_0 is a support of $\xi^{(1)}$, it follows that $x \leq a_0$ and $\mathfrak{a} = \{\xi[S] \mid S \in \mathcal{C}(x)\}$ belongs to I, so that $x \in U_{\mathfrak{a}}$, by definition. By an argument similar to (6.5), we see that $b_{\mathfrak{a}} = b$. This completes the proof of (6.4).

To complete the proof of Proposition 6.10, it suffices to prove that $U_{\mathfrak{a}}$ belongs to $\operatorname{Int}_* A$, for all $\mathfrak{a} \in I$. By the definition of $U_{\mathfrak{a}}$, an element x of A^- belongs to $U_{\mathfrak{a}}$ iff the following three conditions are satisfied:

(i) $x \leq a_0$;
(ii) for all $X \in \mathfrak{a}$, there exists $S \in \mathcal{C}(x)$, satisfying $\xi[S] = X$;
(iii) $\xi[S] \neq Y$ holds for all $Y \in \mathrm{P}(R) - \mathfrak{a}$ and $S \in \mathcal{C}(x)$.

Therefore, we obtain that

$$U_{\mathfrak{a}} = \bigcap\big((a_0]_\bullet \cap \operatorname{Cov}(\xi; X) \mid X \in \mathfrak{a}\big) - \bigcup\big((a_0]_\bullet \cap \operatorname{Cov}(\xi; Y) \mid Y \in \mathrm{P}(R) - \mathfrak{a}\big),$$

which belongs to $\operatorname{Int}_* A$, by Lemma 6.7. ■

By Proposition 6.10, we can state the following definition:

Definition 6.11 Let $\xi\colon A^- \to B$ be a step function. The *adjustment sequence* of ξ is the sequence $\langle \xi^{(n)} \mid n \in \omega \rangle$ defined inductively by $\xi^{(0)} = \xi$, and $\xi^{(n+1)} = (\xi^{(n)})^{(1)}$, for all $n \in \omega$.

As an immediate consequence of Proposition 6.10, we obtain the following:

Corollary 6.12 *Let $\xi\colon A^- \to B$ be a step function. Then the adjustment sequence of ξ is increasing, that is, $\xi^{(n)} \leq \xi^{(n+1)}$ for all n. Furthermore, if $n > 0$, then $\xi^{(n)}$ is an antitone step function.*

7 Capped Tensor Products and Homomorphisms To Ideals

In this section, we establish equivalent conditions under which a tensor product of two lattices with zero is capped. These conditions are stated in terms of adjustment sequences and choice functions (see Section 2.3).

Theorem 4 *Let A and B be lattices with zero. Then the following statements are equivalent:*

(i) *$A \otimes B$ is a capped tensor product.*
(ii) *Let $\varphi\colon \langle A^-; \vee \rangle \to \langle \mathrm{Id}\, B; \cap \rangle$ be a semilattice homomorphism, let $\xi\colon A^- \to B$ be a step function. If ξ is a choice function for φ, then there exists a step function $\eta\colon A^- \to B$ such that the following conditions hold:*

  (ii_1) *$\xi(a) \leq \eta(a) \in \varphi(a)$, for all $a \in A^-$.*

  (ii_2) *The map η is a semilattice homomorphism from $\langle A^-; \vee \rangle$ to $\langle B; \wedge \rangle$.*

(iii) *The adjustment sequence of any step function $\xi\colon A^- \to B$ is eventually constant.*

Proof Throughout this proof, we shall denote by $\varepsilon\colon A \vec{\otimes} B \to A \bar{\otimes} B$ the canonical isomorphism (see Section 2).

(i) *implies* (ii). We claim that

$$H = \bigvee \bigl(a \otimes \xi(a) \mid a \in A^- \bigr) \in A \bar{\otimes} B$$

belongs to $A \otimes B$. Indeed, it follows from Lemma 6.1(i) that the equality

$$\bigvee (a \otimes b \mid a \in \xi^{-1}\{b\}) = \bigvee (a \otimes b \mid a \in \operatorname{Max} \xi^{-1}\{b\})$$

holds, for all $b \in \xi[A^-] - \{0\}$. Thus, the equality

$$H = \bigvee \bigl(a \otimes \xi(a) \mid b \in \xi[A^-] - \{0\}, a \in \operatorname{Max} \xi^{-1}\{b\} \bigr) \in A \otimes B \tag{7.1}$$

holds, and the right hand side of (7.1) is a finite join, by Lemma 6.1(ii). Hence H belongs to $A \otimes B$.

Since $A \otimes B$ is a capped tensor product, H is a finite union of pure tensors. Thus all the values of the map $\alpha = \varepsilon^{-1}(H)$ are principal ideals of B, say, $\alpha(a) = (\eta(a)]$, for all $a \in A^-$. Since $\alpha \in A \vec{\otimes} B$, the map η is a homomorphism from $\langle A^-; \vee \rangle$ to $\langle B; \wedge \rangle$. Furthermore, $\langle a, \xi(a) \rangle \in H$ for all $a \in A^-$, thus $\xi \leq \eta$. Put $K = \varepsilon(\varphi)$. For all $a \in A^-$, the pair $\langle a, \xi(a) \rangle$

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belongs to K; thus $H \subseteq K$. It follows that $\varepsilon^{-1}(H) \leq \varepsilon^{-1}(K)$, that is, $\eta(a) \in \varphi(a)$, for all $a \in A^-$. Thus η is a choice function for φ.

(ii) *implies* (iii). Let $\xi\colon A^- \to B$ be any step function. We associate with ξ its adjustment sequence, $\langle \xi^{(n)} \mid n \in \omega \rangle$. For all $a \in A^-$, we define $\varphi(a)$ as the ideal of B generated by the set $\{\xi^{(n)}(a) \mid n \in \omega\}$. By Corollary 6.12, all the maps $\xi^{(n)}$ are antitone, for $n > 0$, thus $x \leq y$ implies $\varphi(x) \supseteq \varphi(y)$, for $x, y \in A$. Furthermore, if $x, y \in A^-$, then, since $\{x, y\} \in \mathcal{C}(x \vee y)$, the inequality $\xi^{(n+1)}(x \vee y) \geq \xi^{(n)}(x) \wedge \xi^{(n)}(y)$ holds, for all n. Therefore, $\varphi(x \vee y) \supseteq \varphi(x) \cap \varphi(y)$; since φ is antitone, $\varphi(x \vee y) = \varphi(x) \cap \varphi(y)$. Thus φ is a semilattice homomorphism from $\langle A^-; \vee \rangle$ to $\langle \operatorname{Id} B; \cap \rangle$. Since ξ is a choice function for φ, there exists, by assumption, a step function $\eta\colon A^- \to B$ such that $\xi \leq \eta$ and η is a choice function for φ.

For all $a \in A^-$, by the definition of φ, there exists $n > 0$ such that $\eta(a) \leq \xi^{(n)}(a)$. For $n > 0$, since the $\xi^{(n)}$ are antitone and since η is a step function, it follows from Lemma 6.3, applied to $\mathcal{D} = \{\xi^{(n)} \mid n > 0\}$, that there exists $m > 0$ such that $\eta \leq \xi^{(m)}$. On the other hand, $\xi \leq \eta$ and η is a homomorphism from $\langle A^-; \vee \rangle$ to $\langle B; \wedge \rangle$, thus, by Lemma 6.8, $\xi^{(n)} \leq \eta^{(n)} = \eta$, for all $n \in \omega$. It follows that $\eta = \xi^{(m)} = \xi^{(m+1)}$.

(iii) *implies* (i). Let $H \in A \otimes B$. We prove that H is a finite union of pure tensors. Write

$$H = \bigvee (a_i \otimes b_i \mid i < n),$$

where $n \in \omega$ and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$. Consider the function $\xi\colon A^- \to B$ given by the formula (6.1). By Lemma 6.4, ξ is an antitone step function. By assumption, there exists $m \in \omega$ such that $\xi^{(m)} = \xi^{(m+1)}$. Put $\eta = \xi^{(m)}$. By Lemma 6.8, η is a homomorphism from $\langle A^-; \vee \rangle$ to $\langle B; \wedge \rangle$. In particular, the set

$$K = \bigcup \bigl(a \otimes \eta(a) \mid a \in A^-\bigr)$$

is a bi-ideal of $A \times B$. Thus, since $\xi \leq \eta$ and $\xi(a_i) \geq b_i$, for all $i < n$, K contains H.

Put $\varphi = \varepsilon^{-1}(H)$. By Lemma 6.4, $\langle x, \xi(x) \rangle \in H$, for all $x \in A^-$, thus ξ is a choice function for φ. Since $\varphi \in A \vec{\otimes} B$, all the $\xi^{(n)}$, $n \in \omega$, are choice functions for φ. In particular, η is a choice function for φ. This means that $\langle a, \eta(a) \rangle \in H$, for all $a \in A^-$, that is, K is contained in H. Finally, $K = H$.

Since η is both a step function and a homomorphism from $\langle A^-; \vee \rangle$ to $\langle B; \wedge \rangle$, the set $\eta^{-1}\{b\}$ has, by Lemma 6.1, a largest element, say, a_b, for all $b \in \eta[A^-] - \{0\}$. It follows that

$$H = K = \bigcup (a_b \otimes b \mid b \in \eta[A^-] - \{0\}),$$

a finite union of pure tensors. ∎

8 Amenability and $(\mathrm{T}_\vee)$

In this section, we characterize amenable lattices. We start with a sufficient condition.

Proposition 8.1 *Let A be a finite lattice. If A satisfies $(\mathrm{T}_\vee)$, then A is amenable.*

Proof Let A be a finite lattice satisfying $(\mathrm{T}_\vee)$. Since A satisfies $(\mathrm{T}_\vee)$, $\mathrm{J}(A) = \{p_1, \dots, p_n\}$ so that $p_i \mathrel{D} p_j$ implies that $i > j$, for all i, j in $\{1, \dots, n\}$. We verify Condition (iii) of Theorem 4 for A. So let B be any lattice with zero, and let $\xi\colon A^- \to B$ be any map. We prove that the adjustment sequence of ξ is eventually constant. By replacing ξ by $\xi^{(1)}$, we may assume, without loss of generality, that ξ is antitone.

Claim *Let* $0 < j < i \leq n+1$. *Then* $\xi^{(i)}(p_j) = \xi^{(j)}(p_j)$.

Proof of Claim We prove the Claim by induction on i. The Claim is vacuously true for $i = 0$. Let $0 < j < i+1 \leq n+1$, and let us assume that the induction hypothesis holds for i.

By definition,

$$\xi^{(i+1)}(p_j) = \xi^{(i)}(p_j) \vee \bigvee\Bigl(\bigwedge \xi^{(i)}[I] \mid I \in \mathcal{M}(p_j)\Bigr). \tag{8.1}$$

If $j \leq 2$, then $\mathcal{M}(p_j) = \varnothing$ (because if $\langle p, I\rangle$ is a minimal pair, then I has at least two elements), thus, by the induction hypothesis,

$$\xi^{(i+1)}(p_j) = \xi^{(i)}(p_j) = \xi^{(j)}(p_j).$$

If $j > 2$, then let $I \in \mathcal{M}(p_j)$ and let k satisfy $p_k \in I$. In particular, $k < j$, thus $k < i$, and so by the induction hypothesis,

$$\xi^{(i)}(p_k) = \xi^{(i-1)}(p_k) = \xi^{(k)}(p_k).$$

Therefore, $\bigwedge \xi^{(i)}[I] = \bigwedge \xi^{(i-1)}[I]$. Thus, applying (8.1) to $i-1$, we obtain that $\bigwedge \xi^{(i)}[I] \leq \xi^{(i)}(p_j)$. Using (8.1) and the induction hypothesis,

$$\xi^{(i+1)}(p_j) = \xi^{(i)}(p_j) = \xi^{(j)}(p_j),$$

completing the proof of the Claim. ■

By applying the Claim for $i = n+1$, we obtain that $\xi^{(n)}$ and $\xi^{(n+1)}$ agree on $\mathrm{J}(A)$, thus $\xi^{(n)} = \xi^{(n+1)}$. We have verified Theorem 4(iii) for A, so A is amenable. ■

Now we are ready for the characterization of amenable lattices:

Theorem 5 *Let A be a lattice with zero. Then the following conditions are equivalent:*

(i) A is amenable.
(ii) A is locally finite and every finite sublattice of A satisfies $(\mathrm{T}_\vee)$.

Proof (ii) *implies* (i). Let L be a lattice with zero. By Proposition 8.1, $U \otimes L$ is a capped tensor product, for every finite $\{0\}$-sublattice U of A. By Proposition 4.6, $A \otimes L$ is a capped tensor product.

(i) *implies* (ii). It suffices to prove that for every finitely generated amenable lattice A with zero, A is finite and it satisfies $(\mathrm{T}_\vee)$. Let X be a finite generating subset of A, and let $h\colon \mathrm{F}(X) \twoheadrightarrow A$ be the canonical surjective homomorphism. We shall consider the lower limit table $\langle \beta_n \mid n \in \omega \rangle$ associated with h and X. The maps β_n can be computed using the formulas (3.1) and (3.2).

Now we shall use the fact that $A \otimes \mathrm{F}(X)$ is a capped tensor product. We consider the step function $\xi\colon A^- \to \mathrm{F}(X)$ defined by (6.1), see Lemma 6.4. That is,

$$\xi(a) = \bigvee (x \in X \mid a \leq x),$$

for all $a \in A^-$. Let $w \mapsto w^{\mathrm{d}}$ denote the dualization map on $\mathrm{F}(X)$. We obtain, in particular, that $\xi(a) = \beta_0(a)^{\mathrm{d}}$.

Now we consider the adjustment sequence, $\langle \xi^{(n)} \mid n \in \omega \rangle$, of ξ. Let $n \in \omega$, and let us assume that we have proved that $\xi^{(n)}(a) = \beta_n(a)^{\mathrm{d}}$, for all $a \in A^-$. We compute $\xi^{(n+1)}(a)$, for $a \in A^-$:

$$\begin{aligned}\xi^{(n+1)}(a) &= \xi^{(n)}(a) \vee \bigvee \Big(\bigwedge \xi^{(n)}[S] \mid S \in \mathcal{C}(a) \Big) \\ &= \beta_n(a)^{\mathrm{d}} \vee \bigvee \Big(\bigwedge \beta_n[S]^{\mathrm{d}} \mid S \in \mathcal{C}(a), \bigvee S \leq h(1) \Big)\end{aligned}$$

(the condition $\bigvee S \leq h(1)$ is satisfied, because $h(1) = 1$)

$$= \beta_{n+1}(a)^{\mathrm{d}},$$

the last step by (3.2). It follows that $\xi^{(n)}(a) = \beta_n(a)^{\mathrm{d}}$, for all $n \in \omega$ and all $a \in A^-$. However, since $A \otimes \mathrm{F}(X)$ is a capped tensor product, there exists, by Theorem 4, $n \in \omega$ such that $\xi^{(n)} = \xi^{(n+1)}$. It follows that $\beta_n = \beta_{n+1}$. By Proposition 3.9, A is finite and h is a lower bounded homomorphism. By Proposition 3.8, A satisfies $(\mathrm{T}_\vee)$. ■

The smallest finite lattice that does not satisfy $(\mathrm{T}_\vee)$ is the diamond M_3. Since M_3 is a quotient of $\mathrm{F}(3)$, one obtains, by Corollary 3.7 of [10], the following:

Corollary 8.2 *Neither $M_3 \otimes \mathrm{F}(3)$ nor $\mathrm{F}(3) \otimes \mathrm{F}(3)$ is a lattice.*

Corollary 8.3 *There exists a countable sequence $\langle S_n \mid n \in \omega \rangle$ of finite amenable lattices such that the product $\prod_{n \in \omega} S_n$ is not amenable.*

Proof Denote by x_0, x_1, and x_2 the generators of the free lattice $\mathrm{F}(3)$ on three generators. The subset $S_n = S_n(3)$ (see Section 2.2) of $\mathrm{F}(3)$ is a finite $\{\vee, 0\}$-subsemilattice of $\mathrm{F}(3)$, thus it is a lattice. Furthermore, by the end of the proof of Lemma 2.77 in [4], all S_n are lower bounded. Therefore, by Proposition 3.8, all S_n satisfy $(\mathrm{T}_\vee)$. By Proposition 8.1, all S_n are amenable.

On the other hand, the diagonal map embeds $\mathrm{F}(3)$ into the reduced product $L = \prod_{\mathcal{F}} \langle S_n \mid n \in \omega \rangle$, where $\mathcal{F}$ denotes the Fréchet filter on ω. Suppose that $S = \prod_{n \in \omega} S_n$

is amenable. Since L is a quotient of S and F(3) embeds into L, it follows from Theorem 2 that F(3) is also amenable, a contradiction by Corollary 8.2. ■

In fact, the proof above shows that $M_3 \otimes S$ is not a lattice.

Example 8.4 The proof of Corollary 8.3 gives immediately a locally finite *amenable* lattice S with zero such that M_3 belongs to the variety generated by S (in fact, S generates the variety **L** of all lattices): consider the semilattice direct sum $\bigoplus_{n\in\omega} S_n$, where the S_n are the finite lattices in the proof of Corollary 8.3. This shows that the hypothesis of Corollary 5.2 that B is *finite* cannot be weakened to B being locally finite.

In particular, the class of amenable lattices is not a variety.

9 Discussion

The most central open question is stated first:

Problem 1 Let A be a lattice with zero. If, for every lattice L with zero, $A \otimes L$ is a lattice, is A amenable?

By Theorem 3, the answer to Problem 1 is positive for a locally finite lattice A. We believe that in the general case the answer is in the negative.

Problem 2 Does there exist a nontrivial, simple, amenable lattice with zero?

As we will show in Corollary 9.2, there is no nontrivial simple amenable lattice with a largest element.

We recall that a lattice L is *join-semidistributive*, if it satisfies the following condition:

$$(\mathrm{SD}_\vee) \qquad x \vee z = y \vee z \text{ implies that } x \vee z = (x \wedge y) \vee z, \quad \text{for all } x, y, z \in L.$$

Proposition 9.1 *Let S be a simple lattice with at least three elements. If S satisfies* $(\mathrm{SD}_\vee)$, *then S does not have a largest element.*

Proof If S has a largest element, then it has a maximal ideal, say, I. Then I is a *prime ideal* of S. Indeed, if x, $y \notin I$, then, by the maximality of I, there exists $u \in I$ such that $x \vee u = y \vee u = 1$. By $(\mathrm{SD}_\vee)$, $(x \wedge y) \vee u = 1$, thus $x \wedge y \notin I$ (otherwise, $1 \in I$, a contradiction).

So I defines a lattice homomorphism from S to the two-element chain. Since S has at least three elements, the kernel of this homomorphism is a non-trivial congruence of S, which contradicts the simplicity of S. ■

It is known that every finite lattice satisfying $(\mathrm{T}_\vee)$ satisfies $(\mathrm{SD}_\vee)$, see, for example, Theorem 2.20 in [4]. Since $(\mathrm{SD}_\vee)$ is preserved under direct limits, we obtain the following corollary:

Corollary 9.2 *Let S be a simple amenable lattice with zero. If S has at least three elements, then S does not have a largest element.*

To formulate the next problem, let us introduce an additional terminology.

Definition 9.3 Let A be a lattice with zero and let $\mathbf{C}$ be a class of lattices. Then A is $\mathbf{C}$-*amenable*, if $A \otimes L$ is a lattice, for every lattice with zero L in $\mathbf{C}$.

Problem 3 Let $\mathbf{V}$ be a variety of lattices. Is the class of finite $\mathbf{V}$-amenable lattices *decidable*?

For example, if $\mathbf{M}$ is the variety of all *modular* lattices, then M_3 is $\mathbf{M}$-amenable. It would be desirable to obtain a combinatorial characterization of $\mathbf{M}$-amenable lattices, as we did in this paper for $\mathbf{L}$-amenable lattices, where $\mathbf{L}$ is the variety of all lattices. On the other extreme, if $\mathbf{D}$ is the variety of all distributive lattices, every lattice with zero is $\mathbf{D}$-amenable.

Problem 4 Let $\mathbf{A}$ denote the variety of all *Arguesian* lattices. For a finite lattice A, prove that A is $\mathbf{M}$-amenable iff it is $\mathbf{A}$-amenable.

By using a lattice constructed in [2], we can prove that M_4 is not $\mathbf{M}$-amenable, see [11] for details. Furthermore, the corresponding counterexample is a lattice of subspaces of a vector space, thus it is Arguesian. So one may expect, for every non $\mathbf{M}$-amenable lattice A, the existence of a lattice L of subspaces of some vector space such that $A \otimes L$ is not a lattice.

We have proved in Theorem 5 that every amenable lattice is locally finite. This result cannot be relativized to arbitrary varieties, as, for example, any lattice with zero is $\mathbf{D}$-amenable.

Problem 5 Is every $\mathbf{M}$-amenable lattice locally finite?

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