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Varieties of topological groups

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We introduce the concept of a variety $\underline{\underline{V}}$ of topological groups and of a free topological group $F(X, \underline{\underline{V}})$ of $\underline{\underline{V}}$ on a topological space X as generalizations of the analogous concepts in the theory of varieties of groups. Necessary and sufficient conditions for $F(X, \underline{\underline{V}})$ to exist are given and uniqueness is proved. We say the topological group $F_M(X)$ is moderately free on X if its topology is maximal and it is algebraically free with X as a free basis. We show that $F_M(X)$ is a free topological group of the variety it generates and that if $F_M(X)$ is in $\underline{\underline{V}}$ then it is topologically isomorphic to a quotient group of $F(X, \underline{\underline{V}})$. It is also shown how well known results on free (free abelian) topological groups can be deduced. In the algebraic theory there are various equivalents of a free group of a variety. We examine the relationships between the topological analogues of these. In the appendix a result similar to the Stone-Čech compactification is proved.

1. Introduction

The concept of a free topological group was introduced by Markov [5]. In a similar manner we introduce topological analogues of other concepts

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involved in the theory of varieties of abstract groups. Before giving a summary of each section, we note that varieties of topological groups have not been considered before in published papers. Thus nearly all of our results are new. Further, some of the results on free topological groups, originally proved by Markov [5] using norms, have been deduced here from more general results, the proofs of which use elementary methods not involving norms.

In §2 the concepts of a variety of topological groups and of a free topological group of a topological variety are introduced. A theorem (Theorem 2.6), which gives a necessary and sufficient condition for the latter to exist, is proved. Uniqueness of it is also proved (Theorem 2.9).

In §3 topological varieties of a special type are studied. This specialization is justified because firstly the results obtained are significant and secondly some important topological varieties are of this type. In particular the variety of all topological groups is included. The results of this section are, then, generalizations of results, on free topological groups, in [5].

In §4 we indicate how many of the well known results of [5] on free (free abelian) topological groups can be deduced from the results of §2 and §3.

Neumann [6] has various equivalents of the free group of a variety of abstract groups. In §5 we examine the relationships between the topological analogues of these. In fact these analogues are found not to be equivalent. This is one of the most noticeable differences between the algebraic and topological theories.

In the Appendix a result similar to the Stone-Čech compactification is proved. The reason for including it in a paper of this type is that its proof is very similar to the proof of Theorem 2 of [3] on which the main theorem (Theorem 2.6) of §2 relies.

The notation and terminology conform to that of [4] and [6] except the following. By a group, an isomorphism and a space we mean a topological group, a topological isomorphism and a topological space respectively. By an abstract group, a relatively free abstract group, an algebraic relator and an algebraic variety (of abstract groups) we mean the respective concepts of [6] when the words "algebraic" and "abstract" have been deleted. A free abstract group F of the algebraic variety $\underline{\underline{\bar{V}}}$ with basis $\bar{X}$ will be denoted by $F(\bar{X}, \underline{\underline{\bar{V}}})$. If X is an arbitrary topological space then $\bar{X}$ denotes the corresponding set. If G is a topological group then $\bar{G}$ denotes the corresponding abstract group. $\prod_{\gamma \in \Gamma} S_\gamma$ (S_γ topological spaces), $|X|$ and $\text{cl}.X$ denote the cartesian product of the spaces S_γ with the product topology, the cardinal of X and the closure of X respectively.

Note that whenever we refer to a topology on an abstract group we shall mean a topology compatible with the group structure. We conclude the introduction with the important point that unlike earlier papers dealing with free topological groups, we do consider topological groups which are not Hausdorff.

2. Varieties and their free groups

DEFINITION 2.1 A non-empty class $\underline{\underline{V}}$ of groups is said to be a variety if it has the properties:

(a) if G is a subgroup of $\prod_{\gamma \in \Gamma} G_\gamma$, where Γ is any index set and G_γ is in $\underline{\underline{V}}$ for $\gamma \in \Gamma$, then G is in $\underline{\underline{V}}$,

(b) if H is a quotient group of any G in $\underline{\underline{V}}$, then H is in $\underline{\underline{V}}$.

REMARK 2.2 Property (a) is equivalent to (i) closure under cartesian products together with (ii) closure under subgroups.

REMARK 2.3 We see that a variety determines an algebraic variety; the latter is simply the class of abstract groups which with some topology appear in the former. (This is indeed an algebraic variety by 15.51 of [6].)

Note that the symbols $\underline{\underline{V}}$, $\underline{\underline{W}}$ and so on, will always denote varieties. Further the algebraic variety determined by $\underline{\underline{V}}$ will be denoted by $\underline{\underline{\bar{V}}}$.

EXAMPLE 2.4 We list here some important varieties,

(a) the class of all groups,

(b) the class of all abelian groups,

(c) the class $\underline{\underline{V}}$, where G is in $\underline{\underline{V}}$ if and only if $\bar{G}$ is in $\underline{\underline{\bar{W}}}$,

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a fixed algebraic variety,

(d) the class $\underline{\underline{V}}$, where G is in $\underline{\underline{V}}$ if and only if G has the indiscrete topology and $\overline{G}$ is in $\underline{\underline{W}}$, a fixed algebraic variety.

Clearly in (c) and (d) $\overline{\underline{\underline{V}}} = \underline{\underline{W}}$. Also (a) and (b) are special cases of (c).

DEFINITION 2.5 Let $F \in \underline{\underline{V}}$. Then F is said to be a free group of $\underline{\underline{V}}$ on the space X , denoted by $F(X , \underline{\underline{V}})$, if it has the properties:

(a) X is a subspace of F ,

(b) X generates F algebraically,

(c) For any continuous mapping ϕ of X into any group H in $\underline{\underline{V}}$, there exists a continuous homomorphism Φ of F into H such that $\Phi(x) = \phi(x)$ on X .

THEOREM 2.6 *Let* X *be a completely regular space and* $\underline{\underline{V}}$ *a variety. Then* $F(X , \underline{\underline{V}})$ *exists if (and only if) there is a* $G \in \underline{\underline{V}}$ *which has* X *as a subspace.*

Proof Let S be the family of all groups $\{H\}$ where H is in $\underline{\underline{V}}$ and $|H| \leqq \max\{|X| , \aleph_0\} = m$.

Noting that Theorem 2 of [3] is true even when R is not Hausdorff, we can apply it here to (S , X) . Therefore there exists an $F \in S$ with the property that for any continuous mapping ϕ of X into any H in S , there exists a continuous homomorphism Φ of F into H such that $\Phi(x) = \phi(x)$ on X . Also X generates F algebraically.

Let ϕ_1 be any continuous mapping of X into any H in $\underline{\underline{V}}$. The subgroup H_1 of H generated by $\phi_1(X)$ is in $\underline{\underline{V}}$ and $|H_1| \leqq m$. Thus H_1 is in S . Then ϕ_1 is a continuous mapping of X into H_1 . Thus there exists a continuous homomorphism Φ of F into H_1 and therefore of F into H , such that $\Phi(x) = \phi(x)$ on X . Hence F is an $F(X , \underline{\underline{V}})$.

LEMMA 2.7 *Let* $G \in \underline{\underline{V}}$. *Then there exists an* A *in* $\underline{\underline{V}}$ *, where* A *has the indiscrete topology and is algebraically isomorphic to* G .

Proof Let C and D be the Cartesian product and the direct product respectively of a denumerable number of copies of G . Then D is dense in C (Theorem 6.2 of [2]). Therefore C/D has the indiscrete topology.

Clearly C, D and C/D are in $\underline{\underline{V}}$.

Take the diagonal copy of G in C, $\{(x, x, x, \ldots) : x \in G\} = H$. Let f be the natural mapping of C onto C/D. Then f is a homomorphism. Now $f(a) \neq f(b)$ for $a \neq b$ and $a, b \in H$. Then $f(H) = A \leq C/D$. Therefore A is in $\underline{\underline{V}}$ and f is an algebraic isomorphism of H onto A. Thus A is algebraically isomorphic to G, and since $A \subseteq C/D$, A has the indiscrete topology.

THEOREM 2.8 *If* $F(X, \underline{\underline{V}})$ *exists, then* $\overline{F(X, \underline{\underline{V}})} \cong F(\overline{X}, \overline{\underline{\underline{V}}})$.

Proof Let $\overline{\phi}$ be any mapping of $\overline{X}$ into $\overline{G} \in \overline{\underline{\underline{V}}}$. By Lemma 2.7, A is in $\underline{\underline{V}}$ where A has the properties stated in that lemma. Let τ be an algebraic isomorphism from G to A. Then $\tau\phi$ is a continuous mapping from X into A (since A has the indiscrete topology). Therefore there exists a continuous homomorphism Φ of $F(X, \underline{\underline{V}})$ into A such that $\Phi | X = \tau\phi$. Then $\tau^{-1}\Phi$ is a homomorphism of $\overline{F(X, \underline{\underline{V}})}$ into $\overline{G}$ such that $(\tau^{-1}\Phi) \mid \overline{X} = \phi$.

THEOREM 2.9 *Let* X *be an arbitrary space. If* F_1 *is in* $\underline{\underline{V}}$ *and has properties (a) and (c) of Definition 2.5 and the property:*

> *(b') the smallest closed subgroup of* F_1 *that contains* X *is* F_1 *itself (that is* X *generates* F_1 *topologically),*

then there is an isomorphism τ *of* $F(X, \underline{\underline{V}})$ *onto* F_1 *such that* $\tau(x) = x$ *on* X.

Proof By Theorem 2.6 $F(X, \underline{\underline{V}})$ exists. The remainder of the proof is analagous to the proof of Theorem 8.9 of [2].

Theorem 2.9 is indeed a generalization of Theorem 8.9 of [2].

COROLLARY 2.10 *Let* $\underline{\underline{V}}$ *contain the additive group* R *of reals (with the usual topology). Then* $F(X, \underline{\underline{V}})$ *exists for any Tychonoff space* X.

Proof Any Tychonoff space can be embedded in a cartesian product G of copies of R (p. 118 of [4]). Now G is in $\underline{\underline{V}}$ and the result then follows from Theorem 2.6.

(The proof of the following lemma arose from a discussion with John Ellwood.)

LEMMA 2.11 *Let* F *be a group which is generated algebraically by* $\overline{X}$. *If* Φ *is a homomorphism from* F *into a group* G *such that* $\Phi | X$ *is open then* Φ *is open.*

Proof Let O be an open set in aX, $a \in F$. Then $O = aO_1$, where O_1 is open in X. Therefore $\Phi(O) = \Phi(a)\,\Phi(O_1)$. Now $\Phi(O_1)$ is open in G, since $\Phi|X$ is open. Thus $\Phi(O)$ is open. Hence O open in aX, implies $\Phi(O)$ open.

Now $F = \bigcup_{a\in F} aX$. Let A be an open set in F. Then $\Phi(A) = \Phi(A \cap \bigcup_{a\in F} aX) = \bigcup_{a\in F} \Phi(A \cap aX)$. However $\Phi(A \cap aX)$ is open in G (by the first paragraph). Thus $\Phi(A)$ is open.

THEOREM 2.12 *Let X be in $\underline{\underline{V}}$. Then $F(X, \underline{\underline{V}})$ exists and has a quotient group isomorphic to X.*

Proof By Theorem 2.6 $F(X, \underline{\underline{V}})$ exists. Now consider the identity mapping $\phi : X\ (\subseteq F(X, \underline{\underline{V}})) \to X$. This is continuous and therefore can be extended to a continuous homomorphism Φ of $F(X, \underline{\underline{V}})$ onto X. Clearly ϕ is an open mapping. Thus by Lemma 2.11, Φ is open. Hence Φ is an open continuous homomorphism of $F(X, \underline{\underline{V}})$ onto X and the theorem follows.

We conclude this section with a remark which can easily be verified: $F(X, \underline{\underline{V}})$ has the discrete topology if and only if it has X as an open subset.

3. Some special varieties

DEFINITION 3.1 Let F be an arbitrary group and $\underline{\underline{V}}(F)$ the intersection of all varieties containing F. Then $\underline{\underline{V}}(F)$ is said to be the variety generated by F. (Clearly this is indeed a variety.)

It is readily seen that if F is $F(X, \underline{\underline{W}})$ for some space X and variety $\underline{\underline{W}}$, then F is $F(X, \underline{\underline{V}}(F))$.

DEFINITION 3.2 The group F is said to be moderately free on the space X, and will be denoted by $F_M(X)$, if,

(i) $\overline{F}$ is a relatively free abstract group with $\overline{X}$ as a free generating set,

(ii) the topology of F is the strongest group topology (on $\overline{F}$) which will induce the same topology on $\overline{X}$.

THEOREM 3.3 *If F is $F_M(X)$ then F is $F(X, \underline{\underline{V}}(F))$.*

Proof Let ϕ be an arbitrary continuous mapping from X to any G

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in $\underline{\underline{V}}(F)$. Now by Theorem 2.8 $\overline{F}$ is $F(\overline{X}, \overline{\underline{\underline{V}}(F)})$. Therefore there exists a homomorphism Φ of F into G such that $\Phi|X = \phi$. We show now that Φ is continuous.

Let $\{U\}$ be a basis for open neighbourhoods of 1 in T , the given topology on F . Let $\{V\}$ be the collection of inverse images of open neighbourhoods of 1 in G . The system $\{U \cap V\}$ is a basis for open neighbourhoods of 1 for a group topology $\mathcal{T}$. (See Theorem 4.5 of [2].) Obviously $\mathcal{T}$ induces the given topology on X , since ϕ is continuous and $\mathcal{T} \supseteq T$. But since F is $F_M(X)$, $T \supseteq \mathcal{T}$. Thus $\mathcal{T} = T$ and hence Φ is continuous.

We have been unable to find a non-trivial necessary and sufficient condition for an arbitrary group F to be a free group of the variety it generates. However Theorem 3.3 gives a sufficient condition.

In this section we shall confine ourselves to moderately free groups. While this line of study may seem rather restrictive, the results obtained are significant and include results of Markov [5] on free topological groups. Further justification comes from the fact that the free groups of varieties of the type mentioned in Example 2.4(c) are moderately free.

LEMMA 3.4 *Let G be in $\underline{\underline{V}}$ and have generating space X . Then G is a continuous algebraic isomorphic image of a quotient group H of $F(X, \underline{\underline{V}})$ where H has X as a generating space also.*

Proof Clearly $F(X, \underline{\underline{V}})$ exists. Consider the identity mapping $\phi : X (\subseteq F(X, \underline{\underline{V}})) \to X (\subseteq G)$. This is a continuous mapping of X into G . Therefore ϕ can be extended to a continuous homomorphism Φ of F onto G . (The homomorphism Φ is onto since $\Phi(F) \supseteq X$ and $\overline{X}$ generates $\overline{G}$ algebraically.) Therefore G is algebraically isomorphic to a quotient group H of $F(X, \underline{\underline{V}})$.

Now H has a stronger topology than G . Let θ be the natural mapping of $F(X, \underline{\underline{V}})$ onto H , and $\theta(X) = X'$. Then θ maps X one to one onto X' . The topology on X' is weaker than that on X since $\Phi|X : X \to X'$ is continuous. However X is a subspace of G , therefore X' has a stronger topology than X . Thus X and X' are homeomorphic.

THEOREM 3.5 *Let $F_M(X)$ be in $\underline{\underline{V}}$. Then $F_M(X)$ is isomorphic to a quotient group of $F(X, \underline{\underline{V}})$.*

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Proof By Lemma 3.4, $F_M(X)$ is a continuous algebraic isomorphic image of the quotient group H of $F(X, \underline{\underline{V}})$. Also H has X as a generating space.

Clearly, by Definition 3.2 (ii) and the above statement, H and F are isomorphic.

4. Free groups

DEFINITION 4.1 Let X be a completely regular space and $\underline{\underline{V}}$ the variety of all (all abelian) groups. Then $F(X, \underline{\underline{V}})$ is said to be the free (free abelian) group on X.

The existence and uniqueness of the free (free abelian) group on a Tychonoff space, that is Theorem 1 and Theorem 2 (Theorem 20 and 21 of [5]) follow from Corollary 2.10 and Theorem 2.9 respectively. (Note that (F_3) of [5] has been extended to the case when G is an arbitrary group, not necessarily Hausdorff.)

Theorem 3 and Theorem 21 of [5] are special cases of Theorem 2.8. Theorem 5 and Theorem 23 of [5] can be deduced from Theorem 2.12. Finally, Theorem 24 and Corollary 9 of [5] follow from Theorem 3.5.

THEOREM 4.2 *Let* $\overline{\underline{\underline{V}}}$ *be an algebraic variety of exponent zero, and* $\underline{\underline{W}}$ *be the largest variety such that* $\overline{\underline{\underline{W}}} = \overline{\underline{\underline{V}}}$. *If* X *is a Tychonoff space, then* $F(X, \underline{\underline{W}})$ *exists. Further* $F(X, \underline{\underline{W}})$ *is isomorphic to* $F/V(F)$ *where* $V(F)$ *is algebraically a verbal subgroup of* F, *the free group on* X.

Proof Clearly $F(X, \underline{\underline{W}})$ exists by Corollary 2.10, and $F(X, \underline{\underline{W}})$ is moderately free on X. The remainder of the theorem then follows from Theorem 3.5.

It can be proved easily that $F(X, \underline{\underline{W}})$ is disconnected.

The following lemma is a special case of Lemma 3.4.

LEMMA 4.3 *Let* G *be a group with generating space* X. *Then* G *is a continuous algebraic isomorphic image of a quotient group* H *of the free group on* X, H *having* X *as a generating space also.*

THEOREM 4.4 *If* $\overline{A}$ *is a fully invariant abstract subgroup of* $\overline{F}$, *where* F *is the free group on* X, *then* F/A *is moderately free.*

Proof $\overline{F/A}$ is algebraically relatively free. If $A \neq F$ then

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$A \cap X = \phi$, $A \cap X^{-1}X = A \cap XX^{-1} = \{1\}$.

Let f be the natural mapping $F \to F/A$ and $f(X) = X'$. By the above paragraph f maps X one to one onto X' . Clearly X' has a weaker topology than X . Let F' be the free group on X' . Then $\overline{F}$ and $\overline{F'}$ are algebraically isomorphic. Since X' has a weaker topology then X , F' has a weaker topology than F . Now F/A (by Lemma 4.3) is a continuous algebraic-isomorphic image of F'/B (some B).

The topology on F/A is the strongest such that the natural mapping of F' to F/A is continuous. (For suppose there is a stronger topology with this property. Then we have continuous mappings $F \to F' \to F/A$ (new topology). This is a contradiction to F/A being a quotient group of F .)

Thus F/A is isomorphic to F'/B . But every group algebraically isomorphic to F/A is a continuous image of F'/B . (This follows from Lemma 4.3 by noting that H is dependent only on the algebraic structure of G and the topology on X .) Hence the topology on F/A is the strongest topology on the group F/A which will induce the given topology on X' . The result follows.

THEOREM 4.5 *Let* $\underline{\underline{V}}$ *be the variety of all abelian groups. If* X *is any Tychonoff space, then* X *is a closed subspace of* $F(X , \underline{\underline{V}})$. *Further* $F(X , \underline{\underline{V}})$ *is a Hausdorff space.*

Proof By Corollary 2.10, $F(X , \underline{\underline{V}})$ exists. Suppose the reduced word $y = x_1^{a_1} \ldots x_n^{a_n}$, $x_i \in X$, $i = 1 , \ldots , n$, is an accumulation point of X , but is not in X . Let R be the additive group of reals (with the usual topology).

Define a continuous mapping ϕ of X into R by $\phi(X) = 1$. Then there exists a continuous homomorphism Φ of $F(X , \underline{\underline{V}})$ into R such that $\Phi|X = \phi$. Since y is an accumulation point of X , $\Phi(y) = \sum_{i=1}^{n} a_i = 1$. Without loss of generality, then, $a_n \leqq -1$.

Put $Y = \{x : x \in R , 0 \leqq x \leqq 1\}$. Since X is a Hausdorff space, there exists an open subset A of X such that $x_n \in A$ and

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$x_i \notin A$, $i = 1, \ldots, n-1$. Now since X is completely regular, there exists a continuous mapping ψ of X into Y (with the usual topology) such that $\psi(x_n) = 0$ and $\psi(X - A) = 1$. Then ψ is a continuous mapping of X into R . Therefore there exists a continuous homomorphism Ψ of $F(X, \underline{\underline{V}})$ into R such that $\Psi|X = \psi$.

Clearly $\Psi(X) = \psi(X) \subseteq Y$ and $\Psi(y) = \sum_{i=1}^{n-1} a_n = 1 - a_n \geq 2$. Then $\Psi(y)$ is not an accumulation point of $\Psi(X)$ and hence y is not an accumulation point of X . Thus X is a closed subspace of $F(X, \underline{\underline{V}})$. It follows immediately that $F(X, \underline{\underline{V}})$ is Hausdorff.

5. Relatively free groups

Neumann has various equivalents of "the free abstract group of an algebraic variety", namely 13.11, 13.21, 13.22 and 13.23 of [6]. In this section we examine the relationships between the topological analogues of these.

DEFINITION 5.1 A group F is said to be relatively free if it possesses a generating space X such that every continuous mapping of X into F can be extended to a continuous endomorphism of F . X is said to be a free generating space of F .

Clearly a free group of a variety is relatively free.

DEFINITION 5.2 Let X be a generating space for the group F . Then $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, $x_i \in X$, $\varepsilon = \pm 1$ is said to be a law if $\phi(x_1)^{\varepsilon_1} \ldots \phi(x_n)^{\varepsilon_n} = 1$ for any continuous mapping ϕ of X into F .

DEFINITION 5.3 A subgroup A of the group B is said to be fully invariant if $f(A) \subseteq A$ for f an arbitrary continuous endomorphism of B .

Note that if A is a fully invariant subgroup of B then $\bar{A}$ is an algebraic normal subgroup of $\bar{B}$. Also cl.A is a fully invariant subgroup of B .

We now formulate the analogues of 13.11, 13.21, 13.22 and 13.23 of [6].

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(5.4) G is relatively free with generating space X .

(5.5) G has a generating space X such that every algebraic relator of X is a law in G .

(5.6) G has a representation $G \cong F/A$ as the quotient group of the free group F by a fully invariant subgroup A of F , where F and G have the same generating space.

(5.7) G has a representation $G \cong F/R$ such that every continuous endomorphism of the free group F induces the natural endomorphism of G , where G and F have the same generating space and R is a normal subgroup of F . (The induced endomorphism, if it exists, is clearly continuous.)

THEOREM 5.8 *Property (5.7) is equivalent to (5.6), which implies (5.4) which in turn implies (5.5).*

Proof

(i) Clearly (5.7) is equivalent to (5.6).

(ii) Suppose (5.6) is satisfied. Let X be the free generating space of F and f be the natural mapping of F onto F/A . Clearly $Y = \{Ax : x \in X\}$ is a generating space of F/A .

Let ϕ be an arbitrary continuous mapping of Y into F/A . Define a mapping γ of X into F by $\gamma(x) = y$, $x \in X$, $y \in F$ if $\phi(Ax) = Az$ and y is a fixed coset representative of Az . Let O be an open set in F , then $\gamma^{-1}(O) = f^{-1}(\phi^{-1}(f(O)))$. Since ϕ is continuous and f is open and continuous, γ is continuous.

Therefore γ can be extended to a continuous endomorphism Γ of F . Since A is fully invariant, Γ induces an endomorphism Φ of F/A and Φ is an extension of ϕ . Now Γ is continuous and f is open and continuous, thus $\Phi^{-1}(A) = f(\Gamma^{-1}(f^{-1}(A)))$ implies Φ is continuous. Hence (5.4) holds.

(iii) Suppose 5.4 is satisfied. Given an arbitrary continuous mapping $\phi : X \to G$ we can define a many-valued function $\Phi : G \to G$ by $\Phi(x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}) = \phi(x_1)^{\varepsilon_1} \ldots \phi(x_n)^{\varepsilon_n}$. Clearly Φ is single-valued if it is single-valued at 1 . But (5.4) holds,

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thus Φ is single-valued at 1 and therefore the result follows.

COROLLARY 5.9 *If G is a quotient group of a relatively free group F by a fully invariant subgroup of F, then G is relatively free.*

To show that the only relationships between the properties (5.4), (5.5), (5.6) and (5.7) are those expressed in Theorem 5.8 we give the following example.

EXAMPLE 5.10 Let $H(d)$ and $H(n)$ be the additive abstract group of integers with the topologies $T(d)$ and $T(n)$ respectively, where $T(d)$ is the discrete topology and $T(n)$ is the topology which has as a basis the set of all cosets of the subgroups $gp\{n\}$, $gp\{2n\}$, ... It is readily verified (4.21 of [2]) that $T(n)$ is a non-discrete Hausdorff group topology.

Consider $H(n)$, $n > 2$. Let the mapping ϕ take 1 onto n. This certainly is a continuous mapping of the generating space X of $H(n)$ into $H(n)$. Let the obvious homomorphic extension of ϕ be Φ. Then $\Phi^{-1}[gp\{2n\}] = gp\{2\}$. Thus Φ is not continuous. Hence $H(n)$ is not relatively free with generating space X. However since every group which is algebraically relatively free has property (5.5), $H(n)$ has this property. Hence (5.5) does not imply (5.4).

An auxiliary result is that not every Hausdorff topology on an algebraically relatively free group gives rise to a relatively free group.

Another interesting point arises from the above discussion. Obviously $F(X, \underline{\underline{V}}(H(n)))$ exists and is algebraically isomorphic to $H(n)$ but has a strictly stronger topology. Thus the variety generated by a group can contain a group algebraically isomorphic to the former group but having a strictly stronger topology.

Consider $H(1)$. Clearly this is a relatively free group which is not isomorphic to a quotient group of the free group on X, that is $H(d)$. Hence property (5.4) does not imply (5.6).

While we have shown that (5.5) does not imply (5.4), Theorem 5.11 shows that (5.5) together with another property implies (5.4). The proof of Theorem 5.11 is similar to the second part of the proof of Theorem 3.3.

THEOREM 5.11 *If G has property (5.5) and the given topology on G is the strongest topology which will induce the same topology on X, then G is relatively free.*

Note that (5.5) is equivalent to the property that every continuous mapping $X \to G$ can be extended to an endomorphism of G.

Using Lemma 4.3 and Theorem 5.11 we can prove the following result.

COROLLARY 5.12 *Let G be a group with generating space X. If G has property (5.5) then the group H mentioned in Lemma 4.3 is relatively free.*

We conclude with a Theorem which is very easily proved.

THEOREM 5.13 *(i) If the relatively free group F has a discrete generating space X, then $\bar{F}$ is a relatively free abstract group.*

(ii) If F satisfies (i) and A is a fully invariant subgroup of F, then $\bar{A}$ is a fully invariant abstract subgroup of $\bar{F}$.

(iii) If the relatively free group F has the discrete topology, then it is moderately free.

Appendix

The following lemma is easily proved using the elementary theory of filter bases.

LEMMA *If X is any subspace of a Hausdorff space H, then* $|\mathrm{cl}.X| \leq 2^{2^{|X|}}$.

The following theorem is a result similar to the Stone-Čech compactification. In fact the latter can be proved similarly.

THEOREM *Given any topological space X and any cardinal m, there exists a compact space $F(m, X)$ with the following properties:*

(i) X is a subspace of $F(m, X)$,

(ii) X is dense in $F(m, X)$,

(iii) for any continuous mapping ϕ of X into any compact space G, where $|G| \leq m$, there exists a continuous mapping Φ of $F(m, X)$ into G, such that $\Phi(x) = \phi(x)$ on X,

(iv) *for any continuous mapping* ϕ *of* X *into any compact Hausdorff space* H *, there exists a continuous mapping* Φ *of* $F(m, X)$ *into* G *such that* $\Phi(x) = \phi(x)$ *on* X .

Proof Let ϕ be any continuous mapping $X \to H$, H a compact Hausdorff space. Then without loss of generality we can assume $|H| \leqq 2^{2^{|x|}}$. For consider $\text{cl}.\phi(X)$. Its cardinality, by the above lemma, satisfies $|\text{cl}.\phi(X)| \leqq 2^{2^{|\phi(x)|}} \leqq 2^{2^{|x|}}$. Also $\text{cl}.\phi(X)$ is a compact Hausdorff space.

Let S be the family of all homeomorphism classes of compact topological spaces with cardinal $\leqq \max\{m , 2^{2^{|x|}}\}$. Let $\{(G_i , \phi_i)\}_{i \in I}$ consist of all pairs (G_i , ϕ_i) where $G_i \in S$ and ϕ_i is a continuous mapping of X into G_i . Let $G_0 = \prod_{i \in I} G_i$.

Let ν be a mapping $X \to G_0$ defined by $[\nu(x)]_i = \phi_i(x)$. Then ν is a homeomorphism of X onto $\nu(X)$. (There exists a space H in S which contains X as a subspace, for instance the one point compactification of X . Now use Lemma 5 on p.116 of [4] where the mapping which (a) distinguishes points and (b) distinguishes points and closed sets is the mapping of X into H which takes X identically onto X .)

Thus we can regard X as a subspace of G_0 . Let F be the closure of $\nu(X)$ in G_0 . Then F is compact, since G_0 is compact. F is the required space.

Now (i) and (ii) are obvious. Let ϕ be any continuous mapping $X \to G$, $G \in S$. Then $(G , \phi) \equiv (G_\gamma , \phi_\gamma)$ where $\gamma \in I$. Let p_0 be the projection of G_0 on G_γ . Then $\phi = p_0\nu$ or by identifying X with $\nu(X)$, $\phi = p_0|\nu(X)$. Define $\Phi = p_0|F$. Then Φ is a continuous mapping of F into G (since p_0 is a continuous mapping) and $\Phi(x) = \phi(x)$ on X .

The final theorem includes the results of [7]. However it can be

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proved as above, by noting that the one point compactification of a T_0 space is T_0.

THEOREM *Given any* T_0 *space* X *any cardinal* m, *there exists a compact* T_0 *space* $F(m, X)$ *satisfying (i), (ii) and (iv) of the above theorem and (iii)' below:*

(iii)' for any continuous mapping ϕ *of* X *into any compact* T_0 *space* G, *where* $|G| \leqq m$, *there exists a continuous mapping* Φ *of* $F(m, X)$ *into* G *such that* $\Phi(x) = \phi(x)$ *on* X.

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