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ON THE SPACE OF MATRICES OF GIVEN RANK

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1. Introduction

Let V and W be finite dimensional real vector spaces, $k \geqq 0$ an integer. We write $L(V, W)$ for the space of all linear maps $V \to W$ and $L_k(V, W)$ for the subspace of maps with kernel of dimension k; in particular, $L_0(V, W)$ is the open subspace of injective linear maps. Thus $L_k(\mathbb{R}^n, \mathbb{R}^n)$ is the space of $n \times n$-matrices of rank $n-k$ in the title. We also need the notation $G_k(V)$ for the Grassmann manifold of k-dimensional subspaces of V.

In this note we shall investigate the homotopy theory of the smooth fibre bundle

$$\pi: L_k(V, W) \to G_k(V)$$

obtained by mapping an element of $L_k(V, W)$ to its kernel. The basic structure of the bundle is recalled in Section 2. In Section 3 we study the stable homotopy type and establish a stable splitting theorem as a consequence of Miller's results [5] on Stiefel manifolds. To state a special case of the theorem we introduce the notation: $L_k(V, W)_+$ for the pointed space obtained by adjoining a disjoint basepoint to $L_k(V, W)$, $G_{k,l}(\mathbb{R}^n)$ for the flag manifold $O(n)/O(k) \times O(l) \times O(n-(k+l))$ (for $k, l \geqq 0$, $n \geqq k+l$). We think of a point of $G_{k,l}(\mathbb{R}^n)$ as a pair of orthogonal subspaces of $\mathbb{R}^n$ of dimension k, l respectively, and write ζ_k, η_l for the canonical k- and l-plane bundles. Then we have:

Proposition 1.1. *For $0 \leqq k \leqq n$, the space $L_k(\mathbb{R}^n, \mathbb{R}^n)_+$ splits stably as a wedge of Thom spaces:*

$$\bigvee_{0 \leqq l \leqq n-k} G_{k,l}(\mathbb{R}^n)^{\Lambda^2 \eta_l \oplus (\zeta_k \oplus \eta_l)}.$$

In Section 4 we ask when the fibre bundle π is fibre homotopy trivial. Our answer is only partial, but includes:

Proposition 1.2. *For $0 < k < n$, the bundle π: $L_k(\mathbb{R}^n, \mathbb{R}^n) \to G_k(\mathbb{R}^n)$ is fibre homotopy trivial if and only if*

either (*a*) $(n, k) = (2, 1), (4, 3)$ *or* $(8, 7)$,

or (*b*) $k = 1$ *and* $n > 2$.

In case (*a*), *but not in case* (*b*), *the bundle is trivial as smooth fibre bundle.*

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2. The fibre bundle

To understand the structure of the bundle π, we write ζ (or sometimes ζ_k) for the canonical k-plane bundle over $G_k(V)$ naturally embedded as a sub-bundle $\zeta \subseteq V$ of the trivial bundle V. (It is often convenient to use the same notation for a vector space V and the trivial vector bundle $B \times V \to B$ over a space B.) Then we can identify $\pi: L_k(V, W) \to G_k(V)$ with the bundle $L_0(V/\zeta, W) \to G_k(V)$ whose fibre at a point $K \in G_k(V)$ is the space $L_0(V/K, W)$ of vector space monomorphisms $V/K \to W$. An element $f \in L_k(V, W)$ is identified with the induced quotient map: $V/\ker(f) \to W$ at $K = \ker(f)$.

Next we want to replace the bundle π by a homotopy equivalent bundle with compact fibres. To that end we now assume that the vector spaces V and W are equipped with (positive-definite) inner products. Write $O(V, W)$ for the Stiefel manifold of isometric linear maps $V \to W$. As is well known, $O(V, W)$ is homotopy equivalent to $L_0(V, W)$. More precisely, we have:

Proposition 2.1. *Write $\mathfrak{p}(V)$ for the vector space of self-adjoint operators $V \to V$. Then there is a natural diffeomorphism:*

$$O(V, W) \times \mathfrak{p}(V) \to L_0(V, W)$$

given by

$$(f, a) \mapsto f \circ \exp(a).$$

For $V = W$ this is the standard factorization of an invertible matrix as a product of an orthogonal and a positive-definite matrix. The general proof is the same. Notice that (2.1) gives a natural tubular neighbourhood of the submanifold $O(V, W)$ of $L(V, W)$ and natural stable trivialization $\tau O(V, W) \oplus \mathfrak{p}(V) = L(V, W)$ of the tangent bundle.

From now on we shall abbreviate $G_k(V)$ to B and identify V/ζ with the orthogonal complement $\zeta^\perp$ of ζ in V. Functors on vector spaces are extended to vector bundles in the usual way. Then the naturality of (2.1) gives a diffeomorphism over B:

$$L_0(\zeta^\perp, W) \to O(\zeta^\perp, W) \times_B \mathfrak{p}(\zeta^\perp). \tag{2.2}$$

We write:

$$\tilde{E} = L_0(\zeta^\perp, W), \quad E = O(\zeta^\perp, W).$$

Thus $\tilde{E} \to B$ is our original fibre bundle π, and this is fibre homotopy equivalent to the compact fibre bundle $E \to B$.

Remark 2.3. Under the diffeomorphisms (2.2) $O(\zeta^\perp, W)$ corresponds to the subspace of $L_k(V, W)$ consisting of elements f whose adjoint f^* satisfies: $f = ff^*f$.

Lemma 2.4. *The bundle $E \to B$ admits a cross-section if and only if* $\dim V \leqq \dim W$.

This is easy. We include the proof because it introduces notation required later. The

Grassmann manifold $G_1(V)$ will usually be written as the projective space $P(V)$ and ζ_1 as the Hopf bundle H. There are two embeddings:

$$P(\mathbb{R}^{n-k+1}) \xrightarrow{i} G_k(\mathbb{R}^n) \xleftarrow{j} P(\mathbb{R}^{k+1}) \tag{2.5}$$

(for $0<k<n$) given as follows. A 1-dimensional subspace L of $\mathbb{R}^{n-k+1}$ is mapped by i to $L\oplus\mathbb{R}^{k-1}\subseteq\mathbb{R}^n$, and a 1-dimensional subspace L of $\mathbb{R}^{k+1}$ by j to the orthogonal complement of $L\oplus\mathbb{R}^{n-k-1}$ in $\mathbb{R}^n$. Note that:

$$i^*\zeta=H\oplus\mathbb{R}^{k-1}, \quad j^*\zeta^\perp=H\oplus\mathbb{R}^{n-k-1}. \tag{2.6}$$

Proof of (2.4). Take $V=\mathbb{R}^n$ and suppose that $E\to B$ has a section, which we can regard as a bundle monomorphism: $\zeta^\perp\to W$. The pull-back via j is a monomorphism $H\oplus\mathbb{R}^{n-k-1}\to W$ over $P(\mathbb{R}^{k+1})$ with orthogonal complement σ, say. Since $w_k\sigma\neq 0$, $\dim W\geqq(n-k)+k=n$.

For the remainder of the paper we shall assume that $\dim V\leqq\dim W$ and write W as an orthogonal direct sum $U\oplus V$ with $\dim U=m$, $\dim V=n$. This splitting gives an obvious section of $E\to B$.

3. A stable splitting

We consider the stable homotopy type of the fibrewise one-point compactifications $\tilde{E}_B^+\to B$ and $E_B^+\to B$ obtained by adjoining a point at infinity to each fibre. (See, for example, [4].) Recall first that Miller in [5] established a stable splitting

$$O(V,U\oplus V)^+\simeq\bigvee_{0\leqq l\leqq n} G_l(V)^{\mathfrak{o}(\zeta_l)\oplus(U\otimes\zeta_l)}. \tag{3.1}$$

Here the superscript "+" again denotes one-point compactification, and $\mathfrak{o}(\zeta_l)$ is the vector bundle with fibre at a point $K\in G_l(V)$ the Lie algebra of the orthogonal group $O(K)$ of the vector space K. Miller's constructions can be performed equivariantly with respect to the action of the orthogonal groups $O(U)$ and $O(V)$, as in [1]. The result thus extends directly to a splitting theorem for a bundle of Stiefel manifolds by replacing U and V by vector bundles. We apply this to $E\to B$ to conclude:

Proposition 3.2. *There is a natural stable splitting over B:*

$$E_B^+\simeq\bigvee_{0\leqq l\leqq n-k}{}_B\; G_l(\zeta^\perp)_B^{\mathfrak{o}(\eta_l)\oplus(U\oplus\zeta)\otimes\eta_l}$$

as a wedge over B of Thom spaces over B.

For clarity we have here used η_l for the canonical l-plane bundle over the Grassmann bundle $G_l(\zeta^\perp)$. The concept of "Thom space over B" should be self-explanatory: the fibre

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over $K \in B$ is the Thom space

$$G_l(K^{\perp})^{\mathfrak{o}(\eta_l) \oplus (U \oplus K) \otimes \eta_l}.$$

By collapsing the section B at infinity in E_B^+ to a point we obtain $E_B^+/B = E^+$ and (3.2) gives a splitting of E^+. Now E^+ is homotopy equivalent to the space $L_k(V, U \oplus V)_+$ obtained by adjoining a disjoint basepoint to $\tilde{E}$ (rather than compactifying).

Corollary 3.3. *There is a natural stable splitting:*

$$L_k(V, U \oplus V)_+ \simeq \bigvee_{0 \leqq l \leqq n-k} G_{k,l}(V)^{\mathfrak{o}(\eta_l) \oplus (U \oplus \zeta_k) \otimes \eta_l}.$$

As in the special case (1.1), $G_{k,l}(V)$ is the space of pairs (K, L) of orthogonal subspaces of V of dimension k, l respectively. The case $k = 0$ is Miller's original theorem.

4. The question of triviality

In this section we ask when the bundle $\tilde{E} \to B$ is fibre homotopy trivial, or, more restrictively, trivial as smooth fibre bundle. Notice that, if $\tilde{E} \to B$ is fibre homotopy trivial, then so is $E \to B$, and $E_B^+ \to B$ is certainly stably fibre homotopy trivial. If $\tilde{E} \to B$ is trivial in the strong sense, then so is $\tilde{E}_B^+ \to B$, and both $E_B^+ \to B$ and $\tilde{E}_B^+ \to B$ are stably fibre homotopy trivial. It is stable triviality that we investigate first.

Proposition 4.1.

(i) *If $E_B^+ \to B$ is stably fibre homotopy trivial, then the sphere-bundle $(\mathfrak{o}(\zeta^{\perp}) \oplus (U \oplus \zeta) \otimes \zeta^{\perp})_B^+$ is stably fibre homotopy trivial.*

(ii) *If $\tilde{E}_B^+ \to B$ is stably fibre homotopy trivial, then so is $((U \oplus V) \otimes \zeta^{\perp})_B^+$.*

Proof of (i). Write F for a fibre of $E \to B$; it is a Stiefel manifold of dimension N, say. Since F is connected and admits a framing, there is a stable map $e: S^N \to F^+$ which induces an isomorphism of integral homology groups in dimension N. Writing ξ for the vector bundle $\mathfrak{o}(\zeta^{\perp}) \oplus (U \oplus \zeta) \otimes \zeta^{\perp}$ over B, let $p: E_B^+ \to \xi_B^+$ be the projection onto the top factor, $l = n - k$, in the decomposition (3.2).

Now suppose that we have a stable trivialization $t: B \times F^+ \to E_B^+$ over B. Then $p \circ t \circ (1 \times e): B \times S^N \to \xi_B^+$ is a (stable) fibre homotopy equivalence, by Dold's lemma, since it induces a homology isomorphism in each fibre.

This completes the proof of (i). The proof of (ii) is similar, using the equivalence: $\tilde{E}_B^+ \simeq E_B^+ \wedge_B (\mathfrak{p}(\zeta^{\perp}))_B^+$ given by (2.2). Observe that $(\mathfrak{o}(\zeta^{\perp}) \oplus (U \oplus \zeta) \oplus \zeta^{\perp}) \oplus \mathfrak{p}(\zeta^{\perp})$ is isomorphic to $(U \oplus V) \oplus \zeta^{\perp}$.

Remark 4.2. It is, of course, unnecessary to use the strength of (3.2) to obtain the projection p "onto the top cell" of E. We simply take a tubular neighbourhood (over B) of the standard section $B \to E$. The normal bundle is ξ, and the Pontrjagin–Thom construction gives the required projection p.

Recall that $\tilde{E}\to B$ is the bundle $L_k(V, U\oplus V)\to G_k(V)$ and that $\dim U=m$, $\dim V=n$. We shall discuss the triviality problem under three headings: (a) $1<k<n-1$, (b) $k=n-1$, (c) $k=1$. Standard facts about vector and sphere-bundles over real projective space will be used without comment; they can be found in texts such as [2], [3].

(a) $1<k<n-1$

Proposition 4.3. *If* $1<k<n-1$, *then* $\tilde{E}\to B$ *is not fibre homotopy trivial.*

This is an easy corollary of (4.1)(i). Indeed, we shall show that if $E_B^+\to B$ is stably fibre homotopy trivial, $0<k<n$, then

$$-m-k+1\equiv 0 \pmod{a(n-k+1)}$$
$$m+k+1\equiv 0 \pmod{a(k+1)}, \tag{4.4}$$

where $a(r)$ is the Hurwitz–Radon number, the order of $[H]-1$ in $KO^0(P(\mathbb{R}^r))$. The proposition then follows; for, if $k>1$ and $n-k>1$, then both $a(k+1)$ and $a(n-k+1)$ are divisible by 4.

Proof of (4.4). As above set $\xi=\mathfrak{o}(\zeta^\perp)\oplus(\mathbb{R}^m\oplus\zeta)\otimes\zeta^\perp$. Then, by (4.1)(i), ξ_B^+ is stably fibre homotopy trivial. The congruences (4.4) are just the conditions that the restriction of ξ_B^+ to each of the subspaces $P(\mathbb{R}^{n-k+1})$ and $P(\mathbb{R}^{k+1})$ as in (2.5) is stably fibre homotopy trivial.

We give the details in the first case; the second is similar. It is convenient to think of $\mathfrak{o}(\zeta^\perp)$ as the exterior square $\lambda^2(\zeta^\perp)$. Then we have, by (2.6),

$$[i^*\zeta]=\lambda^2(n-k+1-[H])+(m+k-1+[H])(n-k+1-[H])$$

in $KO^0(P(\mathbb{R}^{n-k+1}))$. Using the identity $\lambda^2(x+y)=\lambda^2x+xy+\lambda^2y$ we obtain:

$$[i^*\xi]-\dim\xi=(-m-k+1)([H]-1).$$

So the sphere-bundle associated to $i^*\xi$ is stably fibre homotopy trivial if and only if $a(n-k+1)$ divides $-m-k+1$.

(b) $k=n-1$

In this case, by taking orthogonal complements we can identify $B=G_{n-1}(V)$ with $G_1(V)=P(V)$ and $\zeta^\perp$ with the Hopf line bundle H. The bundles $\tilde{E}\to B$ and $E\to B$ become $L_0(H, U\oplus V)\to P(V)$ and $O(H, U\oplus V)\to P(V)$, or, equivalently, the complement of the zero-section in the vector bundle $L(H, U\oplus V)\to P(V)$ and the unit sphere bundle. If the sphere-bundle is stably fibre homotopy trivial then $a(n)$ must divide $m+n$. Conversely, if $m+n\equiv 0 \pmod{a(n)}$, then the vector bundle $(m+n)H$ is trivial.

Proposition 4.5. *If* $1<k=n-1$, *then* $\tilde{E}\to B$ *is fibre homotopy trivial if and only if* $m+n\equiv 0 \pmod{a(n)}$. *When this condition holds, both* $\tilde{E}\to B$ *and* $E\to B$ *are trivial as smooth bundles.*

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(c) $k=1$

The final case is the most difficult. We are considering the bundles $L_0(H^\perp, U\oplus V)\to P(V)$ and $O(H^\perp, U\oplus V)\to P(V)$, where $H^\perp$ is the orthogonal complement of H in V.

Proposition 4.6. *For $k=1<n$, $\tilde{E}\to B$ is trivial as bundle if and only if $n=2$, 4 or 8 and $m\equiv 0 \pmod{a(n)}$. In these cases $E\to B$ is also trivial.*

Proof. We first establish the necessity of the condition. If $\tilde{E}\to B$ is trivial, then it is certainly fibre homotopy trivial and (4.4) gives the restriction: $m\equiv 0 \pmod{a(n)}$. But now both clauses (i) and (ii) of (4.1) apply, so that $\mathfrak{p}(H^\perp)_B^+$ must be stably fibre homotopy trivial, that is: $n\equiv 0 \pmod{a(n)}$.

For the converse, observe that $L_0(H^\perp, U\oplus V)$ is naturally identified with $L_0(H^\perp\otimes H, (U\oplus V)\otimes H)$ by taking the tensor product with the identity on the line bundle H. But $H^\perp\otimes H$ is the tangent bundle of $P(V)$ and this is trivial if $n=2$, 4 or 8. If $m+n\equiv 0 \pmod{a(n)}$, then $(U\oplus V)\otimes H$ is trivial. This establishes the triviality of $\tilde{E}\to B$, and the same argument shows that $E\to B$ is also trivial.

We complete the proof of (1.2) by verifying:

Proposition 4.7. *If $k=1$ and $m=0$, then $E\to B$ is trivial as bundle.*

Proof. We can give an explicit trivialization: $P(V)\times SO(V)\to O(H^\perp, V)$ of the bundle $E\to B$ by mapping (K,g), where K is a 1-dimensional subspace of V and g is an element of the special orthogonal group of V, to the composition $K^\perp\subseteq V\to V$ of g with the inclusion.

It remains to examine the question of the triviality or fibre homotopy triviality of $E\to B$ for $k=1$ and $m>0$. We have been unable to give an answer even in the first interesting case $n=3$. Our present knowledge is collected in the final proposition. Part (i) is based on a suggestion of D. Hacon.

Proposition 4.8. *For $k=1$, $m>0$, $n=3$:*

(i) *the bundle $E\to B$ is trivial if $m=4$;*

(ii) *the bundle $E\to B$ is not fibre homotopy trivial if $m+4$ is not a power of 2;*

(iii) *the bundle $E_B^+\to B$ is stably fibre homotopy trivial if and only if $m\equiv 0 \pmod 4$.*

Proof of (i). We take U to be the space $\mathbb{H}$ of quaternions, V the space of pure quaternions, and regard $U\oplus V$ (or $V\oplus U$) as the space of pure Cayley numbers. Then the group G_2 of automorphisms of the Cayley numbers acts orthogonally on $U\oplus V$. The action is transitive on the Stiefel manifold of orthonormal 2-frames in $U\oplus V$, and the stabilizer of any 2-frame in V is the subgroup $Sp(1)$ which fixes the whole of V.

We obtain an explicit trivialization: $P(V)\times G_2/Sp(1)\to O(H^\perp, U\oplus V)$, along the lines of (4.7), by sending $(K, gSp(1))$ to the composition $K^\perp\subseteq V\subseteq U\oplus V\to U\oplus V$ of g and the inclusion.

Proof of (ii). Suppose that the bundle $O(H^{\perp}, \mathbb{R}^{m+3}) \to P(\mathbb{R}^3)$ is fibre homotopy trivial. Then so is its restriction to the subspace $P(\mathbb{R}^2)$. This restricted bundle can be identified with $O(\mathbb{R} \oplus H, \mathbb{R}^{m+3}) \to P(\mathbb{R}^2)$, because $2H$ is trivial, or with the mapping torus of the involution T of the Stiefel manifold of 2-frames in $\mathbb{R}^{m+3}$ which changes the sign of the second vector. Since the bundle is fibre homotopy trivial, T must be homotopic to the identity. The condition on m now follows from a theorem of James ([**3**, (23.10)]. (The same theorem implies that $E \to B$ is never fibre homotopy trivial in the cases $k=1$, $m>0$, $n=5$ or 9.)

Proof of (iii). We show that each of the components in the decomposition (3.2) is stably trivial if $4|m$. Only the middle term $P(H^{\perp})_B^{(U \oplus H) \otimes \eta}$, where η is the Hopf line bundle over the projective bundle $P(H^{\perp})$, causes difficulty. Observe first that $P(H^{\perp})$ can be embedded in the trivial bundle $B \times P(V)$ by inclusion of $H^{\perp}$ in V; the bundle $U \otimes \eta$ over $P(H^{\perp})$ is the restriction of the bundle $U \otimes H$ over $B \times P(V)$, and this is trivial because m is divisible by $a(n)$. So we must show that $P(H^{\perp})_B^{H \otimes \eta}$ is stably trivial.

Now the cofibre sequence over B:

$$P(H)_B^+ \to P(H^{\perp} \oplus H)_B^+ \to P(H^{\perp})_B^{H \otimes \eta},$$

given by the inclusion of H in $H^{\perp} \oplus H$, is split by the projection $P(H^{\perp} \oplus H) \to B = P(H)$. But the bundle $P(H^{\perp} \oplus H)_B^+$ is trivial, since $H^{\perp} \oplus H = V$. By splitting the trivial bundle we obtain a stable equivalence:

$$P(H^{\perp})_B^{H \otimes \eta} \vee_B (B \times S^0) \underset{B}{\simeq} (B \times P(\mathbb{R}^{n-1})^H) \vee_B (B \times S^0).$$

Inclusion and projection gives a stable map $P(H^{\perp})_B^{H \otimes \eta} \to B \times P(\mathbb{R}^{n-1})^H$ which is an equivalence on fibres and hence a stable fibre homotopy equivalence. (Alternatively, decompose both sides of the equivalence:

$$O(H^{\perp}, \mathbb{R}^3) \simeq B \times O(\mathbb{R}^2, \mathbb{R}^3), \text{ (4.7).)}$$

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