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ON SPECTRAL DECOMPOSITION OF IMMERSIONS OF FINITE TYPE

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Let $x : M \to E^m$ be an immersion of finite type. In this paper we study the following two problems: (1) When is the spectral decomposition of the immersion x linearly independent? (2) When is the spectral decomposition orthogonal? Several results in this respect were obtained.

1. INTRODUCTION

A submanifold M of a Euclidean m-space E^m is said to be *of finite type* [**1, 2**] if each component of its position vector field x can be written as a finite sum of eigenfunctions of the Laplacian Δ of M (with respect to the induced metric), that is, if

$$x = c + x_1 + x_2 + \cdots + x_k \tag{1.1}$$

where c is a constant vector, $x_1, x_2, \ldots, x_k$ are non-constant maps satisfying $\Delta x_i = \ell_i x_i$, $i = 1, \ldots, k$. If in particular all eigenvalues $\{\ell_1, \ell_2, \cdots, \ell_k\}$ are mutually different, then M is said to be of *k-type*. If we define a polynomial P by

$$P(t) = \prod_{i=1}^{k} (t - \ell_i), \tag{1.2}$$

then $P(\Delta)(x - c) = 0$. Conversely, if M is compact and if there exists a constant vector c and a nontrivial polynomial P such that $P(\Delta)(x - c) = 0$, then M is of finite type [**1**, pp.255-258]. If M is not compact, then the existence of a nontrivial polynomial P such that $P(\Delta)(x - c) = 0$ does not imply that M is of finite type in general. However,

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we remark in Section 4 that the existence of such a polynomial P guarantees that M is of finite type when either M is of 1-dimensional or the polynomial P has exactly k distinct roots where $k = \deg P$.

The class of 1-type submanifolds M in E^m has been classified by Takahashi [12]. In fact, he showed that the submanifolds M in E^m for which

$$(1.3) \qquad \Delta x = \ell x$$

are precisely either the minimal submanifolds of $E^m(\ell = 0)$ or the minimal submanifolds of hyperspheres S^{m-1} in E^m (the case when $\ell \neq 0$, actually $\ell > 0$).

As a generalisation of Takahashi's result, Garay [9, 10] studied the hypersurfaces M^n in E^{n+1} for which

$$(1.5) \qquad \Delta x = Ax,$$

where A is a diagonal matrix

$$(1.5)' \qquad A = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_{n+1} \end{pmatrix}, \quad \mu_i \in \mathbb{R}, \ i = 1, 2, \cdots, n+1.$$

In [5], Dillen, Pas and Verstraelen observed that Garay's condition is not coordinate-invariant and they considered the submanifolds in E^m for which

$$(1.6) \qquad \Delta x = Ax + B$$

where $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$. This setting generalises T. Takahashi's condition, following Garay's idea, in a way which is independent of the choice of coordinates. In [10], Garay proved that if a hypersurface M in E^{n+1} satisfies his condition, it is either minimal in E^{n+1} or it is a hypersphere or it is a spherical cylinder (see, also [11]). In [8], Dillen, Pas and Verstraelen proved that a surface in E^3 satisfies their condition if and only if it is an open part of a minimal surface, a sphere or a circular cylinder.

In the first part of this article we obtain precise relations between the spectral decomposition (1.1) of an immersion $x : M \to E^m$ and condition (1.5) of Garay and condition (1.6) of Dillen, Pas and Verstraelen. Some applications will be given in this respect. In the second part we obtain a complete classification of hypersurfaces in E^{n+1} satisfies condition (1.6) which generalises the main results of [8, 9, 10].

2. Spectral Decompositions

In the following, for simplicity, we assume that the eigenvalues $\{\ell_1, \dots, \ell_k\}$ associated with the spectral decomposition (1.1) *are mutually distinct*. For each ℓ_i we put $V(\ell_i) = \{f \in C^\infty(M) \mid \Delta f = \ell_i f_i\}$.

Lemma 2.1. *Let $x : M \to E^m$ be an immersion of finite type. Then for any $i \in \{1, \dots, k\}$ there exist linearly independent vectors $c_{ij} \in E^m$ and linearly independent functions $f_{ij} \in V(\ell_i)$, $j = 1, \dots, m_i$ such that*

$$x_i = \sum_{j=1}^{m_i} f_{ij} c_{ij}, \qquad i = 1, 2, \dots, k. \tag{2.1}$$

Proof: Since $\Delta x_i = \ell_i x_i$, there exist vectors a_{ij} $(j = 1, \dots, n_i)$ in E^m and functions φ_{ij} $(j = 1, \dots, n_i)$ in $V(\ell_i)$ such that

$$x_i = \sum_{j=1}^{n_i} a_{ij} \varphi_{ij}. \tag{2.2}$$

Let $E_i = \operatorname{Span}\{a_{i1}, \dots, a_{in_i}\}$ and $c_{i1}, \dots, c_{im_i}$ a basis of E_i. Since $V(\ell_i)$ is a vector space, (2.2) implies

$$x_i = \sum_{j=1}^{m_i} c_{ij} f_{ij} \tag{2.3}$$

for some functions $f_{ij} \in V(\ell_i)$, $j = 1, \dots, m_j$. We claim that $f_{i1}, \dots, f_{im_i}$ are linearly independent functions in $V(\ell_i)$. This can be easily seen as follows. In fact, if not, then one of $f_{i1}, \dots, f_{im_i}$ is a linear combination of the others. Without loss of generality, we may assume that

$$f_{i1} = \sum_{j=2}^{m_i} b_j f_{ij}, \qquad b_j \in \mathbb{R}. \tag{2.4}$$

Then we have

$$x_i = \sum_{j=2}^{m_i} (c_{ij} + b_j c_{i1}) f_{ij}, \tag{2.5}$$

which implies that $\dim E_i < m_i$. This is a contradiction. □

We need the following.

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DEFINITION 2.1: Let $x : M \to E^m$ be an immersion of k-type. Assume that the spectral decomposition of x is given by (1.1). Then the immersion x is said to be *linearly independent* [3] if the set $\{c_{ij} \mid i = 1, \dots, k; j = 1, \dots, m_i\}$ is linearly independent, where c_{ij} are given by Lemma 2.1. The immersion x is said to be *orthogonal* [3] if the subspaces $E_1, \dots, E_k$ are mutually orthogonal, where $E_i = \text{Span}\{c_{i1}, \dots, c_{im_i}\}$, $i = 1, \dots, k$.

THEOREM 2.2. *Let $x : M \to E^m$ be an immersion of finite type. Then the immersion of x is linearly independent if and only if x satisfies $\Delta x = Ax + B$ for some $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$.*

PROOF: Let $x : M \to E^m$ be an immersion of finite type. Without loss of generality, we may assume x to be full.

($\Longleftarrow$) Assume that x satisfies Dillen-Pas-Verstraelen's condition, that is, there exist $A \in \mathbb{R}^{m\times m}$ and $B \in R^m$ such that $\Delta x = Ax + B$. Then, by (1.1) and $\Delta x_i = \ell_i x_i$, we obtain

$$Ac + B + (Ax_1 - \ell_1 x_1) + \cdots + (Ax_k - \ell_k x_k) = 0. \tag{2.6}$$

Since $\Delta(Ax_i) = A(\Delta x_i) = \ell_i Ax_i$, (2.6) implies

$$\ell_1^j(Ax_1 - \ell_1 x_1) + \cdots + \ell_k^j(Ax_k - \ell_k x_k) = 0, \quad j = 1, 2, \cdots. \tag{2.7}$$

Because $\ell_1, \dots, \ell_k$ are assumed to be mutually distinct, (2.7) yields

$$Ax_i = \ell_i x_i, \qquad i = 1, 2, \dots, k. \tag{2.8}$$

Combining (2.1) and (2.8) we get

$$\sum_{j=1}^{m_i} (Ac_{ij} - \ell_i c_{ij}) f_{ij} = 0, \qquad i = 1, 2, \dots, k. \tag{2.9}$$

From the linear independence of $f_{i1}, \dots, f_{im_i}$ (see Lemma 2.1) we obtain $Ac_{ij} = \ell_i c_{ij}$. Since eigenvectors belonging to distinct eigenspaces of A are independent, the immersion is linearly independent.

($\Longrightarrow$) Assume that the immersion x is linearly independent. For each x_i, let x_i be expressed as (see Lemma 2.1)

$$x_i = \sum_{j=1}^{m_i} f_{ij} c_{ij}, \tag{2.10}$$

where $c_{i1}, \dots, c_{im_i}$ are independent vectors in E^m and $f_{i1}, \dots, f_{im_i}$ are independent functions in $V(\ell_i)$. By definition, $\{c_{ij} \mid i = 1, \dots, k;\ j = 1, \dots, m_i\}$ are linearly independent. We put

$$S = \left(c_{11}, \dots, c_{1m_1}, \dots, c_{k1}, \dots, c_{km_k}\right). \tag{2.11}$$

Since the immersion x is assumed to be full, we have $\sum m_i = m$ and the matrix S is nonsingular. Let D be the diagonal $m \times m$ matrix given by

$$D = \operatorname{diag}(\ell_1, \dots, \ell_1, \dots, \ell_k, \dots, \ell_k), \tag{2.12}$$

where ℓ_i repeats m_i times. We put $A = SDS^{-1}$ and $B = -Ac$. Then by direct computation we obtain $\Delta x = Ax + B$. ∎

Theorem 2.3. *Let $x : M \to E^m$ be an immersion of finite type. Then the immersion x is orthogonal if and only if $\Delta x = Ax + B$ for some symmetric matrix $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$.*

Proof: Without loss of generality we may assume x being full. Assume that there exist a symmetric matrix $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$ such that $\Delta x = Ax + B$. Let c_{ij}, $i = 1, \dots, k$, $j = 1, \dots, m_i$ be the vectors given in Lemma 2.1. Then as in the proof of Theorem 2.2, we have $Ac_{ij} = \ell_i c_{ij}$. Since A is symmetric, distinct eigenspaces of A are mutually orthogonal. Thus the immersion x is orthogonal.

Conversely, if the immersion x is orthogonal, one may choose a Euclidean coordinate system such that $x_1 \in \operatorname{Span}\{\varepsilon_1, \dots, \varepsilon_{m_1}\}, \cdots, x_k \in \operatorname{Span}\{\varepsilon_{m-m_k+1}, \dots, \varepsilon_m\}$, where $\{\varepsilon_1, \dots, \varepsilon_m\}$ is the canonical orthonormal basis of E^m. It is easy to see that with respect to this coordinate system, $\Delta x = Dx - Dc$, where D is the diagonal matrix given by (2.12). Thus, with respect to the original coordinate system, we have $\Delta x = Ax + B$, for some symmetric matrix $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$. ∎

Remark 2.4: It is easy to see that if $\Delta x = Ax + B$ for some *symmetric* matrix $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$, then, with respect to a suitable coordinate system of E^m, it satisfies Garay's condition ((1.5) together with (1.5)$'$.)

From Theorem 2.2 we obtain easy the following

Corollary 2.5. *Every k-type curve C which lies fully in E^{2k} satisfies Dillen-Pas-Verstraelen's condition (1.6).*

Proof: This corollary follows from Theorem 2.2 and that the fact that each eigenspace of Δ of C is of dimension $\leqslant 2$. So, if a k-type curve C lies fully in E^{2k}, then the immersion is linearly independent. ∎

From Theorem 2.3 we obtain the following new characterisation of W-curves.

COROLLARY 2.6. *Let C be a curve in E^m. Then C is a W-curve if and only if the immersion of C in E^m satisfies $\Delta x = Ax + B$ for some symmetric matrix $A \in \mathbb{R}^{m\times m}$ and $B \in E^m$.*

PROOF: Let $x : C \to E^m$ be an immersion of a curve C into E^m. If $\Delta x = Ax+B$ for some symmetric matrix $A \in \mathbb{R}^{m\times m}$ and $B \in E^m$, then $\Delta H = AH$. So, if P denotes the characteristic polynomial of A, then by the Cayley-Hamilton theorem $P(A) = 0$ and thus $P(\Delta)H = 0$. Therefore, by applying Proposition 4.1, C is of finite type. Thus, we may apply Theorem 2.3 to conclude that the spectral decomposition

$$x = c + x_1 + \cdots + x_k \tag{2.13}$$

is orthogonal. Thus x can be expressed as the following form:

$$x = c + \sum_{i=1}^{k} (a_i \cos \ell_i s + b_i \sin \ell_i s), \tag{2.14}$$

where $a_i, b_i \in E^m$ and $\ell_1, \dots, \ell_k$ are mutually distinct non-negative real numbers. Since the spectral decomposition (2.13) is orthogonal, $E_i = \operatorname{Span}\{a_i, b_i\}$, $i = 1, \dots, k$ are mutually orthogonal. Thus, by using the condition $\langle x'(s), x'(s)\rangle = 1$, we may conclude that either $|a_i| = |b_i|$ and $a_i \perp b_i$ or $\ell_i = 0$, for each i. Therefore C is a W-curve in E^m. The converse is trivial. □

REMARK 2.7: Combining Corollaries 2.5 and 2.6 and results of [4, 5] we may conclude that there exist infinitely many finite type curves and submanifolds in E^m which satisfy the condition $\Delta x = Ax + B$ for some matrix $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$, but there exist no symmetric matrix $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^m$ with $\Delta x = Ax + B$.

3. CLASSIFICATION OF HYPERSURFACES

In this section we prove the following classification theorem which generalises the main results of [**8**, **9**, **10**].

THEOREM 3.1. *A hypersurface M in E^{n+1} satisfies $\Delta x = Ax + B$ for some $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $B \in \mathbb{R}^{n+1}$ if and only if it is an open portion of a minimal hypersurface, a hypersphere S^n or a spherical cylinder $S^\ell \times E^{n-\ell}$, $\ell \in \{1, 2, \dots, n-1\}$.*

PROOF: It is easy to see that if M is one of the hypersurfaces mentioned in Theorem 3.1, then there exist $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $B \in \mathbb{R}^{n+1}$ such that $\Delta x = Ax + B$.

Conversely, assume that $\Delta x = Ax + B$ for some $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $B \in \mathbb{R}^{n+1}$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and E^{n+1}, respectively. Let H, h and D be the mean-curvature vector, the second fundamental form and the normal

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connection of M in E^{n+1}, respectively. By taking covariant derivative of both sides of $\Delta x = Ax + B$ and applying the formula of Weingarten, we may obtain

$$(3.1)\qquad A_H X = \frac{1}{n}(AX)^T \quad \text{and} \quad D_X H = -\frac{1}{n}(AX)^N,$$

where A_H is the Weingarten map at H, $(AX)^T$ and $(AX)^N$ the tangential and the normal components of AX, respectively. Thus, for any vector fields, X, Y tangent to M, we find

$$(3.2)\qquad n\langle A_H X, Y\rangle = \langle AX, Y\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product of E^{n+1}. By taking covariant derivative of both sides of (3.2) and applying the formulas of Gauss and Weingarten and (2.2), we obtain

$$(3.3)\qquad n\langle(\nabla_Z A_H)X, Y\rangle = \langle Ah(Z,X), Y\rangle + \langle h(Y,Z), AX\rangle$$

for X, Y, Z tangent to M. Thus, by combining (3.3) and the equation of Codazzi, we get

$$(3.4)\qquad \{n(X\alpha) + \langle AX, \xi\rangle\}SZ = \{n(Z\alpha) + \langle AZ, \xi\rangle\}SX,$$

where ξ is a unit normal vector of M in E^{n+1} and $S = A_\xi$.

If M is minimal, there is nothing to prove. So, we may assume that M is not minimal in E^{n+1}. Put $W = \{p \in M \mid H(p) \neq 0\}$. Then W is a nonempty open subset of M. Let $e_1, \ldots, e_n$ be an orthonormal frame field tangent to W which diagonalises A_H. Then we have $Se_i = \kappa_i e_i, i = 1, \ldots, n$. Thus from (3.4) we find

$$(3.5)\qquad ne_j\alpha = -\langle Ae_j, \xi\rangle, \qquad j = 1, \ldots, n.$$

By using (3.5) and some computations, we may prove that the mean curvature α is constant. Hence, (3.5) yields

$$(3.6)\qquad \langle h(e_i, e_i), Ae_j\rangle = 0, \qquad i \neq j.$$

Let $W_2 = \{p \in W \mid \text{rank } A_H \geqslant 2\}$. Then W_2 is open. From (3.6) we get

$$(3.7)\qquad (Ae_1)^N = \cdots = (Ae_n)^N = 0, \qquad \text{on} \quad W_2,$$

$$(3.8)\qquad AX = nA_H X, \qquad \text{for} \quad X \in TW_2.$$

Case 1. $W_2 \neq \phi$. In this case, by taking exterior derivative of (3.8) with respect to a tangent vector Y, we find

$$(3.9)\qquad n(\nabla_Y A_H)X = Ah(X,Y) - nh(Y, A_H X), \qquad X, Y \in TW_2.$$

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Thus, by taking the scalar product of (3.9) with H, we obtain

$$\langle Ah(X,Y), H\rangle = n\langle A_H X, A_H Y\rangle, \qquad X,Y \in TW_2, \tag{3.10}$$

$$\langle A_H X, Y\rangle\langle A\xi,\xi\rangle = n\langle A_H^2 X, Y\rangle \tag{3.11}$$

which implies $(n\kappa_i - \langle A\xi,\xi\rangle)\kappa_i = 0$ for any eigenvalue κ_i of A_H. Therefore, either A_H is proportional to the identity map or A_H has exactly two distinct eigenvalues 0 and $\kappa(\neq 0)$. If the first case occurs, each connected component of W_2 is an open part of a hypersphere S^n of E^{n+1}. Thus, by continuity of A_H, the whole hypersurface M is an open portion of S^n. If the second case occurs, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the eigenspaces of A_H with eigenvalues of 0 and κ, respectively. From (3.9) we find

$$(\nabla_U A_H)X = 0, \qquad \text{for} \quad X \in \mathcal{D}_i, \quad U \in \mathcal{D}_j, \quad i \neq j, \tag{3.12}$$

and

$$(\nabla_Y A_H)X = 0, \qquad \text{for} \quad X,Y \in \mathcal{D}_i \quad \text{with} \quad X \perp Y. \tag{3.13}$$

Since the multiplicity of κ is $\geqslant 2$ on W_2, (3.13) implies that κ is a nonzero constant on the nonempty open subset of W_2 where the multiplicity of κ is maximal. So, by continuity, $W_2 = M$ and $\mathcal{D}_1$ and $\mathcal{D}_2$ define two distributions on M. Moreover, by using (3.12) and (3.13), we may also prove that both distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable and their maximal integrable submanifolds are totally geodesic in M. Therefore, locally, M is the Riemannian product of two Riemannian manifolds M_1 and M_2, where M_1 and M_2 are maximal integrable submanifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Since $h(X,U) = 0$ for $X \in \mathcal{D}_1$ and $U \in \mathcal{D}_2$, a lemma of Moore implies that M is locally the product of two Euclidean submanifolds. Since M is a hypersurface of E^{n+1}, we further see that M is open portion of a spherical cylinder $E^{n-\ell} \times S^\ell$, $2 \leqslant \ell < n$.

CASE 2. $W_2 = \phi$. Since M is not minimal, $W \neq \phi$. So, there exist orthonormal frame fields $e_1, \dots, e_n$ on W such that

$$A_H = \begin{pmatrix} \mu & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad \mu \neq 0 \tag{3.14}$$

with respect to $e_1, \dots, e_n$. From (3.3) and (3.6) we have

$$Ae_2 = \cdots = Ae_n = 0, \tag{3.15}$$

$$(\nabla_X A_H)e_1 = (\nabla_{e_1} A_H)X = (\nabla_X A_H)Y \tag{3.16}$$

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for $X, Y \in \mathcal{D}_2$, where $\mathcal{D}_1 = \mathrm{Span}\{e_1\}$ and $\mathcal{D}_2 = \mathrm{Span}\{e_2, \dots, e_n\}$. By applying (3.16) we may prove that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable and their maximal integrable submanifolds are totally geodesic. Therefore, by applying deRham's decomposition theorem, a lemma of Moore and (3.14), we conclude that locally W is the product of a plane curve and a linear $(n-1)$-subspace, say $C \times E^{n-1} \subset E^2 \times E^{n-1}$. Since the coordinates of $C \times E^{n-1}$ can be expressed as $x(s, u_2, \dots, u_n) = (f(s), y(s), u_2, \dots, u_n)$, where s is the arc length of C, the condition $\Delta x = Ax + B$ implies the immersion $y : C \to E^2$ satisfies $\Delta y = Ey + D$ for some $E \in \mathbb{R}^{2\times 2}$ and $D \in \mathbb{R}^2$. So the mean curvature vector H_y of y satisfies $\Delta H_y = EH_y$. Let P denote the characteristic polynomial of E. Then $P(\Delta)H_y = P(E)H_y = 0$ by Cayley-Hamilton's theorem. Thus, by applying Proposition 4.1, C is a finite type curve in E^2. Therefore, by applying Theorem 3 of [6], C is an open portion of a circle. So, by continuity, M is an open portion of a circular cylinder $S^1 \times E^{n-1}$ in E^{n+1}. ∎

4. Some Further Results and Remarks

As we mentioned in the Introduction, for a compact submanifold M in E^m, the existence of a nontrivial polynomial P such that $P(\Delta)(x - c) = 0$ for some $c \in E^m$ (or $P(\Delta)H = 0$) guarantees M being of finite type. In this section, we would like to point out that the same result holds for some important cases for *noncompact* submanifolds, too.

Proposition 4.1. *Let C be a curve in E^m parametrised by arclength s. If there is a nontrivial polynomial P of one variable such that $P(\Delta)H = 0$, then C is of finite type.*

Proof: If there is a nontrivial polynomial (over $\mathbb{R}$) such that $P(\Delta)H = 0$, then the immersion $x = x(s)$ satisfies an ordinary differential equation with constant coefficients. Thus, $x(s)$ takes the following form:

$$x(s) = \sum_i e^{\mu_i s} \left\{ \sum_t \left(a_0^{i_t} + \dots + a_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}}\right) \cos(\ell_{i_t} s) + \sum_t \left(b_0^{i_t} + \dots + b_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}}\right) \sin(\ell_{i_t} s) \right\} \tag{4.1}$$

where $\mu_i, \ell_{i_t} \in \mathbb{R}$ and $a_j^\ell, b_j^\ell \in E^m$. Thus, we have

$$x'(s) = \sum_i e^{\mu_i s} \left\{ \sum_t (B_{i_t} \cos(\ell_{i_t} s) + C_{i_t} \sin(\ell_{i_t} s)) \right\}, \tag{4.2}$$

where

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$$B_{i_t} = \mu_i\Big(a_0^{i_t} + \cdots + a_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}}\Big) + \Big(a_1^{i_t} + \cdots + \ell_{i_t} a_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}-1}\Big) + \ell_{i_t}\Big(b_0^{i_t} + \cdots + b_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}}\Big), \tag{4.3}$$

$$C_{i_t} = \mu_i\Big(b_0^{i_t} + \cdots + b_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}}\Big) + \Big(b_1^{i_t} + \cdots + \ell_{i_t} b_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}-1}\Big) - \ell_{i_t}\Big(a_0^{i_t} + \cdots + a_{\ell_{i_t}}^{i_t} s^{\ell_{i_t}}\Big). \tag{4.4}$$

Let $\mu_N = \max_i\{\mu_i\}$ and $V_N = \{j \mid \mu_j = \mu_N\}$. If $\mu_N > 0$, then, by using (4.2) and $\langle x'(s), x'(s)\rangle = 1$, we may obtain

$$\sum_{t\in V_N} \left(B_{N_t} \cos(\ell_{N_t} s) + C_{N_t} \sin(\ell_{N_t} s)\right) = 0. \tag{4.5}$$

So, by the independence of $\cos(\ell_{N_t} s)$, $\sin(\ell_{N_t} s)$, $t \in V_N$, we conclude $B_{N_t} = C_{N_t} = 0$ for $t \in V_N$. From (4.3) and (4.4), we conclude that

$$a_0^{N_t} = \cdots = a_{\ell_{N_t}}^{N_t} = b_0^{N_t} = \cdots = b_{\ell_{N_t}}^{N_t} = 0$$

which yields a contradiction. Therefore, $\mu_N \leqslant 0$, that is, $\mu_i \leqslant 0$ for any i. Similarly, we may prove that $\mu_i \geqslant 0$ for any i. Consequently, x takes the form:

$$x(s) = \sum_i \{(a_0^i + \cdots + a_{\ell_i}^i s^{\ell_i}) \cos(\ell_i s) + (b_0^i + \cdots + b_{\ell_i}^i s^{\ell_i}) \sin(\ell_i s)\}. \tag{4.6}$$

Let $k = \max_i\{\ell_i\}$. Then we may rewrite (4.6) as follows.

$$x(s) = \sum_i \left(a_0^i \cos(\ell_i s) + b_0^i \sin(\ell_i s)\right) + \sum_i \left(a_1^i \cos(\ell_i s) + b_1^i \sin(\ell_i s)\right)s + \cdots + \sum_i \left(a_k^i \cos(\ell_i s) + b_k^i \sin(\ell_i s)\right)s^k, \tag{4.7}$$

where the coefficient of s^k is nonzero. If $k > 0$, then by comparing the coefficients of s^{2k} from the equation $\langle x'(s), x'(s)\rangle = 1$, we obtain

$$\sum_i \ell_i\left(-a_k^i \sin(\ell_i s) + b_k^i \cos(\ell_i s)\right) = 0,$$

which implies either $\ell_i = 0$ or $a_k^i = b_k^i = 0$. From this we conclude that $x(s)$ takes the following form:

$$x(s) = a_0 + \cdots + a_k s^k + \sum_i \left(b_i \cos(\ell_i s) + c_i \sin(\ell_i s)\right). \tag{4.8}$$

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However, by applying the condition $\langle x'(s), x'(s)\rangle = 1$ again, we have $k \leqslant 1$ for (4.8). ∎

Proposition 4.2. *Let $x : M \to E^m$ be an immersion. If there exists a polynomial P such that $P(\Delta)H = 0$, then either M is of infinitely type or is of k-type with $k \leqslant \deg P$.*

Proof: Let $P = t^d + c_1 t^{d-1} + \cdots + c_n$ be a polynomial such that $P(\Delta)H = 0$. Suppose M is of k-type with finite k. Then we have the spectral decomposition

$$x = c + x_1 + \cdots + k_k \tag{4.9}$$

with $\Delta x_i = \ell_i x_i$, where $\{\ell_1, \ldots, \ell_k\}$ are mutually distinct. Since $\Delta x = -nH$, $n = \dim M$, (4.9) implies

$$-n\Delta^j H = \ell_1^{j+1} x_1 + \cdots + \ell_k^{j+1} x_k, \qquad j = 0, 1, 2, \ldots . \tag{4.10}$$

Thus, by $P(\Delta)H = 0$, we find

$$\ell_1 P(\ell_1) x_1 + \cdots + \ell_k P(\ell_k) x_k = 0. \tag{4.11}$$

By applying Δ^j to (4.11) we obtain

$$\ell_1^{j+1} P(\ell_1) x_1 + \cdots + \ell_k^{j+1} P(\ell_k) x_k = 0, \qquad j = 0, 1, 2, \ldots . \tag{4.12}$$

Since $\ell_1, \ldots, \ell_k$ are mutually distinct, (4.12) yields $P(\ell_1) = \cdots = P(\ell_k) = 0$. Therefore, $k \leqslant \deg P$. ∎

Proposition 4.3. *Let $x : M \to E^m$ be an immersion. If there exist a vector $c \in E^m$ and a polynomial $P(t) = \prod_{i=1}^{k} (t - \ell_i)$ with mutually distinct $\ell_1, \ldots, \ell_k$ such that $P(\Delta)(x - c) = 0$, then M is of finite type.*

Proof: Consider the following linear system:

$$\begin{aligned} x - c &= x_1 + x_2 + \cdots + x_k, \\ \Delta x &= \ell_1 x_1 + \ell_2 x_2 + \cdots + \ell_k x_k, \\ &\vdots \\ \Delta^{k-1} x &= \ell_1^{k-1} x_1 + \ell_2^{k-2} x_2 + \cdots + \ell_k^{k-1} x_k. \end{aligned} \tag{4.13}$$

Since $\ell_1, \ldots, \ell_k$ are mutually distinct, we may solve for $x_1, \ldots, x_k$ in terms of $x - c$, $\Delta x, \ldots, \Delta^{k-1} x$ to obtain

$$\prod_{j \neq i} (\ell_j - \ell_i) x_i = \sigma_{i,k-1}(x - c) - \sigma_{i,k-2} \Delta x + \cdots + (-1)^{k-2} \sigma_{i,1} \Delta^{k-2} x + (-1)^{k-1} \Delta^{k-1} x, \tag{4.14}$$

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where $\sigma_{i,j}$ is the j-th elementary symmetric function of $\ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_k$. In other words, we have

$$\sigma_{i,k-1} = \prod_{j \neq i} \ell_j, \quad \sigma_{i,k-2} = \prod_{\substack{j \neq \ell \\ j,\ell \neq i}} \ell_j \ell_\ell, \quad \dots, \quad \sigma_{i,1} = \sum_{j \neq i} \ell_j. \tag{4.15}$$

From the hypothesis of the theorem we have

$$\Delta^k x - \sigma_1 \Delta^{k-1} x + \sigma_2 \Delta^{k-2} x - \cdots + (-1)^{k-1} \sigma_{k-1} \Delta x + (-1)^k \sigma_k (x - c) = 0, \tag{4.16}$$

where σ_j is the j-th elementary symmetric function of $\ell_1, \ell_2, \dots, \ell_k$. From (4.13), (4.14) and (4.16), we may obtain

$$\prod_{j \neq i} (\ell_j - \ell_i) \Delta x_i = \ell_i \prod_{j \neq 1} (\ell_j - \ell_i) x_i, \qquad i = 1, \dots, k. \tag{4.17}$$

Since $\ell_1, \dots, \ell_k$ are mutually distinct, (4.17) yields $\Delta x_i = \ell_i x_i$ and by (4.13) we have $x = c + x_1 + \cdots + x_k$. This shows that M is of finite type. □

REMARK 4.4: Proposition 4.2 and 4.3 remain true if M is a pseudo - Riemannian submanifold of a pseudo-Euclidean space.

REMARK 4.5: For further results concerning linearly independent and orthogonal immersions, see [**3**]. For examples, by applying the representation theory of Lie groups, the first-named author proved in [**3**] that every equivariant isometric immersion $x : M \to E^m$ from any compact Riemannian homogeneous space M into E^m is an orthogonal immersion, moreover, M is immersed into the *adjoint hyperquadric* Q of E^m (in the sense of [**3**]) by x as a minimal submanifold of Q.

REMARK 4.6: By applying Proposition 4.3, the first-named author and Li proved in [**7**] that every 3-type hypersurface of a hypersphere S^{n+1} in E^{n+2} has non-constant mean curvature.

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