LOCAL FRAMES

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ABSTRACT. Following some work by N. Ashby and myself, it is shown how the Fermi construction of local inertial frames can be generalized, in the slow motion approximation, to the neighbourhood of the earth. This allows a clear and simple description of the relativistic effects for the motion of an earth satellite. A proposal by I. Ciufolini to measure the relativistic precession is reviewed.

1. GEOMETRIC FRAMES

A determined effort should be made to drop our mental habit to visualize and work with spatial frames as rigid, cartesian frames; and to think about time as a unique, uniformly flowing variable. Space-time coordinates are arbitrary ways to label events, usually chosen on the basis of mathematical convenience. Before the introduction of the metric, space-time is like a jelly fish: its marks are of no use to identify events. The objects of physics are not coordinates, by measurements and their relationships; frames of references are abstractions, introduced to identify quantitatively and operationally motion with respect to standard, conventionally chosen objects. That every measurement, in particular measurements of angles, distances, velocities and time intervals, are relative has been greatly stressed by E. Mach.

In our commonly accepted framework of general relativity however, the relative character of space-time measurements appears only indirectly: material objects (and the boundary and initial conditions for Einstein's field equations) act as sources and determine the metric structure; it is through this abstract structure that measurements are defined. Thus, the basic question in the analysis of actual space-time measurements is, what is their geometric meaning? To make this identification one must confirm that the measurement tecniques and their results are consistent with the properties of the corresponding abstract quantity. For example, distance measurements by the round trip light time must use atomic clocks and fulfil the required transformation laws when the instrument is set in motion. Of course,

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this identification presupposes that laboratory instruments "work properly", that is to say, fulfil the strong equivalence principle. This condition is needed, for example, to aver that atomic time is proper time.

A full understanding of local frames and local gravitational dynamics requires understanding and familiarity with the local geometric structure of space-time. Because of the (weak) equivalence principle a sufficiently small region of space-time can be approximated by a Minkowsky manifold, in which gravitational effects are small and show up only through tides, to wit, differences in gravitational force. This is the basis of Fermi's construction of local inertial coordinates in the neighbourhood of a freely falling body, which I now review (Fermi 1922).

Given a time-like geodesic l and an event P, one constructs the space geodesic λ (P) from P to l, intersecting it orthogonally at P_o. If P is not too far, such geodesic is unique. Denote by n^μ the unit vector tangent to λ (P) at P_o and by r the geodesic distance from P to P_o. Let also $\Lambda^\mu_{(i)}$ (i and other latin indices = 1,2,3; greek indices include the time index 0) be a triad of orthonormal vector at l, orthogonal to l and parallely propagated along it. The Fermi coordinates are the invariant quantities

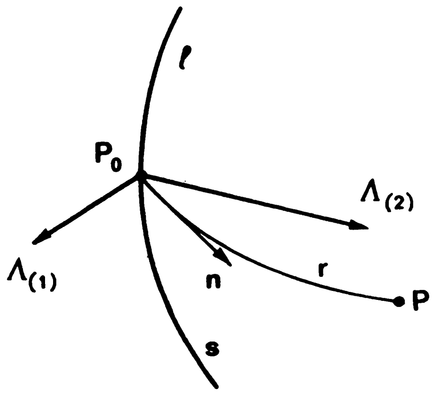

Fig.1. The geometry of the Fermi coordinates.

$$x^\mu = (s, x^i = r\,\Lambda_{(i)}{}^\mu n_\mu), \tag{1}$$

where s is the proper time along l. In this frame, in the neighbourhood of l, the metric deviates from flatness by terms of order

$$h = O(r^2K) = O(r^2M/R^3), \tag{2}$$

where K is the space-time curvature. In the solar system K = M/R is expressed in terms of the sun's mass M and distance R. More precisely, one has the following metric:

$$g_{oo} = -1 - R_{omon}\, x^m x^n + O(r^3)$$
$$g_{oi} = - 2/3\, R_{omin}\, x^m x^n + O(r^3) \qquad (3)$$
$$g_{ij} = \delta_{ij} = -1/3\, R_{imjn}\, x^m\, x^n + O(r^3),$$

where the Riemann tensor appears in Fermi components.

The Fermi coordinates of a nearby test body fulfil the equation of geodesic deviation

$$\frac{dx^i}{ds^2} = R^i{}_{0j0}\, x^j, \qquad (4)$$

the relativistic generalization of the tidal equations; the absence of centrifugal and Coriolis foces justify the adjective inertial. The fact that the relative acceleration is proportional to the relative coodinates is indeed a test that the frame is a Fermi frame and provides a way to measure the curvature of space-time.

2. EARTH-BOUND INTERTIAL FRAMES

Inspite of its elegance and physical appropriateness Fermi's construction is of no use in the most important case, the motion of earth satellites. Of course it is impossible to choose a frame of reference in which the gravitational force vanishes in the whole neighourbood of the earth. In newtonian physics one can, and indeed usually does, eliminate only the gravitational force of the sun and the other planets: in an earth-bound, non-rotating frame their effect is purely tidal. In general relativity the gravitational field depends nonlinearly from the sources and it is not possible to do this in a rigorous way.

The equations currently used to describe the relativistic motion of earth satellities are those of the general n-body problem in the Slow Motion Approximation (SMA), in a frame where the overall center of gravity is at rest. This approximation is determined to within gauge transformations, corresponding to small, but arbitrary changes in the coordinates X^μ ; the coordinates have no geometrical meaning. The difference between the acceleration of a satellite and that of the earth has many and complicated relativistic corrections without any obvious physical interpretation. As a consequence, the study of the relativistic corrections for the moon and LAGEOS is done only by numerical integration (sec, e.g., Moyer 1971) and is beset by difficulties and uncertainties.

N. Ashby and myself have shown how to approximatively generalize Fermi's construction of inertial frames to the case in which the earth is present. Assume the bodies to be divided in two groups: the external group (mainly the sun), with characteristic mass M and distance R and the local group (mainly the earth), with characteristic mass m << M and distance r << R; consistently with the SMA we confine ourselves

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to distances r such that $O(m/r) = O(M/R) = V^2$. To construct an earth-bound Fermi frame we must define an external metric with respect to which the earth follows a geodesic. The separation between external and local gravitational field is trivial for the components oi and ij of the metric which, in our postnewtonian SMA, are additive functions of the sources; the problem lies with the time-time component. A result by Eddington and Clark (1938), later shown independently also by Bertotti (1954) (see also Misner, Thorne and Wheeler 1973, Exercise 39.15), shows the way to do this in the case in which there is only one local body. In the approximation in which the bodies are point-like consider the metric obtained by dropping all the terms which are singular or undetermined at the position of the local body. This is the external metric $G_{e\mu\nu}$. The world line l of the local body is a geodesic with respect to this metric. This reduces in fact the relativistic many body problem in the postnewtonian approximation to the geodesic principle. Note also that the external metric depends upon the local mass m: for example, in the calculation of the retardation effect of the external gravitational potential U_e one must take into account the acceleration produced by m on the external sources. The full metric

$$G_{\mu\nu} = G_{e\mu\nu} + H_{\mu\nu} \tag{5}$$

differs from $G_{e\mu\nu}$ by local terms proportional to m, m^2 and mM.

The stage is now set for detailed calculations. By means of space geodesic with respect to the external metric construct the space Fermi coordinates x^i. The spatial hypersurfaces spanned by x^i, however, should not be labelled with the proper time s_e of the external metric, but with an appropriate variable s corresponding to the time shown by a proper clock on the surface of the earth. I will come back to this point later. We now apply the coordinate transformation

$$X^\mu \rightarrow x^\mu = (s, x^i)$$

to the full metric, to obtain

$$g_{\mu\nu}(x) = G_{\alpha\beta}(X) \frac{\partial X^\alpha}{\partial x^\mu} \frac{\partial X^\beta}{\partial x^\nu}. \tag{6}$$

This is of course more easily said then done (Ashby and Bertotti 1985); but the result is striking for its simplicity and its physical meaning. Four groups of terms, of the types

$$\frac{m}{r} V^2, \quad \frac{m}{r}\left(\frac{x \cdot V}{r}\right)^2, \quad \frac{m}{r} U_e, \quad \frac{m}{r} x\nabla U_e , \tag{7}$$

cancel. Here V^i is the ordinary velocity of the earth and U_e the external newtonian potential. The metric (5) contains the expression of the external metric in Fermi coodinates; this expression, denoted by $g_{e\mu\nu}$, can be calculated on the basis of eq.(3) from the external curvature tensor at 1. We have, to the appropriate order in V,

$$g_{oo} = (1+\Delta)\,[g_{eoo} + 2m/r - 2(m/r)^2 - (5m/3r)\,U_{e,ij}\,x^i x^j]$$

$$g_{oi} = g_{eoi} \tag{8}$$

$$g_{ij} = g_{eij} + 2\,\delta_{ij}\,m/r.$$

The small parameter

$$\Delta(s) = \frac{ds_e}{ds} - 1 \tag{9}$$

accounts for the difference between the external proper time s and the physical time s. If s is identified with the proper time of a clock at a constant distance a from 1, its value is to a good approximation given by the constant 2m/a.

In the metric (8) we find the Schwarzschild field of the earth (which now is at rest) up to the correct approximation, which includes in g_{oo} the main non-linear correction. The external metric g_{eoo}, contains fine groups of terms which have been evaluated explicitely in Ashby and Bertotti (1985) for the case in which there is only one external body, the sun. The relativistic corrections have been computed neglecting cubic terms in the small parameter r/R.

1. Ordinary tides, of order $M r^2/R^3$.

2. Nonlinear corrections to the solar tides, of order $M^2 r^2/R^4$.

3. A solar gravitomagnetic term, of order $Mr^2\,V/R^3$, in the 0i component, arising from the fact that in this frame the sun moves with speed V.

4. Terms quadratic in the solar velocity, of order $Mr^2\,V^2/R^3$.

5. A term arising from the relativistic precession of the Fermi axes $\Lambda_{(i)}$ produced by the moving sun, of order $Mr^2 Vs/R^4$.

Finally, the last term in g_{oo}, of order Mmr/R^3 is a characteristic interaction between the earth potential and the solar tidal potential. Note this terms and the relativistic corrections 2., 3., and 4. are, in order of magnitude, generically $(r/R)^2$ smaller than the main relativistic corrections to the earth's field (of order $(m/r)^2 = O(V^4)$). Their observability is difficult; we are studying this problem for the moon.

The simple and intuitive from of eq. (8) makes it possible to derive it also directly, without going through the coordinate transformation (7) (Ashby and Bertotti (1984)). The only really new term is the interaction; it can be computed by solving explicitely the oo field equation to the appropriate order.

In my opinion the geometric meaning and the simplicity gained with this generalized Fermi frame suggests that the dynamics of earth satellities should be described by eq. (8) and implemented in appropriate software. Note also that it provides an invariantly defined time variable for the whole neighbourhood of the earth. This time is a good candidate as a global reference for the timing systems on the ground and in space.

3. ABSOLUTE ROTATION

There are, broadly speaking, two kinds of ways to define rotation: dynamically and kinematically. Our Fermi axes are defined dynamically, as those with respect to which the equations of motion have a given, simple form. One could say, they measure "absolute" rotation with respect to an abstract, underlying geometric structure. Kinematical rotation is measured with respect to a given body, like the earth or the distant matter in the Universe (as one does with VLBI). The agreement between dynamically defined rotation in empty space and rotation with respect to the Universe is the gist of Mach's Principle and is, to an exceedingly good approximation, a consequence of general relativity and the current cosmological models. In reality the construction and the comparison between different rotating frames of reference is a complex and delicate matter (sse, e.g., Bertotti et al. 1984).

A more accurate construction of earth-bound dynamical frames is a worth-while and challenging task; the measurement of the relativistic precession of an "absolute" direction determined in this way with respect to distant matter is an important and unsolved problem in experimental gravitation. The Stanford gyroscope experiment (Anderson et al. 1982) premises well; I would like to discuss now a very recent and interesting proposal using nondedicated satellites.

There is a contribution to the "absolute" precession due to the sun, which drays around our Fermi axes with the angular velocity

$$\Omega_{ij} = 3\,M\,(V_j R_i - V_i R_j)/2R^3 \tag{10}$$

(de Sitter - Fokker precession; it affects, of course, also a gyroscope, see Misner, Thorne and Wheeler 1973, eq (40.34).) Since in our frame the orbit of an earth satellite is planar, aside from tidal effects, this orbital plane moves with respect to distant matter with the angular velocity (10). As a consequence, the node of the satellite advances by about 16 marcsec/y. There is also another relativistic precession due to the earth angular momentum (Lense - Thirring effect); it is also prograde and for LAGEOS it amounts to 31 marcsec/y.

The relativistic displacement of the node is well above the actual accuracy with which we can measure its position (for a related measurement with LAGEOS, see Yoder et al. 1983); but the uncertainty in the newtonian precession of the node due to the earth's oblateness is about one order of magnitude larger than our effect, about 10^{-6} *126 °/y = 450 marcsec/y. This classical contribution, which is proportional to an odd function of the cosine of the inclination,

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could be eliminated with a special satellite on a polar orbit (Van Patten and Everitt 1976); but Ciufolini (1985) has come out with a better and simpler solution. He remarks that the classical nodal precession of two satellites in circular orbits and the same semimajor axis, but with supplementary inclinations, are equal and opposite; hence the middle point between the two nodes defines an "absolute" direction. It is therefore sufficient to have, beside the present LAGEOS with an inclination of 110°, another one with an inclination of 70°. The accuracy one needs for the inclination and the semimayor axis are, respectively, 1° and 16 Km and appear feasable. Of course the position of the nodes and hence of their middle point is measured with respect to the earth; the measurement of the relativistic precession requires a corresponding accuracy in the rotation rate of the earth.

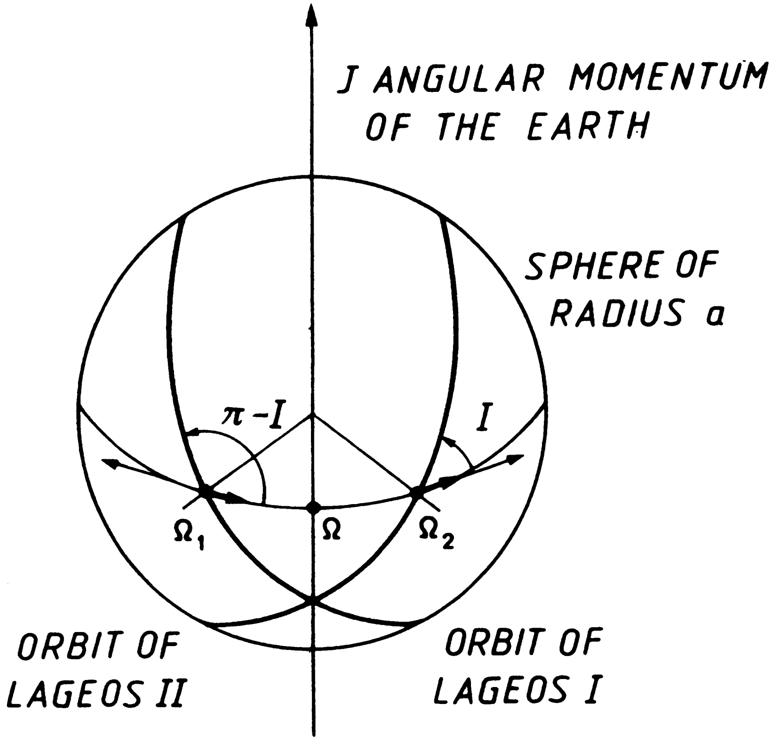

Fig.2. The nodes of two satellites with the same orbit, except for the inclinations, which are supplementary, precess by the same amount in opposite senses. Hence their mid point is an absolute direction.

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