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THE COHOMOLOGY OF THE UNITS IN CERTAIN Z_p-EXTENSIONS

BY

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Dedicated to the memory of R. A. Smith

ABSTRACT. For K/k a Z_p-extension with Galois group Γ, Iwasawa in [4], poses the question of determining the cohomology groups $H^n(\Gamma, E)$ of the unit group E of K. In this article we compute the cohomology of the units (up to finite groups) for a certain class of Z_p-extensions.

Let k be a finite extension of the field Q of rational numbers, and fix p, a prime number. Suppose that K/k is a Z_p-extension with Galois group $\Gamma = \text{Gal}(K/k)$. For integers $n \geq 0$, let k_n be the n^{th} layer of K, $(k_0 = k)$, and denote by Γ_n the Galois group $\text{Gal}(K/k_n)$ so that $\Gamma_n = \Gamma^{p^n}$. Let E_n be the group of units of k_n and $E = \cup_{n\geq 0} E_n$ be the unit group of K. Then E is a discrete Γ-module, and in [4], Iwasawa poses the question of determining the cohomology groups

$$H^n(\Gamma, E) = \varinjlim H^n(\Gamma/\Gamma_n, E_n).$$

Since Γ is a free pro-p-group, $H^n(\Gamma, A) = 0$ for $n \geq 3$ for every discrete Γ-module A, so that this question is of interest only for $n = 1, 2$. Iwasawa ([4], prop. 2) proves

$$H^1(\Gamma, E) \sim (Q_p/Z_p)^r, H^2(\Gamma, E) \approx (Q_p/Z_p)^{r-1}$$

for some integer r, $1 \leq r \leq u$, where $u = u(K/k)$ is the number of prime ideals of k ramified in K. Here for abelian groups A, B we write $A \sim B$ if there is a map $\phi : A \to B$ with finite kernel and cokernel. If $\{A_n\}$ and $\{B_n\}$ are two sequences of finite groups then we write $A_n \sim B_n$ if there are homomorphisms $\phi_n : A_n \to B_n$ whose kernels and cokernels have orders bounded independently of n.

In [4], Iwasawa gives several conditions on the Z_p-extension K/k under which he shows that $r = u$. In this article we compute the invariant r for certain Z_p-extensions which will be described in §2.

1. Firstly, we note that

$$H^2(\Gamma, E) = \varinjlim H^2(\Gamma/\Gamma_n, E_n)$$

and since Γ/Γ_n is a cyclic group, we have

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$$H^2(\Gamma/\Gamma_n, E_n) \simeq E_0/N_n(E_n)$$

where $N_n = N_{n,0}$ is the norm map from k_n to $k_0 = k$. We note that these are isomorphisms of abelian groups which commute with the inflation maps of the direct limit (but which will *not* necessarily commute with the action of the Galois group Δ in §3).

If we denote by P_n, I_n, C_n the group of principal ideals, the group of (fractional) ideals, and the ideal class group of k_n, respectively, then we have the exact sequence

$$0 \to P_n \to I_n \to C_n \to 0.$$

We obtain the exact cohomology sequence

$$0 \to P_n^\Gamma \to I_n^\Gamma \to C_n^\Gamma \to H^1(\Gamma, P_n) \to 0$$

where $H^1(\Gamma, P_n) \approx E_0 \cap N_n(k_n^*)/N_n(E_n)$, and for a Γ-module A, A^Γ denotes the elements of A fixed by Γ.

Let S_n be the set of prime ideals of k_n ramified in K/k_n so that S_n is a finite set of primes of k_n (dividing p) and in fact S_n is the set of all primes of k_n lying over a prime in S_0. Let $\langle S_n \rangle$ be the subgroup of the ideal class group C_n generated by the primes in S_n and denote by C'_n the "S-class group" of k_n, i.e.,

$$C'_n = C_n/\langle S_n \rangle.$$

It then follows that

$$0 \to \langle S_n \rangle^\Gamma \to C_n^\Gamma \to (C'_n)^\Gamma$$

is exact, and we have the following lemma.

LEMMA. *If $(C'_n)^\Gamma \sim 0$, then $E_0 \cap N_n(k_n^*)/N_n(E_n) \sim 0$ and $C_n^\Gamma \sim I_n^\Gamma/P_n^\Gamma$.*

PROOF. It is clear that $(C'_n)^\Gamma \sim 0$ implies that $C_n^\Gamma \sim \langle S_n \rangle^\Gamma$. But it is also clear that $\langle S_n \rangle^\Gamma \sim I_n^\Gamma/P_n^\Gamma$ is (up to groups of bounded order) the subgroup of C_n represented by ideals invariant under Γ. But then it follows that $H^1(\Gamma, P_n) \simeq E_0 \cap N_n(k_n^*)/N_n(E_n)$ must have order which is bounded for all n.

We now describe a result proved in [2], [7] which is sufficient for $(C'_n)^\Gamma \sim 0$.

THEOREM. *Let K/k be a $\mathbf{Z}_p$-extension, $k/\bar{k}$ a finite abelian extension such that $K/\bar{k}$ is a Galois extension. Suppose:*
(I) *The set $S = S_0$ of all primes of k ramified in K consists of all the prime divisors of a single prime ideal p of $\bar{k}$.*
(II) *p has local degree equal to one.*
(III) *The decomposition group D of p in* $\mathrm{Gal}(k/\bar{k})$ *acts trivially on* Γ.
Then $(C'_n)^\Gamma \sim 0$.

The hypotheses of this theorem are satisfied, for example, by the cyclotomic $\mathbf{Z}_p$-extension of any field k abelian over $\mathbf{Q}$ $(=\bar{k})$. In this context the result is due to Greenberg [3].

2. In this section we give a set of $\mathbf{Z}_p$-extensions for which we can compute $H^n(\Gamma, E)$.

They will satisfy the hypotheses of Theorem 1. In addition we require that the field $\bar{k}$ in Theorem 1 is either $\boldsymbol{Q}$ or an imaginary quadratic field. This will allow us to describe the units of k, up to finite groups, as a cyclic module over $\mathbf{Z}[\Delta]$, where $\Delta = \mathrm{Gal}(k/\bar{k})$. Also for such fields k, Leopoldt's conjecture is known to be true.

Let $\bar{k} = \boldsymbol{Q}$ or an imaginary quadratic field, and let $k/\bar{k}$ be an abelian extension with Galois group $\Delta = \mathrm{Gal}(k/\bar{k})$. Suppose that p is a prime satisfying $\delta^{p-1} = 1$ for all $\delta \in \Delta$.

CASE I. If $\bar{k} = \boldsymbol{Q}$ and k is a complex abelian extension of $\boldsymbol{Q}$, denote by $\hat{\Delta}$ the group of characters of Δ with values in the $(p-1)^{st}$ roots of unity of $\boldsymbol{Z}_p$. Let $J \in \Delta$ be the automorphism obtained by restricting complex conjugation to k. It is known ([1], [5]) that for each character $\chi \in \hat{\Delta}$ such that $\chi(J) = -1$ or $\chi = \chi_0$ (the principal character) there is a uniquely defined $\boldsymbol{Z}_p$-extension $K\chi/k$ such that $K\chi/\boldsymbol{Q}$ is a Galois extension. In fact $\mathrm{Gal}(K\chi/\boldsymbol{Q})$ is a semidirect product $\Delta \cdot \Gamma$ with $\Gamma = \mathrm{Gal}(K\chi/k) \simeq \boldsymbol{Z}_p$ and Δ acts on Γ via χ, i.e., $\delta(\gamma) = \bar{\delta}\gamma\bar{\delta}^{-1} = \gamma^{\chi(\delta)}$ for each $\gamma \in \Gamma$, $\delta \in \Delta$, where $\bar{\delta}$ is any lift of $\delta \in \Delta$ to $\mathrm{Gal}(K\chi/\boldsymbol{Q})$. Hence K_{χ_0}/k is the cyclotomic $\boldsymbol{Z}_p$-extension of k, and $K\chi/\boldsymbol{Q}$ is non-abelian for $\chi \neq \chi_0$. In order to satisfy hypothesis III of the theorem, §1, we consider only those extensions $K\chi/k$ such that $D \subseteq \ker \chi$, where D is the decomposition group of p in Δ.

CASE II. If $\bar{k} = \boldsymbol{Q}$ and k is a totally real abelian extension of $\boldsymbol{Q}$, then k has exactly one $\boldsymbol{Z}_p$-extension, the cyclotomic $\boldsymbol{Z}_p$-extension.

If $\bar{k}$ is an imaginary quadratic field, then for hypothesis II of Theorem 1 we require that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in $\bar{k}$, and we consider the unique $\boldsymbol{Z}_p$-extension of k, $K_{\mathfrak{p}}/k$, which is ramified only at $\mathfrak{p}$. Then $K_{\mathfrak{p}}$ is abelian over $\bar{k}$ with Galois group $\mathrm{Gal}(K_{\mathfrak{p}}/\bar{k}) \simeq \Gamma \times \Delta$.

3. We calculate the cohomology groups

$$H^n(\Gamma, E) \qquad n = 1, 2$$

for the $\boldsymbol{Z}_p$-extensions of §2 using the results of [2, §1], and [6] (see also [5]). Since

$$H^n(\Gamma, E) = \lim_{\rightarrow} H^n(\Gamma/\Gamma_n, E_n)$$

and since $H^2(\Gamma/\Gamma_n, E_n) \approx E_0/N_n(E_n)$ (as abelian groups) we may use the lemma of §1 to conclude for the $\boldsymbol{Z}_p$-extensions considered here

$$H^2(\Gamma/\Gamma_n, E_n) \sim E_0/E_0 \cap N_n(k_n^*).$$

As Δ acts on $\Gamma = \mathrm{Gal}(K\chi/k)$ via the character $\chi (\chi = \chi_0$ in Case II) it follows that, for $\phi \in \hat{\Delta}$,

$$H^n(\Gamma/\Gamma_n, E_n)_\phi \approx H^{n+2}(\Gamma/\Gamma_n, E_n)_{\phi\chi}.$$

Also the groups $(E_0/E_0 \cap N_n(k_n^*))_\phi$ are computed in [1, §1] in Case I, and the calculation in Case II is similar. We tabulate the results in Proposition 1.

PROPOSITION 1. *Let K/k be one of the $\boldsymbol{Z}_p$-extensions defined in §2. Then*

$$
\begin{array}{llll}
\text{Case I} & H^0(\Gamma/\Gamma_n, E_n)_\phi \sim \mathbf{Z}/p^n\mathbf{Z} & \phi(J) = +1, \quad \phi \neq \chi_0 \\
& & D \subseteq \ker \phi \\
& \sim 0 & \text{otherwise} \\
& H^2(\Gamma, E)_\phi \sim \mathbf{Q}_p/\mathbf{Z}_p & \phi(J) = \chi(J), \quad \phi \neq \chi \\
& & \phi|D = \chi|D \\
& \sim 0 & \text{otherwise} \\
\text{Case II} & H^0(\Gamma/\Gamma_n, E_n)_\phi \sim \mathbf{Z}/p^n\mathbf{Z} & \phi \neq \chi_0, \ D \subseteq \ker \phi \\
& \sim 0 & \text{otherwise} \\
& H^2(\Gamma, E_n)_\phi \sim \mathbf{Q}_p/\mathbf{Z}_p & \phi \neq \chi_0, \ D \subseteq \ker \phi \\
& \sim 0 & \text{otherwise}
\end{array}
$$

We now give the D-decomposition for $H^1(\Gamma, E)$. It follows from the results in [2, §2; 5; 6] that $D \subseteq \ker \chi$ implies that C_n^Γ has known Δ-decomposition up to groups whose order are bounded independently of n. Lemma 1 also gives the exactness of the following sequence up to groups of bounded orders

$$\sim 0 \to P_n^\Gamma/P_0 \to I_n^\Gamma/I_0 \to C_n^\Gamma \to \sim 0.$$

However

$$I_n^\Gamma/I_0 \sim \mathbf{Z}/p^n\mathbf{Z}[\Delta/D]$$

and in Case I

$$
\begin{array}{ll}
(C_n^\Gamma)_\phi \sim \mathbf{Z}/p^n\mathbf{Z} & \phi \in V, \ \phi \neq \chi, \ \phi|D = \chi|D \\
\sim 0 & \text{otherwise}
\end{array}
$$

where $V = \{\phi \in \hat{\Delta} | \phi(J) = -1 \text{ or } \phi = \chi_0\}$. In Case II the truth of Leopoldt's conjecture implies that $C_n^\Gamma \sim 0$. Hence in both cases this gives a calculation of

$$P_n^\Gamma/P_0 \simeq H^1(\Gamma, E).$$

PROPOSITION 2. *For the $\mathbf{Z}_p$-extensions defined in §2 we have:*

$$
\begin{array}{lll}
\text{Case I} & H^1(\Gamma, E)_\phi \sim \mathbf{Q}_p/\mathbf{Z}_p & \text{if } \ \phi = \chi \\
& & \text{or } \phi(J) = +1, \ \phi|D = \chi_0|D, \ \phi \neq \chi_0 \\
& \sim 0 & \text{otherwise} \\
\text{Case II} & H^1(\Gamma, E)_\phi \sim \mathbf{Q}_p/\mathbf{Z}_p & \phi|D = \chi_0|D \\
& \sim 0 & \text{otherwise}
\end{array}
$$

(Note in Case I if $\chi = \chi_0$, then $H^1(\Gamma, E)_\phi \sim \mathbf{Q}_p/\mathbf{Z}_p$ if and only if $\phi(J) = +1$ and $D \subseteq \ker \phi$.)

We see that the integer r defined by Iwasawa in Case II always satisfies $r = u(K/k)$. However in Case I there are many examples in which we find that $r = u/2$ if $J \notin D$.

H. Yamashita has also written a paper on this subject (Tohoku Math. J. **36** (1984), pp. 75–80).

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