ON TWO–PARAMETER LINEARIZING TRANSFORMATIONS FOR UNIFORM TREATMENT OF TWO–BODY MOTION

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Abstract. Differential changes of time involving two parameters are considered. Universal expressions for dynamical variables of interest in Keplerian motion allow us to reduce the integration of the time transformations to that of integrands depending on an eccentric–like universal anomaly. Elliptic integrals and functions are required to complete the integration.

Key words: reparametrizing transformations, generalized anomalies, uniform treatment of Keplerian systems, universal functions, elliptic functions.

1. Introduction

For the integration of perturbed Kepler problems within the framework of the elliptic–type two–body motion, Ferrándiz & Ferrer (1986) and Ferrándiz *et al.* (1987) replaced the physical time t by a fictitious time τ as the independent variable. Thus, they generalized previous research by Belen'kii (1981) and Cid *et al.* (1983) concerning *regularization and linearization* of equations governing perturbed Keplerian systems. They used the method of linearization by time transformations on introducing a *family of generalized anomalies* defined via a differential relation in which the linearizing function depends on *two homogeneous parameters* that can be taken as functions of some orbital elements. For pure Kepler problems, they analytically integrated the time transformations, in closed form, by means of elliptic integrals and functions. Their developments were originally intended to facilitate (on the basis of some radial intermediaries) the analytical treatment of artificial Earth satellite orbits under a zonal model of geopotential, and as a preconditioning of the problem prior to numerical integration.

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For analytical step regulation in numerical integration of highly eccentric orbits, E. V. Brumberg (1992) proposed the use of *orbital length of arc* as independent variable, and made to fit his derivation into a pattern resembling that of Ferrándiz and his colleagues. He also pointed out that his transformation is applicable to *any kind of Keplerian motion.*

We aim at a systematic study of changes of time allowing the extension of results established by those authors to a uniform treatment of motion under a *universal formulation* of the two–body problem, in the sense, e. g., of Stumpff (1959), Chapter V, Stiefel & Scheifele (1971), §11, or Battin (1987), §4.5 and §4.6. To be precise, we consider time transformations

$$t \longrightarrow \tau \quad \text{given by} \quad dt \;=\; \Phi(\,r\,;\,\alpha_0\,,\,\alpha_1\,)\,d\tau \;=\; \frac{r^{\,\alpha}}{\sqrt{\alpha_0\,+\,\alpha_1\,r}}\,d\tau\,, \qquad (1)$$

α_0 and α_1 being parameters of the transformation and α is a real number, combined with the Sundman transformation $t \to s : dt = r\,ds$, introducing a *universal eccentric–like anomaly* (see, e. g., Stumpff 1959, §41, Formulae [V; 35] and [V; 38]; Stiefel & Scheifele 1971, §11, Formula [60].)

To this end, we resort to certain families of special functions, the so–called *universal functions* (Battin 1987, §4.5 and §4.6) and the *Stumpff c–functions* (Stumpff 1959, §37, §41; Stiefel & Scheifele 1971, §11), by means of which the integration of the reparametrizing transformation is reduced to that of some algebraic functions. As expected, when integrating the changes of time parameter at issue, elliptic integrals and functions will enter.

2. Universal Functions and Useful Relations

The *Stumpff c–functions* (Stumpff 1959, §37, §41; Stiefel & Scheifele 1971, §11) are a family of transcendental functions whose first members solve, under a unified treatment, the second–order linear differential equation

$$d^2y/ds^2 \;+\; \varrho\,y \;=\; 0 \quad \text{(where } \varrho \text{ is a real parameter.)} \qquad (2)$$

After appropriate changes of dependent and independent variables, this is the form to which Kepler problems are reduced, the parameter ϱ being then related to the value of the energy of the two–body system. For $z = \varrho\,s^2$, the general solution to Eq. (2) is a linear combination of the Stumpff functions $c_0(z)$ and $c_1(z)$. In general, these functions obey the *defining relation*

$$c_n(\,z\,) = \sum_{k=0}^{\infty}(\,-\,1\,)^k\,\frac{z^{\,k}}{(\,2\,k\,+\,n\,)!}\;, \qquad n \;=\; 0\,,\,1\,,\,2\ldots.$$

The power series are convergent for all complex values of z, and define real-valued functions for real values of z. Some calculations with these functions are simplified if the *universal functions* U_n introduced by *Battin* are used:

$$U_n(s,\varrho) \equiv s^n c_n(\varrho s^2)\,, \quad n = 0,1,2\dots \tag{3}$$

Let $L = \mu(1-e)/(2q)$ be the *negative of the energy* of the Keplerian orbit (Stiefel & Scheifele 1971, p. 50.) Take $\varrho = 2L$, and consider (Stiefel & Scheifele 1971, pp. 50–51; Battin 1987, §4.5, §4.6.): $\mu \equiv$ gravitational parameter, $q \equiv$ distance of the pericentre, $e \equiv$ eccentricity, $r \equiv$ modulus of the radius vector, $(x,y) \equiv$ Cartesian coordinates in the orbital plane,

$$r = q + \mu e s^2 c_2\left(2Ls^2\right) = q + \mu e U_2(s,2L); \tag{4}$$

$$x = q - \mu s^2 c_2(2Ls^2) = q - \mu U_2(s,2L), \tag{5}$$

$$y = \sqrt{\mu q(1+e)}\; s c_1(2Ls^2) = \sqrt{\mu q(1+e)}\; U_1(s,2L); \tag{6}$$

$$dt = r\,ds \text{ (Sundman transformation)}\,; \; t = qs + \mu e U_3(s,2L); \tag{7}$$

$$\frac{dU_n}{ds} = U_{n-1}\,, \quad n = 1,2,3,\dots; \tag{8}$$

$$1 = U_0^2 + \varrho U_1^2\,, \quad U_1^2 = U_2(1+U_0) = 2U_2 - \varrho U_2^2\,. \tag{9}$$

3. Universal Two–Parameter Time Transformations

We deal with two–parameter time transformations $t \to \tau$ (e. g., those of Ferrándiz & Ferrer 1986, or Ferrándiz *et al.* 1987) as the change of time in Eq. (1). To extend the results of these authors to universal form, introduce

$$C^2 = \sqrt{\alpha_0^2 + q^2\alpha_1^2}\,, \; a_0 = \alpha_0/C^2, \; a_1 = \alpha_1 q/C^2, \; a_0^2 + a_1^2 = 1\,.$$

We use the *distance of the pericentre* q instead of the semi–major axis or the semi–latus rectum, employed by some authors when dealing with elliptic orbits. Now, the differential relation governing the time transformation is

$$C\,dt = r^\alpha [a_0 + a_1(r/q)]^{-1/2}\, d\tau\,.$$

We combine it with the Sundman transformation (7), along with Formula (4), to obtain

$$\sqrt{a_0 q + a_1 r}\;\, r^{1-\alpha} = \sqrt{(a_0+a_1)\,q + a_1\mu e U_2}\;\,(q + \mu e U_2)^{1-\alpha}\,.$$

After the *change of integration variable* $s \to v$ given by $U_2(s,2L) = v$, using Formulae (8) and (9), the integration of the transformation relating the generalized anomaly τ to s is reduced to that of an integrand in v:

$$d\tau = \frac{C}{\sqrt{q}}\,\frac{\sqrt{(a_0+a_1)\,q + a_1\mu e v}}{(q+\mu e v)^{\alpha-1}\sqrt{v\,[2-(2L)v]}}\,dv\,. \tag{10}$$

The calculations and results hinge on the analysis of the polynomials occurring in (10). Further details and applications will be communicated in a future paper. For pure elliptic motion, Ferrándiz & Ferrer (1986) took $\alpha = 2$, while Ferrándiz *et al.* (1987) considered $\alpha = 3/2$, and integrated their general time transformations in terms of elliptic integrals and functions.

4. A Universal Approach to Arc Length as Time Argument

Let $d\sigma$ denote the arc element along a Keplerian conic–section orbit. From (5) and (6), universal expressions for the Cartesian coordinates in the orbital plane, by taking into account (8) and (9) we obtain a differential relation between the arc length σ and the universal anomaly s:

$$d\sigma = \sqrt{\mu q(1+e) + \mu^2 e^2 U_1^2(s, 2L)}\; ds. \tag{11}$$

The change of integration variable $s \to w$ given by $U_1(s, 2L) = w$ yields

$$d\sigma = \sqrt{q}\, \sqrt{\frac{\mu q(1+e) + \mu^2 e^2 w^2}{q - \mu(1-e) w^2}}\; dw. \tag{12}$$

These developments are intended to extend Brumberg's (1992) study. Concerning the integration of these differential relations, comments like those at the end of Section 3 are in order.

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