A NOTE ON COMMUTATIVE BAER RINGS

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Introduction

The class of commutative rings known as Baer rings was first discussed by J. Kist [4], where many interesting properties of these rings were established. Not necessarily commutative Baer rings had previously been studied by I. Kaplansky [3], and by R. Baer himself [1]. In this note we show that commutative Baer rings, which generalize Boolean rings and p-rings, satisfy the Birkhoff conditions for a variety. Next we give a set of equations characterising this variety involving $+$ and $\cdot$ as binary operations, $-$ and $*$ as unary operations, and 0 as nullary operation. Finally we describe Baer-subdirectly irreducible commutative Baer rings and state the appropriate representation theorem.

1. Preliminaries

We will use the following notations:

For $a \in R$ where R is a commutative ring,

$$(a)_R = aR = \{ab : b \in R\}, \text{ and } (a)_R^* = \{b \in R : ab = 0\}.$$

Braces and parentheses without subscripts have their usual meaning. We can now define a commutative Baer ring: A commutative ring R is a Baer ring iff for any $a \in R$ there is an idempotent $a^* \in R$ such that

$$(a)_R^* = (a^*)_R$$

J. Kist [4] has proved that a commutative Baer ring has no non-zero nilpotents. Also the idempotent generator 0^* of $(0)_R^* = R$ must be a unit 1.

LEMMA. $(a)_R^{**} = \bigcap\{(b)_R^* : b \in (a)_R^*\}$ satisfies $(a)_R^{**} = (1-a^*)_R$.

PROOF. Immediate.

The following result is crucial to the whole paper.

PROPOSITION 1. *In a commutative Baer ring R, for any pair a, b in R,*

$$(a \cdot b)^* = a^* + b^* - a^* \cdot b^*.$$

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PROOF. That $(a \cdot b)_R^* = (ab)_R^{***}$ can be easily checked where, for $S \subseteq R$, the annihilator of S is $(S)_R^* = \bigcap\{(s)_R^* : s \in S\}$.

Also $(ab)_R^{**} = (a)_R^{**} \cap (b)_R^{**}$ and so

$$(ab)_R^* = ((a)_R^{**} \cap (b)_R^{**})_R^*.$$

Now $(a)_R^{**} = (1-a^*)_R$ and similarly for $(b)_R^{**}$. Thus

$$\begin{aligned}(ab)_R^* &= ((1-a^*)_R \cap (1-b^*)_R)_R^* \\ &= (1-a^*-b^*+a^*b^*)_R^* \\ &= (a^*+b^*-a^*b^*)_R \text{ since } a^* \text{ and } b^* \text{ are idempotent.}\end{aligned}$$

And so $\quad (ab)^* = a^*+b^*-a^* \cdot b^*$ as required.

This Lemma can also be deduced from the isomorphism $e \mapsto \mathscr{M}_e$ described by J. Kist [4], p. 46. The present proof is used to avoid introducing the space of minimal prime ideals.

2. The variety

Our investigation began with the idea of treating $a \mapsto a^*$ as a unary operation on commutative Baer rings. This led to asking the questions answered in this section. Call a subring S of a commutative Baer ring R a Baer-subring if $x \in S$ implies $x^* \in S$. Then we have

LEMMA 1. *If R is a commutative Baer ring and S is a Baer-subring of R, then S is a commutative Baer ring.*

PROOF. For any $x \in S$, $(x)_R^* = (x^*)_R$ in R and it is immediate that

$$(x)_S^* = (x^*)_R \cap S = (x^*)_S.$$

Thus the Lemma is proved.

LEMMA 2. *If $\{R_\alpha : \alpha \in A\}$ is a family of commutative Baer rings, and $R = \times_{\alpha \in A} R_\alpha$ is their direct product as commutative rings, we may write $\langle x_\alpha \rangle^* = \langle x_\alpha^* \rangle$ and make R into a commutative Baer ring.*

PROOF. We must prove that $(\langle x_\alpha \rangle)_R^* = (\langle x_\alpha \rangle^*)_R$. Clearly $\langle y_\alpha \rangle \cdot \langle x_\alpha \rangle = \langle 0_\alpha \rangle$ for all $\alpha \in A$ iff $y_\alpha x_\alpha = 0$ for all $\alpha \in A$. But this is equivalent to

$$y_\alpha \in (x_\alpha)_{R_\alpha}^* = (x_\alpha^*)_{R_\alpha} \qquad \text{for all } \alpha \in A,$$

or

$$y_\alpha x_\alpha^* = y_\alpha \qquad \text{for all } \alpha \in A.$$

Thus $\langle y_\alpha \rangle \cdot \langle x_\alpha{}^* \rangle = \langle y_\alpha \rangle$ or, equivalently, $\langle y_\alpha \rangle \in (\langle x_\alpha \rangle^*)_R$. Finally, we check that $\langle x_\alpha \rangle^*$ as defined, is idempotent, and we are through.

Let us call an ideal J of the commutative Baer ring R a Baer-ideal if for any x, y of R with $x-y \in J$ we also have $x^*-y^* \in J$. Then we obtain

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LEMMA 3. *If R is a commutative Baer ring and J is a Baer-ideal of R, then R/J is a commutative Baer ring.*

REMARK. This result is equivalent to defining a Baer-congruence ρ in the obvious manner and proving that the quotient ring R/ρ is still a commutative Baer ring.

PROOF. Suppose J is a Baer-ideal. Then R/J is certainly a commutative ring, and also if $e^2 = e$ in R, $(e/J)^2 = e/J$ in R/J. We must prove that

$$(x/J)\cdot(y/J) = (0/J) \text{ iff } (y/J)(x^*/J) = (y/J)$$

or, equivalently,

$$xy \in J \text{ iff } yx^* - y \in J$$

for $x, y \in R$.

Assume $xy \in J$. Then $(xy-0) \in J$ and, by the definition of a Baer-ideal,

$$(xy)^* - 0^* = (x^* + y^* - x^*y^* - 1) \in J.$$

Multiplying through by y, we obtain $y(x^*+y^*-x^*y^*-1) = yx^*-y \in J$.

For the reverse, assume that $yx^*-y \in J$. Then $-x(yx^*-y) = xy \in J$ and the Lemma is proved.

LEMMA 4. *There exist commutative Baer rings with non-empty carriers.*

PROOF. Immediate. Take any Boolean ring with unit.

THEOREM 1. *If we view commutative Baer rings as algebras $\mathscr{R} = \langle R; +, \cdot, -, *, 0\rangle$ with the definitions of subalgebra, product algebra and quotient algebra given above, then commutative Baer rings form a variety.*

PROOF. This follows immediately, using Lemmas 1–4, from Birkhoff's Theorem. See P. M. Cohn [2] pp. 169–170.

The next step in this work was to find a set of equations defining commutative Baer rings. This proved quite easy, as we see in the next section.

3. The equations

Writing down all the useful identities satisfied by $*$ in a Baer ring gave the following result immediately. Equation (x) is crucial, and is shown elsewhere to characterise Baer rings within a certain class.

THEOREM 2. *Suppose $\mathscr{R} = \langle R; +, \cdot, -, *, 0\rangle$ is an algebra with binary operations $+, \cdot$; unary operations $-, *$; and nullary operation 0; and also that $\mathscr{R}$ satisfies the following equations:*

(i) $(x+y)+z = x+(y+z)$ (ii) $x\cdot(y\cdot z) = (x\cdot y)\cdot z$

(iii) $x+y = y+x$ (iv) $x\cdot y = y\cdot x$

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(v) $x+0 = x$ (vi) $x+(-x) = 0$

(vii) $x \cdot (y+z) = x \cdot y + x \cdot z$ (viii) $x \cdot x^* = 0$

(ix) $x \cdot (x^*)^* = x$ (x) $(x \cdot y)^* = x^* + y^* + (-(x^* \cdot y^*))$.

Then R is a commutative Baer ring where $(x)_R^* = (x^*)_R$ *for the idempotent* x^*.

PROOF. By a sequence of Lemmas.

LEMMA 5. *Equations* (i) *to* (vii) *define a commutative ring* $\langle R; +, \cdot, -, 0\rangle$.

PROOF. This is well known.

We will thus assume that all the usual facts that hold in an arbitrary commutative ring (not necessarily with identity) are valid in $\mathscr{R}$.

LEMMA 6. *The element* $1 \in R$ *given by* $1 =_{\mathrm{Df}} 0^*$ *satisfies the equation* $1 \cdot x = x$.

PROOF. $0^* = (0 \cdot x^*)^* = 0^* + x^{**} - 0^* \cdot x^{**}$ by (x) where $x^{**} =_{\mathrm{Df}} (x^*)^*$. Thus $x^{**} = 0^* \cdot x^{**}$ and so by (x)

$$x = x^{**} \cdot x = (0^* \cdot x^{**}) \cdot x = 0^* \cdot (x^{**} \cdot x) = 0^* \cdot x$$

and $1 = 0^*$ is a multiplicative identity.

LEMMA 7. *If* $x \cdot y = 0$ *then* $x^* \cdot y = y$.

PROOF. $x \cdot y = 0$ implies $x^* + y^* - x^* \cdot y^* = 0^* = 1$. Thus

$$(x^* + y^* - x^* \cdot y^*) \cdot y = 1 \cdot y = y$$

and we obtain $x^* \cdot y = y$ since $y^* \cdot y = 0$ by (viii).

The Lemma follows.

LEMMA 8. *If* $x^* \cdot y = y$ *then* $x \cdot y = 0$.

PROOF. $x^* \cdot y = y$ implies, by (x), $x^{**} + y^* - x^{**} \cdot y^* = y^*$. But this implies that

$$x^{**} = x^{**} \cdot y^*$$

and so, by (x),

$$x = x \cdot x^{**} = x \cdot x^{**} \cdot y^* = xy^*.$$

Finally, applying y, $x \cdot y = x \cdot y^* \cdot y = 0$ and we are through.

LEMMA 9. $x^* \cdot x^* = x^*$.

PROOF. By (viii), $0 = x \cdot x^*$ and so

$$1 = x^* + x^{**} - x^* \cdot x^{**}$$

follows from (x). But this is

$$1 = x^* + x^{**}$$

using (viii). Hence

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$$x^* = x^* \cdot 1 = x^*(x^{**}+x^*) = x^* \cdot x^*$$

again by (viii).

This proves the Lemma.

PROOF OF THEOREM 2. From Lemmas 7 and 8 we see that for $x \in R$, $x \cdot y = 0$ iff $x^* \cdot y = y$. This means that $(x)_R^* = (x^*)_R$ where x^* is, by Lemma 9, idempotent. This completes the proof.

4. Sub-direct unions

It is well known that amongst commutative rings with no non-zero nilpotents, fields are precisely the subdirectly irreducible ones. To emphasise that our notions are all Baer-notions, we use the terms: Baer-subdirectly irreducible and Baer-subdirect union.

LEMMA 10. *A commutative Baer ring R is Baer-subdirectly irreducible iff R is an integral domain.*

PROOF. By well known results, a commutative Baer ring R is Baer-subdirectly irreducible if the intersection J of all the Baer-ideals of R is different from zero. This uses the obvious relation between Baer-ideals and Baer-congruences.

Suppose that R is Baer-subdirectly irreducible and so its $J \neq (0)_R$. Then for $j \neq 0$ in $J, j-0 \in J$ and so $j^*-0^* \in J$ and $j^{**}-0^{**} = j^{**} \in J$. Now j^{**} is an idempotent and R cannot have idempotents other than 0 or 1 and so $j^{**} = 1$. Thus $(j)_R^{**} = R$ and $(j)_R^* = (0)_R$. And so we have proved that $J = R$ and no non-zero element $j \in R$ has non-trivial annihilator i.e. R is an integral domain.

The fact that integral domains are Baer-subdirectly irreducible commutative Baer rings is easily proved by reversing the above.

THEOREM 3. *Any commutative Baer ring is a Baer-subdirect union of a family of integral domains. Conversely, any Baer-subring of a Baer-direct union of integral domains is a commutative Baer ring.*

PROOF. Immediate, using Birkhoff's Theorem and Lemma 9. This subdirect union representation can be given explicitly by

$$\phi : R \to \bigtimes_{M \in \mathscr{M}} R/M, \; x\phi = \langle x(M) \rangle$$

where $\mathscr{M}$ is the set of all minimal prime ideals of R, see [5].

5. Final remarks

One of us (T.P.S.) is shortly publishing some results [5] which discuss many properties of commutative Baer rings involving prime and minimal prime ideals, the algebra of idempotents and related ideas. A number of characterisations of

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commutative Baer rings are given from amongst various classes of commutative rings.

However we have not considered the topic of independence amongst the equations (i)–(x). And the question: 'When is a commutative Baer ring a Baer-subdirect union or a Baer-direct union of fields?' seems interesting. Then there is the problem of describing free commutative Baer rings and so on.

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