9
Multipole analysis

Start by taking the photon momentum to define the z-axis (Fig. 9.1); the generalization follows below. In this case the plane wave can be expanded as [Fe80]

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_l i^l \sqrt{4\pi(2l+1)}\, j_l(kx) Y_{l0}(\Omega_x) \tag{9.1}$$

The vector spherical harmonics are defined by the relations [Ed74]

$$\mathcal{Y}^M_{Jl1} \equiv \sum_{m\lambda} \langle lm1\lambda | l1JM\rangle Y_{lm}(\Omega_x)\mathbf{e}_\lambda \tag{9.2}$$

Note this sum goes over all three spherical unit vectors, $\lambda = \pm 1, 0$. The definition in Eq. (9.2) can be inverted with the aid of the orthogonality properties of the Clebsch–Gordan (C–G) coefficients

$$Y_{lm}\mathbf{e}_\lambda = \sum_{JM} \langle lm1\lambda | l1JM\rangle \mathcal{Y}^M_{Jl1} \tag{9.3}$$

The $\mathbf{e}_\lambda$ are now just fixed vectors; they form a complete orthonormal set.

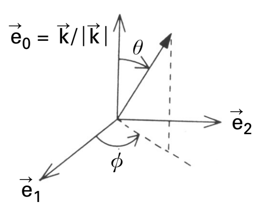

Fig. 9.1. Coordinate system with z-axis defined by photon momentum.

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Therefore any vector can be expanded in spherical components as

$$
\begin{aligned}
\mathbf{v} &= \sum_{\lambda}(\mathbf{v}\cdot\mathbf{e}_{\lambda})\,\mathbf{e}_{\lambda}^{\dagger} = \sum_{\lambda} v_{\lambda}\mathbf{e}_{\lambda}^{\dagger} \\
v_{\pm 1} &= \mp\frac{1}{\sqrt{2}}(v_x \pm i v_y) \qquad\qquad v_0 = v_z
\end{aligned}
\tag{9.4}
$$

As we shall see, the vector spherical harmonics project an irreducible tensor operator (ITO) of rank J from any vector density operator in the nuclear Hilbert space. A combination of Eqs. (9.1) and (9.3) and use of the properties of the C–G coefficients yields[1]

$$
\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l}\sum_{J} i^{l}\sqrt{4\pi(2l+1)}\,j_l(kx)\langle l01\lambda|l1J\lambda\rangle\mathscr{Y}^{\lambda}_{Jl1}(\Omega_x) \tag{9.5}
$$

The C–G coefficient limits the sum on l to three terms $l = J, l = J \pm 1$, and these C–G coefficients can be explicitly evaluated to give for $\lambda = \pm 1$ [Ed74]

$$
\begin{aligned}
\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{J\geq 1} i^{J}\sqrt{\frac{4\pi(2J+1)}{2}}\Bigg\{ &-\lambda\, j_J(kx)\mathscr{Y}^{\lambda}_{JJ1} \\
-i\left[\sqrt{\frac{J+1}{2J+1}}\, j_{J-1}(kx)\mathscr{Y}^{\lambda}_{J,J-1,1} - \sqrt{\frac{J}{2J+1}}\, j_{J+1}(kx)\mathscr{Y}^{\lambda}_{J,J+1,1}\right]\Bigg\}
\end{aligned}
\tag{9.6}
$$

From [Ed74] one has

$$
\begin{aligned}
\nabla\times j_J(kx)\mathscr{Y}^{\lambda}_{JJ1} &= i\left[\left(\frac{d}{dx}-\frac{J}{x}\right) j_J(kx)\sqrt{\frac{J}{2J+1}}\mathscr{Y}^{\lambda}_{J,J+1,1}\right. \\
&\qquad \left. +\left(\frac{d}{dx}+\frac{J+1}{x}\right) j_J(kx)\sqrt{\frac{J+1}{2J+1}}\mathscr{Y}^{\lambda}_{J,J-1,1}\right]
\end{aligned}
\tag{9.7}
$$

The differential operators just raise and lower the indices on the spherical Bessel functions, giving $-kj_{J+1}(kx)$ and $kj_{J-1}(kx)$, respectively. A combination of these results gives for $\lambda = \pm 1$

$$
\begin{aligned}
\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{x}} &= \sum_{J\geq 1}\sqrt{2\pi(2J+1)}\, i^{J}\Big\{-\lambda\, j_J(kx)\mathscr{Y}^{\lambda}_{JJ1}(\Omega_x) \\
&\qquad -\frac{1}{k}\nabla\times[j_J(kx)\mathscr{Y}^{\lambda}_{JJ1}(\Omega_x)]\Big\} \qquad ;\ \lambda = \pm 1
\end{aligned}
\tag{9.8}
$$

[1] Note this is the amplitude for photon *absorption.*

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Note the divergence of both sides of this equation vanishes [Ed74].[2] Now use

$$\mathscr{Y}^{\lambda\dagger}_{JJ1} = -(-1)^{\lambda}\mathscr{Y}^{-\lambda}_{JJ1} \tag{9.9}$$

to arrive at the basic result for photon emission with $\lambda = \pm 1$

$$-e_{\mathrm{p}}\left(\frac{\hbar c^2}{2\omega_k\Omega}\right)^{1/2}\int e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{e}^{\dagger}_{\mathbf{k}\lambda}\cdot\hat{\mathbf{J}}(\mathbf{x})\,d^3x \tag{9.10}$$

$$= e_{\mathrm{p}}\left(\frac{\hbar c^2}{2\omega_k\Omega}\right)^{1/2}\sum_{J\geq 1}(-i)^J\sqrt{2\pi(2J+1)}\,[\hat{T}^{\mathrm{el}}_{J,-\lambda}(k)+\lambda\hat{T}^{\mathrm{mag}}_{J,-\lambda}(k)]$$

The *transverse electric* and *magnetic multipole operators* are defined by

$$\begin{aligned}\hat{T}^{\mathrm{el}}_{JM}(k) &\equiv \frac{1}{k}\int d^3x\left[\nabla\times j_J(kx)\mathscr{Y}^{M}_{JJ1}(\Omega_x)\right]\cdot\hat{\mathbf{J}}(\mathbf{x})\\ \hat{T}^{\mathrm{mag}}_{JM}(k) &\equiv \int d^3x\left[j_J(kx)\mathscr{Y}^{M}_{JJ1}(\Omega_x)\right]\cdot\hat{\mathbf{J}}(\mathbf{x})\end{aligned} \tag{9.11}$$

This important result merits several observations.

In a nucleus both the convection current density arising from the motion of charged particles (e.g. protons) and the intrinsic magnetization density coming from the intrinsic magnetic moments of the nucleons contribute to the electromagnetic interaction. The appropriate interaction hamiltonian should actually be written as

$$\begin{aligned}H' &= -e_{\mathrm{p}}\int\hat{\mathbf{J}}_c(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})\,d^3x - e_{\mathrm{p}}\int\hat{\boldsymbol{\mu}}(\mathbf{x})\cdot[\nabla\times\mathbf{A}(\mathbf{x})]\,d^3x\\ &= -e_{\mathrm{p}}\int\left[\hat{\mathbf{J}}_c(\mathbf{x})+\nabla\times\hat{\boldsymbol{\mu}}(\mathbf{x})\right]\cdot\mathbf{A}(\mathbf{x})\,d^3x\end{aligned} \tag{9.12}$$

To obtain the second line, a vector identity has been employed

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \tag{9.13}$$

The total divergence has been converted to a surface integral far away from the nucleus using Gauss' theorem

$$\int_V \nabla\cdot\mathbf{v}\,d^3x = \int_S \mathbf{v}\cdot d\mathbf{S} \tag{9.14}$$

Finally, the integral over the far-away surface can be discarded for a *localized source*. A second application of this procedure yields the relation

$$\int d^3x\left[\nabla\times j_J(kx)\mathscr{Y}^{M}_{JJ1}\right]\cdot\nabla\times\hat{\boldsymbol{\mu}}(\mathbf{x}) \tag{9.15}$$

$$= \int d^3x\,\hat{\boldsymbol{\mu}}(\mathbf{x})\cdot\nabla\times[\nabla\times j_J(kx)\mathscr{Y}^{M}_{JJ1}] = k^2\int d^3x\,\hat{\boldsymbol{\mu}}(\mathbf{x})\cdot[j_J(kx)\mathscr{Y}^{M}_{JJ1}]$$

[2] The relation to be used is $\vec{\nabla}\cdot[j_J(kx)\vec{\mathscr{Y}}^{M}_{JJ1}] = 0$.

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In arriving at the second equality the relation $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$ has been employed; the term $\nabla \cdot \mathbf{v}$ vanishes here, and in this application the remaining term satisfies the Helmholtz equation $(\nabla^2 + k^2)\mathbf{v} = 0$, as the reader can readily verify. Thus the multipole operators can be rewritten to explicitly exhibit the individual contributions of the convection current and the intrinsic magnetization densities

$$\begin{aligned}
\hat{T}^{\rm el}_{JM}(k) &= \frac{1}{k}\int d^3x \left\{[\nabla \times j_J(kx)\mathscr{Y}^M_{JJ1}]\cdot\hat{\mathbf{J}}_c(\mathbf{x}) + k^2 j_J(kx)\mathscr{Y}^M_{JJ1}\cdot\hat{\boldsymbol{\mu}}(\mathbf{x})\right\} \\
\hat{T}^{\rm mag}_{JM}(k) &= \int d^3x \left\{j_J(kx)\mathscr{Y}^M_{JJ1}\cdot\hat{\mathbf{J}}_c(\mathbf{x}) + [\nabla \times j_J(kx)\mathscr{Y}^M_{JJ1}]\cdot\hat{\boldsymbol{\mu}}(\mathbf{x})\right\}
\end{aligned} \tag{9.16}$$

The $\hat{T}_{JM}$ are now *irreducible tensor operators of rank J in the nuclear Hilbert space*. This can be proven in general by utilizing the properties of the vector density operator $\hat{\mathbf{J}}(\mathbf{x})$ under rotations. It is easier to prove this property explicitly in any particular application. For example, consider the case where the nucleus is pictured as a collection of non-relativistic nucleons, and the intrinsic magnetization density at the point $\mathbf{x}$ is constructed in first quantization by summing over the contribution of the individual nucleons

$$e_{\rm p}\hat{\boldsymbol{\mu}}(\mathbf{x}) = \mu_N \sum_{i=1}^{A} \lambda_i \boldsymbol{\sigma}(i)\delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \tag{9.17}$$

Here λ_i is the intrinsic magnetic moment of the ith nucleon in nuclear magnetons (see below).[3] The contribution to $\hat{T}^{\rm el}_{JM}$, for example, then takes the form

$$\begin{aligned}
&e_{\rm p}\int j_J(kx)\mathscr{Y}^M_{JJ1}\cdot\hat{\boldsymbol{\mu}}(\mathbf{x})\,d^3x = \\
&\qquad \mu_N \sum_{i=1}^{A} \lambda_i j_J(kx_i) \sum_{mq} \langle Jm1q|J1JM\rangle Y_{Jm}(\Omega_i)\sigma_{1q}(i)
\end{aligned} \tag{9.18}$$

Here the definition of the vector spherical harmonics in Eq. (9.2) has been introduced. Each term in this sum is now recognized, with the aid of [Ed74], to be a tensor product of rank J formed from two ITO of rank J and 1, respectively.[4] Thus $\hat{T}^{\rm el}_{JM}$ is evidently an ITO of rank J under commutation with the total angular momentum operator, which in this

[3] One could be dealing with a density operator in second quantization, or expressed in collective coordinates, etc; to test for an ITO, one first constructs the appropriate total angular momentum operator $\hat{\mathbf{J}}$, and then examines the commutation relations (see [Ed74]).

[4] Any spherically symmetric factor does not affect the behavior under rotations.

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case takes the form

$$\hat{\mathbf{J}} = \sum_{i=1}^{A} \mathbf{J}(i) = \sum_{i=1}^{A} [\mathbf{L}(i) + \mathbf{S}(i)] \qquad ; \text{ angular momentum} \tag{9.19}$$

As another example, the convection current in this same picture of the nucleus is

$$\hat{\mathbf{J}}_c(\mathbf{x}) = \sum_{i=1}^{Z} \frac{1}{m_p c} \{\delta^{(3)}(\mathbf{x} - \mathbf{x}_i), \mathbf{p}(i)\}_{\text{sym}} \quad \doteq \quad \sum_{i=1}^{Z} \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \frac{\mathbf{p}(i)}{m_p c} \tag{9.20}$$

The need for symmetrization[5] arises from the fact that $\mathbf{p}(i)$ and $\mathbf{x}_i$ do not commute; the current density arising from the matrix element of this expression takes the appropriate quantum mechanical form $(\hbar/2im_pc)[\psi^* \nabla\psi - (\nabla\psi)^*\psi]$. The last equality in Eq. (9.20) follows since one of the symmetrized terms can be partially integrated in the required matrix elements of the current, using the hermiticity of $\mathbf{p}(i)$ and the observation that $\nabla \cdot \mathbf{A} = 0$ in the Coulomb gauge. Multipoles constructed from the convection current density in Eq. (9.20) are now shown to be ITO by arguments similar to the above.

The *parity* of the multipole operators is [Bl52]

$$\begin{aligned} \hat{\Pi}\, \hat{T}^{\text{el}}_{JM}\, \hat{\Pi}^{-1} &= (-1)^J \hat{T}^{\text{el}}_{JM} \\ \hat{\Pi}\, \hat{T}^{\text{mag}}_{JM}\, \hat{\Pi}^{-1} &= (-1)^{J+1} \hat{T}^{\text{mag}}_{JM} \end{aligned} \tag{9.21}$$

Again the general proof follows from the behavior of the current density $\hat{\mathbf{J}}(\mathbf{x})$ as a polar vector under spatial reflections. It is easy to see this behavior in any particular application. For example, it follows from Eqs. (9.17) and (9.20) if one uses the properties of the individual quantities under spatial reflection: $\sigma_{1q} \rightarrow \sigma_{1q}$; $p_{1q} \rightarrow -p_{1q}$; and $Y_{lm}(-\mathbf{x}/|\mathbf{x}|) = (-1)^l Y_{lm}(\mathbf{x}/|\mathbf{x}|)$. Parity selection rules on the matrix elements of the transverse multipole operators now follow directly.

There is no $J = 0$ term in the sum in Eq. (9.10). This arises from the fact that the vector potential is transverse, and hence there are only transverse unit vectors, or equivalently unit helicities $\lambda = \pm 1$, arising in its expansion into normal modes [see Eqs. (8.2) and (8.4)]. This has the consequence, for example, that there can be no $J = 0 \rightarrow J = 0$ real photon transitions in nuclei.

The *Wigner–Eckart theorem* [Ed74] can now be employed to exhibit the angular momentum selection rules and M-dependence of the matrix

[5] $\{A, B\}_{\text{sym}} \equiv (AB + BA)/2$.

element of an ITO between eigenstates of angular momentum

$$\langle J_f M_f|\hat{T}_{JM}|J_i M_i\rangle = \frac{(-1)^{J_i - M_i}}{(2J+1)^{1/2}}\langle J_f M_f J_i - M_i|J_f J_i J M\rangle\langle J_f||\hat{T}_J||J_i\rangle \quad (9.22)$$

The Clebsch–Gordan (C–G) coefficients provide all the relevant information. They contain the entire M-dependence, and they vanish unless the angular momentum quantum numbers satisfy the triangle inequality, e.g. $|J_i - J_f| \leq J \leq J_i + J_f$. We adopt the convention that this selection rule is built into the reduced matrix elements themselves, and that they are defined to be zero unless the triangle inequality is satisfied.

Note that the required matrix elements of Eq. (9.10) imply $M_f = M_i - \lambda$. This means that the photon carries away the angular momentum λ along the z-axis, which is the direction of emission of the photon in the preceding analysis (Fig. 9.1); thus the *helicity* of the photon (its angular momentum along $\mathbf{k}$) is $\lambda = \pm 1$.

If the target is unpolarized and unobserved, one can simply pick a convenient z-axis along which to quantize, and the photon momentum $\mathbf{k}$ provides such a choice. In that case, the average over initial target orientations $\overline{\sum_i} = (2J_i+1)^{-1}\sum_{M_i}$ and sum over final target orientations $\sum_f = \sum_{M_f}$ can be immediately evaluated with the aid of the Wigner–Eckart theorem and the orthonormality properties of the C–G coefficients to give

$$\frac{1}{2J_i+1}\sum_{M_i}\sum_{M_f}\left|\sum_J (-i)^J\sqrt{2J+1}\langle J_f M_f|\hat{T}_{JM}|J_i M_i\rangle\right|^2$$
$$= \frac{1}{2J_i+1}\sum_J |\langle J_f||\hat{T}_J||J_i\rangle|^2 \quad (9.23)$$

One then proceeds directly to the transition rate given below in Eq. (9.41).

It is useful for the subsequent discussion of angular correlations to first digress and consider the more general situation where the photon is *emitted in an arbitrary direction* relative to the coordinate axes picked to describe the quantization of the nuclear system. The situation is illustrated in Fig. 9.2. The unit vectors describing the photon are assumed to have Euler angles $\{\alpha, \beta, \gamma\}$ with respect to the nuclear quantization axes. The difficulty in achieving this configuration is that the photon axes here are the axes that are assumed to be *fixed in space*, having been determined, for example, by the detection of the photon, and the rotations are to be carried out with respect to these axes.

Now one knows how to carry out a rotation of the nuclear state vector relative to a fixed set of axes. For example, consider the rotation operator that rotates a physical state vector through the angle β relative

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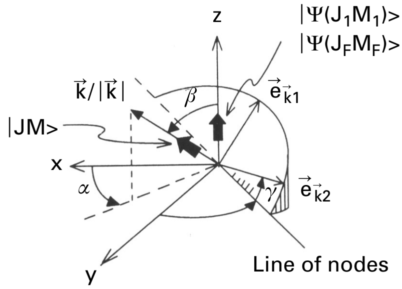

Fig. 9.2. Photon emitted in arbitrary direction relative to quantization axes for nuclear system. Note $\{\alpha, \beta, \gamma\}$ are Euler angles.

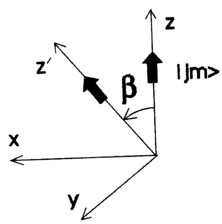

Fig. 9.3. Rotate physical state vector by angle β about y-axis.

to a laboratory-fixed y-axis as indicated in Fig. 9.3. It follows entirely from the defining commutation relations for the angular momentum, that the operator which accomplishes this task is $\hat{R}_{-\beta} \equiv e^{-i\beta\hat{J}_y}$. This is demonstrated as follows. Introduce a new unit vector along the z' direction and dot this into the angular momentum operator

$$
\begin{aligned}
\mathbf{e}_{z'} &= \mathbf{e}_z \cos\beta + \mathbf{e}_x \sin\beta \\
\mathbf{e}_{z'} \cdot \hat{\mathbf{J}} &= \hat{J}_z \cos\beta + \hat{J}_x \sin\beta
\end{aligned}
\tag{9.24}
$$

Now make use of the following identity and basic commutation relations

$$
\begin{aligned}
e^{-i\beta\hat{J}_y}\hat{J}_z e^{i\beta\hat{J}_y} &= \hat{J}_z + (-i\beta)[\hat{J}_y, \hat{J}_z] + \frac{(-i\beta)^2}{2!}[\hat{J}_y, [\hat{J}_y, \hat{J}_z]] \\
&\quad + \frac{(-i\beta)^3}{3!}[\hat{J}_y, [\hat{J}_y, [\hat{J}_y, \hat{J}_z]]] + \cdots \\
[\hat{J}_i, \hat{J}_j] &= i\varepsilon_{ijk}\hat{J}_k
\end{aligned}
\tag{9.25}
$$

One finds

$$
\begin{aligned}
e^{-i\beta\hat{J}_y}\hat{J}_z e^{i\beta\hat{J}_y} &= \hat{J}_z\left(1 - \frac{\beta^2}{2!} + \cdots\right) + \hat{J}_x\left(\beta - \frac{\beta^3}{3!} + \cdots\right) \\
&= \hat{J}_z \cos\beta + \hat{J}_x \sin\beta \\
&= \mathbf{e}_{z'} \cdot \hat{\mathbf{J}}
\end{aligned}
\tag{9.26}
$$

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Thus, from general principles,

$$(\mathbf{e}_{z'} \cdot \hat{\mathbf{J}})e^{-i\beta\hat{J}_y} = e^{-i\beta\hat{J}_y}\hat{J}_z \tag{9.27}$$

Now apply this relation to the state vector $|jm\rangle$ representing a particle with angular momentum j and z-component m, and let $\hat{J}_z$ act on this eigenstate.

$$(\mathbf{e}_{z'} \cdot \hat{\mathbf{J}})[e^{-i\beta\hat{J}_y}|jm\rangle] = m[e^{-i\beta\hat{J}_y}|jm\rangle] \tag{9.28}$$

This is the desired result. The quantity $e^{-i\beta\hat{J}_y}|jm\rangle$ is a rotated eigenstate with angular momentum m along the new z'-axis.

The goal now is to rotate the nuclear state vector $|J_iM_i\rangle$ quantized with respect to the photon axes into a nuclear state vector $|\Psi_i(J_iM_i)\rangle$ correctly quantized with respect to the indicated $\{x, y, z\}$ coordinates. A concentrated effort, after staring at Fig. 9.2, will convince the reader that the following rotations, carried out with respect to the laboratory-fixed photon coordinate system in the indicated sequence, will achieve this end

1. $-\alpha$ about $\mathbf{k}/|\mathbf{k}|$
2. $-\beta$ about $\mathbf{e}_{\mathbf{k}2}$
3. $-\gamma$ about $\mathbf{k}/|\mathbf{k}|$

The rotation operator that accomplishes this rotation is

$$\hat{R}_{+\gamma,+\beta,+\alpha} = \exp\{i\gamma\hat{J}_3\}\exp\{i\beta\hat{J}_2\}\exp\{i\alpha\hat{J}_3\} \tag{9.29}$$

The $\{2, 3\}$ axes are now the laboratory-fixed $\{\mathbf{e}_{\mathbf{k}2}, \mathbf{k}/|\mathbf{k}|\}$ axes. Thus

$$|\Psi_i(J_iM_i)\rangle \;=\; \hat{R}_{\gamma,\beta,\alpha}|J_iM_i\rangle \;=\; \sum_{M_k}\mathscr{D}^{J_i}_{M_kM_i}(\gamma,\beta,\alpha)|J_iM_k\rangle \tag{9.30}$$

Here the rotation matrices have been introduced that characterize the behavior of the eigenstates of angular momentum under rotation [Ed74]. It is clear from Fig. 9.2 that one can identify the usual polar and azimuthal angles that the photon makes with respect to the nuclear coordinate system according to $\beta \leftrightarrow \theta$ and $\alpha \leftrightarrow \phi$; the angle $\gamma \leftrightarrow -\phi$ of the orientation of the photon polarization vector around the photon momentum is a definition of the overall phase of the state vector, and, as such, merely involves a phase convention; the choice here is that of Jacob and Wick [Ja59]. It will be apparent in the final result that this phase is irrelevant. Equation (9.30) expresses the required nuclear state vector as a linear combination of state vectors quantized along the photon axes. Since now only matrix elements between states quantized along $\mathbf{k}$ are required, *all the previous results can be utilized*. The required photon transition matrix element takes the form

$$\langle\Psi_f(J_fM_f)|\hat{H}_{J,-\lambda}|\Psi_i(J_iM_i)\rangle \;=\; \langle J_fM_f|\hat{R}^{-1}_{\gamma,\beta,\alpha}\hat{H}_{J,-\lambda}\hat{R}_{\gamma,\beta,\alpha}|J_iM_i\rangle \tag{9.31}$$

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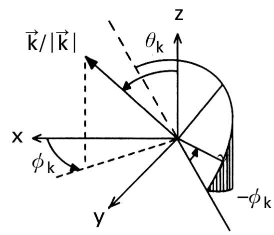

Fig. 9.4. Configuration for transition matrix element describing photon emission and nuclear process $J_iM_i \to J_fM_f$ with nuclear quantization axis along the z-axis.

Here λ is the photon helicity, and $\hat{H}_{J,-\lambda}$ indicates one of the contributions to the operator in Eq. (9.10). Evidently

$$\hat{R}^{-1}_{\gamma,\beta,\alpha} = \hat{R}_{-\alpha,-\beta,-\gamma} \tag{9.32}$$

The definition of an ITO can now be used to simplify the calculation [Ed74]

$$\hat{R}_{-\alpha,-\beta,-\gamma}\hat{H}_{J,-\lambda}\hat{R}^{-1}_{-\alpha,-\beta,-\gamma} = \sum_{M'} \mathscr{D}^J_{M',-\lambda}(-\alpha,-\beta,-\gamma)\hat{H}_{JM'} \tag{9.33}$$

The previous identification of the angles, and a combination of these results, permits one to write the transition matrix element describing the nuclear process $J_iM_i \to J_fM_f$ with the nuclear quantization axis along z and emission of a photon with helicity λ (Fig. 9.4) as

$$\langle\Psi_f(J_fM_f)|\hat{H}'(\mathbf{k}\lambda)|\Psi_i(J_iM_i)\rangle \;=\; \langle J_fM_f|\hat{H}^{\text{em}}_1(\mathbf{k}\lambda)|J_iM_i\rangle \tag{9.34}$$

where the appropriate transition operator is given by

$$\begin{aligned}\hat{H}^{\text{em}}_1(\mathbf{k}\lambda) \;&=\; e_{\text{p}}\left(\frac{\hbar c^2}{2\omega_k\Omega}\right)^{1/2}\sum_{JM}(-i)^J\sqrt{2\pi(2J+1)} \\ &\qquad\times[\hat{T}^{\text{el}}_{JM}(k)+\lambda\hat{T}^{\text{mag}}_{JM}(k)]\;\mathscr{D}^J_{M,-\lambda}(-\phi_k,-\theta_k,\phi_k)\end{aligned} \tag{9.35}$$

The Wigner–Eckart theorem in Eq. (9.22) now permits one to extract all the angular momentum selection rules and M-dependence of the matrix element in Eq. (9.34). All M's now refer to a common set of coordinate axes.[6]

[6] These axes were originally the photon axes with the z-axis along $\mathbf{k}$, but they can now just as well be the nuclear $\{x, y, z\}$ axes in Fig. 9.4; the equivalence of these two interpretations is readily demonstrated by taking out the M-dependence in a C–G coefficient with the aid of the Wigner–Eckart theorem — it is the same in both cases. The two interpretations differ only by an *overall* rotation (with $\hat{R}^{-1}\hat{R}$ inserted everywhere), which leaves the physics unchanged.

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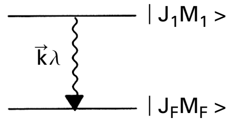

Fig. 9.5. Nuclear transition with real photon emission.

The final $\mathscr{D}^J_{M,-\lambda}$ in Eq. (9.35) plays the role of a "photon wave function," since the square of this quantity gives the intensity distribution in (θ_k, ϕ_k) of electromagnetic radiation carrying off $\{J, -M, \lambda\}$ from the target.

We proceed to calculate the *transition rate* for the process indicated in Fig. 9.5. The total transition rate for an unoriented nucleus is given by the Golden Rule

$$\omega = \frac{2\pi}{\hbar} \sum_f \overline{\sum_i} |\langle J_f M_f ; \mathbf{k}\lambda | H' | J_i M_i \rangle|^2 \delta(E_f + \omega_k - E_i) \tag{9.36}$$

The appropriate sum over final states is given by

$$\sum_f = \frac{\Omega}{(2\pi)^3} \sum_\lambda \sum_{M_f} \int d^3k \tag{9.37}$$

The $\int dk$ allows one to integrate over the energy-conserving delta function $\int dk\, \delta(E_f + \omega_k - E_i) = 1/\hbar c$. The integral over final solid angles of the photon $\int d\Omega_k$ can be performed with the aid of the orthogonality properties of the rotation matrices [Ed74]

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, \mathscr{D}^J_{M,-\lambda}(-\phi, -\theta, \phi)^* \mathscr{D}^{J'}_{M',-\lambda}(-\phi, -\theta, \phi)$$
$$= \frac{4\pi}{2J+1} \delta_{JJ'} \delta_{MM'} \tag{9.38}$$

Note that since λ is the same in both functions, the dependence on the last ϕ (which was the phase convention adopted for the third Euler angle $-\gamma$ in Fig. 9.2) drops out of this expression, as advertised.

The average over initial nuclear states is performed according to $\overline{\sum_i} = (2J_i + 1)^{-1} \sum_{M_i}$. The use of the Wigner–Eckart theorem in Eq. (9.22) and the orthonormality of the C–G coefficients then permits one to perform the required sums over M_f and M_i

$$\sum_{M_f} \sum_{M_i} |\langle J_f M_f J_i - M_i | J_f J_i J M \rangle|^2 = 1 \tag{9.39}$$

The final sum on M gives $\sum_M = 2J + 1$.

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Since the matrix element of one or the other multipoles must vanish by conservation of parity, assumed to hold for the strong and electromagnetic interactions, it follows that

$$|\langle J_f||\hat{T}_J^{\rm el} + \lambda \hat{T}_J^{\rm mag}||J_i\rangle|^2 = |\langle J_f||\hat{T}_J^{\rm el}||J_i\rangle|^2 + |\langle J_f||\hat{T}_J^{\rm mag}||J_i\rangle|^2 \tag{9.40}$$

This expression is now independent of λ, and the sum over final photon polarizations gives $\sum_\lambda = 2$.

A combination of these results yields the total photon transition rate for the process illustrated in Fig. 9.5

$$\omega_{fi} = 8\pi\alpha\, kc\, \frac{1}{2J_i+1} \sum_{J\geq 1} \left\{ |\langle J_f||\hat{T}_J^{\rm el}(k)||J_i\rangle|^2 + |\langle J_f||\hat{T}_J^{\rm mag}(k)||J_i\rangle|^2 \right\} \tag{9.41}$$

The multipole operators are now dimensionless. Equation (9.41) is a very general result. It holds for any localized quantum mechanical system. All that has been assumed about the target is that there is a local electromagnetic current operator. For most nuclear transitions of interest involving real photons, the wavelength is large compared to the size of the nucleus. It is thus important to consider the *long-wavelength reduction* of the multipole operators. This informative analysis is somewhat technical, and in order to not break the thread of the present development, we relegate the details to appendix A.[7]

[7] We leave it as an exercise for the reader to demonstrate that the inclusion of target recoil in the density of final states leads to an additional factor of r on the right side of Eq. (9.41) where $r^{-1} = 1 + \hbar k/M_T c$.