DIASSOCIATIVE GROUPOIDS ARE NOT FINITELY BASED

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In [2] Evans and Neumann raised the question of whether diassociativity in groupoids or loops is equivalent to a finite set of identities and in [3], Neumann still lists the problem as unsolved. It is the purpose of this paper to show that the answer to the question for groupoids is negative.

A groupoid is diassociative if every two generator subgroupoid is associative. The collection of two variable consequences of the associative law define this variety. We use a technique similar to that given by Evans in [1]: Let F be the free groupoid on a and b. Let K be any collection of identities $u(x, y) = v(x, y)$. If a word $w(a, b)$ in F contains a subword, $u(t_1(a, b), t_2(a, b))$, where $u(x, y) = v(x, y)$ is in K, then replacing

$$u(t_1(a, b), t_2(a, b))$$

by

$$v(t_1(a, b), t_2(a, b))$$

is called a K-transformation of $w(a, b)$. Words $w_1(a, b)$ and $w_2(a, b)$ are K-equivalent if there is a sequence of K-transformations $w_1 \to \cdots \to w_2$ from w_1 to w_2. Then w_1 and w_2 are K-equivalent if and only if

$$w_1(x, y) = w_2(x, y)$$

is a consequence of the identities K.

Now let I be any collection of diassociative identities in x and y. Since free semigroups are diassociative, the two sides of any diassociative identity can differ only in the placement of parentheses. Let n be larger than the length of either side of any identity in I. If $w(a, b)$ is an unassociated, or partially associated, finite string of a's and b's, by 'a word of the form $w(a, b)$' we mean a groupoid word constructed by inserting additional parentheses in $w(a, b)$. Let J denote the collection of all diassociative identities in x and y. Then any two words of the forms $(ab^n a)(b)$ and $(ab^n)(ab)$ are J-equivalent. To show that I is not a basis for the variety, we will show that any I-transformation of a word of the form $(ab^n a)(b)$ is again of this form.

Let $u(x, y) = v(x, y)$ be in I where $u(t_1, t_2)$ is a subword of some word of the form $(ab^n a)(b)$. If $u(t_1, t_2) = b$, $v(t_1, t_2)$ must be b, and the replacement of u by v does not change the word. If $u(t_1, t_2)$ is a subword of $ab^n a$, replacing u by v changes the parentheses in the first component but preserves the form $(ab^n a)(b)$. Finally, suppose $u(t_1, t_2) = (ab^n a)(b)$. If $u(x, y) = x$ (or y), then $v(x, y)$ must be $x(y)$ and replacing u by v does not change the word. Suppose that u is a product:

$$u(x, y) = u_1(x, y) \cdot u_2(x, y).$$

Then

$$u(t_1, t_2) = u_1(t_1, t_2) \cdot u_2(t_1, t_2) = (ab^n a)(b)$$

so that $u_1(t_1, t_2) = ab^n a$ and $u_2(t_1, t_2) = b$. Since F is free on $\{a, b\}$, we conclude that $u_2(x, y) = y$ (or analogously, x) where $t_2(a, b) = b$. Then

$$u_1(t_1, t_2) = u_1(t_1, b) = ab^n a.$$

This can be true only if x occurs at both ends of the word $u_1(x, y)$ and a occurs at both ends of the word $t_1(a, b)$. Since $t_1(a, b)$ is a subword of $ab^n a$, it must be either a or $ab^n a$. If $t_1 = a$, $u_1(a, b) = ab^n a$ so that $u_1(x, y) = xy^n x$. Then $u(x, y) = xy^n x \cdot y$, which has length greater than n. If $t_1 = ab^n a$, then $u_1(x, y)$ must be x, and

$$u(x, y) = u_1(x, y) \cdot u_2(x, y) = x \cdot y.$$

Since $u(x, y) = v(x, y)$ holds in the variety, $v(x, y)$ must be $x \cdot y$ and replacing u by v does not change the word.

THEOREM. *No finite collection of groupoid identities is a basis for the variety of diassociative groupoids.*

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References

[1] Trevor Evans, 'The number of semigroup varieties', *Quart. J. Math. Oxford* (2), **19** (1968), 335—336.

[2] Trevor Evans and B. H. Neumann, 'Varieties of groupoids and loops', *J. London Math. Soc.* 28 (1953), 342—350.

[3] B. H. Neumann, *Special Topics in Algebra: Universal Algebra.* Lecture notes prepared by Peter M. Neumann. Courant Institute of Mathematical Sciences, New York University, 1962.

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