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APPLICATIONS OF CONVERGENCE SPACES

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Convergence notions are used extensively in the areas of probability and statistics. Many times proofs can be simplified by considering an appropriate convergence structure on the space and using well-known results from the theory of convergence spaces; for example, compactness arguments are sometimes simplified by using a generalized Ascoli theorem in the convergence space setting. The theory of convergence spaces is also used to generalize some results in probability and statistics.

0. Preliminaries

Many of the convergence notions studied in probability and statistics are sequential and are sometimes not determined by a topology. Even if they are described by a topology, it is sometimes more convenient to study these notions in the setting of a convergence space. Convergence spaces given here are defined sequentially. Several authors have recently studied convergence spaces from a filter point of view. The necessary definitions and terminology are given below; however, the reader is referred to Novák [11] for further details concerning sequential convergence spaces and Kent [8] for results in the filter setting.

A *convergence structure* on a set X satisfies the following axioms:

(1) $x_n \to x$ whenever $x_n = x$, $n \geq 1$;

(2) $x_{n_k} \to x$ whenever $x_n \to x$;

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(3) $x_n \to x$, $x_n \to y$ implies that $x = y$.

The arrow denotes convergence. A convergence structure merely specifies the convergent sequences and limits. The set X along with the convergence structure is called a *convergence space*. These ideas date back to Fréchet [6].

A point x belongs to the closure of a subset A of X provided there is a sequence in A which converges to x . The set of all points of closure of A is denoted by $\text{cl } A$. The closure operator in a convergence space is in general not idempotent. A subset A of X is called *closed* whenever $\text{cl } A = A$. The set of all closed subsets of a convergence space form the closed subsets for a topology on X . This associated topological space is denoted by λX . A function $f : X \to Y$ between two convergence spaces is said to be *continuous* if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X . It is not difficult to show in this case that $f : \lambda X \to \lambda Y$ is also continuous.

A convergence space is called *separable* whenever it contains a countable subset A such that $\text{cl } A = X$. A subset A is called *relatively compact* whenever each sequence in A has a convergent subsequence and *compact* whenever A is closed and relatively compact.

Let $C(X)$ $(C^*(X))$ denote the set of all continuous (bounded continuous) real-valued functions from the convergence space X into the real line R . Consider the following convergence structure on $C(X)$: $f_n \to f$ iff $f_n(x_n) \to f(x)$ in R whenever $x_n \to x$ in X . For reasons given later, a different convergence structure is defined on $C^*(X)$: $f_n \to f$ iff $f_n(x_n) \to f(x)$ in R whenever $x_n \to x$ in X and also (f_n) is uniformly bounded on X . These two convergence spaces are denoted by $C_c(X)$ and $C_c^*(X)$. This type of convergence is sometimes called continuous convergence. It seems to have been first studied by Hahn [7] and later by Cook and Fischer [4] in the filter setting. A subset H of $C_c(X)$ is called *equicontinuous* whenever $f(x_n) \to f(x)$ uniformly in $f \in H$, provided $x_n \to x$ in X .

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1. The space of probability measures

The following generalizes the Ascoli theorem to the convergence space setting. The proof is omitted since it resembles the usual one for the separable metric space case.

PROPOSITION 1.1. *Let X be a separable convergence space and let H be a subset of $C_c(X)$. Then H is relatively compact in $C_c(X)$ iff*

(1) *$H(x)$ is a bounded subset of R for each $x \in X$,*

(2) *H is equicontinuous.*

Let X be a metric space and let $\mathcal{B}$ denote the Borel σ-field; that is, the smallest σ-field containing the open subsets of X. *The metric space X is considered to be a convergence space, where the convergent sequences coincide with those determined by the metric.* It would be more accurate to say that X is a metrizable convergence space. Let $M(X)$ denote the set of all probability measures defined on $(X, \mathcal{B})$. The convergence structure on $M(X)$ is defined as follows: $P_n \to P$ iff $\int f_n dP_n \to \int f dP$ whenever $f_n \to f$ in $C_c^*(X)$. It can be shown that this satisfies the axioms for a convergence structure, which is the reason for requiring convergent sequences in $C_c^*(X)$ to be uniformly bounded. Let the set $M(X)$ with this convergence structure be denoted by $M_c(X)$. Note that the map $\omega : M_c(X) \times C_c^*(X) \to R$, defined by $\omega(P, f) = \int f dP$, is jointly continuous. Another desirable property is the following.

PROPOSITION 1.2. *Let X be a metric space. Then the map $\phi : M_c(X) \to C_c\left(C_c^*(X)\right)$, defined by $\phi(P)(f) = \int f dP$, is a closed embedding.*

Proof. If $P \in M(X)$, then by the dominated convergence theorem, $\phi(P) : C_c^*(X) \to R$ is continuous, and so $\phi(P) \in C\left(C_c^*(X)\right)$. Let $\phi(P) = \phi(Q)$; then $\int f dP = \int f dQ$ for each $f \in C^*(X)$ and it follows (for example, see Billingsley [2], Theorem 1.3) that $P = Q$. Hence ϕ is one-to-one. The continuity of ϕ and ϕ^{-1} follows easily from the definitions and so ϕ is an embedding.

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Suppose that $\phi(P_n) \to h$ in $C_c(C_c^*(X))$. Let (f_n) be a sequence in $C^*(X)$ such that $f_n \downarrow 0$; then by a theorem of Dini, $f_n \to 0$ in $C_c^*(X)$, and since h is continuous, $h(f_n) \to h(0) = 0$ in R . Hence by the Daniell representation theorem (for example, see Ash [1], 4.2.9, p. 175) there is a $P \in M(X)$ such that $\phi(P) = h$ and so the range of ϕ is a closed subspace of $C_c(C_c^*(X))$. //

A study of continuous convergence in $C(X)$ in the filter setting is given by Feldman [5], for X a convergence space in the filter sense; however, one must be careful in translating these results to the sequential setting. The following can be deduced from Feldman [5, Theorem 3]. Let X be a separable metric space; then $C_c(X)$ is a separable convergence space. It can be shown from this that $C_c^*(X)$ is also a separable convergence space.

Recall that a subset H of $M(X)$ is called *tight* if for each $\varepsilon > 0$ there is a compact subset K of X such that $P(K) > 1 - \varepsilon$ whenever $P \in H$. Proposition 1.1 provides a straightforward method for proving compactness for many spaces in probability and statistics.

PROPOSITION 1.3 (Prohorov). *Let X be a separable metric space and let H be a tight subset of $M(X)$. Then H is a relatively compact subset of $M_c(X)$.*

Proof. From Proposition 1.2, it suffices to show that $\phi(H)$ is a relatively compact subset of $C_c(C_c^*(X))$. Let $f \in C^*(X)$; then $\phi(H)(f)$ is a bounded subset of R . Suppose that $f_n \to f$ in $C_c^*(X)$ and $|f_n| \le M$, $n \ge 1$. Given $\varepsilon > 0$, choose a compact subset K of X such that $P(X-K) < \varepsilon/4M$ for each $P \in H$. Since $f_n \to f$ uniformly on K , then let n_0 be such that $|f_n(x)-f(x)| < \varepsilon/2$ for each $x \in K$, $n \ge n_0$. Then

$$\left|\int f_n dP - \int f dP\right| \le \int_K |f_n - f| dP + \int_{X-K} |f_n - f| dP < \varepsilon$$

for each $P \in H$, $n \ge n_0$. It follows that $\phi(H)$ is an equicontinuous subset of $C_c(C_c^*(X))$. From the previously mentioned result that $C_c^*(X)$

is a separable convergence space, it follows by Proposition 1.1 that $\phi(H)$ is relatively compact. //

Let $M_\omega X$ denote the usual weak convergence on $M(X)$; that is, $P_n \to P$ iff $\int f dP_n \to \int f dP$ for each $f \in C^*(X)$.

PROPOSITION 1.4. *Let X be a metric space; then $M_\omega X = M_c X$.*

Proof. Certainly continuous convergence implies weak convergence. Conversely, let $P_n \to P$ in $M_\omega X$ and let $f_n \to f$ in $C^*(X)$ with $|f_n| \le M$. Let $Q_n = P_n f_n^{-1}$ and $Q = Pf^{-1}$ be the induced probability measures on $(R, \mathcal{B})$. Then from Billingsley [2, Theorem 5.5, p. 34], $Q_n \to Q$ in $M_\omega R$. Since (f_n) is uniformly bounded, then the identity function on R is uniformly integrable re (Q_n) , so $\int x dQ_n \to \int s dQ$ (for example, see Loève [10, p. 183]). Hence $\int f_n dP_n \to \int f dP$, so $P_n \to P$ in $M_c(X)$. //

2. The space of test functions

The domain for a test in statistics is generally taken to be a Euclidean subspace; however, our discussion is relative to the metric space setting. Let X denote a metric space and $\mathcal{B}$ the σ-field generated by all the open subsets of X . Let T denote the set of all tests on $(X, \mathcal{B})$; that is, $\phi \in T$ provided $\phi : X \to [0, 1]$ is a measurable function on $(X, \mathcal{B})$.

Let μ be a σ-finite measure on $(X, \mathcal{B})$ and let Y denote the set of all μ-integrable functions on $(X, \mathcal{B})$. Define convergence in Y to be L'-convergence; that is, $f_n \to f$ iff $\int |f_n - f| d\mu \to 0$. More precisely, points in Y are equivalence classes and Y is a metric space. Define the following convergence structure in T : $\phi_n \to \phi$ iff $\int \phi_n f d\mu \to \int \phi f d\mu$ for each f in Y . Let T with this convergence structure be denoted by T_μ . The space T_μ is compact whenever X is separable (for example, see Lehmann [9, Theorem 3, p. 354]) and forms the

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basis for proving the existance of certain optimal test in statistics. Proposition 1.1 provides a straightforward alternative method for showing that T_μ is compact. As in the case for Y, points in T_μ are actually equivalence classes; that is, $\phi \sim \psi$ iff $\phi = \psi$ almost everywhere $[\mu]$, or equivalently, $\int \phi f d\mu = \int \psi f d\mu$ for each $f \in Y$.

PROPOSITION 2.1. *Let X be a metric space and μ a σ-finite measure on $(X, \mathcal{B})$. The map $\alpha : T_\mu \to C_c(Y)$, defined by $\alpha(\phi)(f) = \int f\phi d\mu$, is a closed embedding.*

Proof. The fact that α is an embedding is routine to check. Suppose that $\alpha(\phi_n) \to h$ in $C_c(Y)$; then $h(f) = \lim_n \int f\phi_n d\mu$ for each $f \in Y$. Hence h is a bounded linear functional on Y and from the Riesz representation theorem there is a $g \in L^\infty$ such that $h(f) = \int fg d\mu$ for each $f \in Y$. It is easy to show that $0 \le g \le 1$ almost everywhere $[\mu]$, so $g \in T_\mu$ and hence $h = \alpha(g)$. Thus $\alpha(T_\mu)$ is a closed subspace of $C_c(Y)$. //

A straightforward application of Propositions 1.1 and 2.1 gives an alternative proof of the following result.

PROPOSITION 2.2. *Let X be a separable metric space and μ a σ-finite measure on $(X, \mathcal{B})$. Then the space T_μ is compact.*

Let us consider an appropriate test space whenever $P \subset M(X)$. Again let T denote the set of all tests on $(X, \mathcal{B})$ and define $\phi \sim \psi$ iff $\int \phi dP = \int \psi dP$ for each $P \in \mathcal{P}$. This is an equivalence relation in T and two tests are equivalent whenever they have the same power functions on $\mathcal{P}$. The set of equivalence classes with the following convergence structure is denoted by $T_\mathcal{P}$: $\phi_n \to \phi$ iff for each $P \in \mathcal{P}$, $\int \phi_n dP \to \int \phi dP$. If μ is a σ-finite measure on $(X, \mathcal{B})$ and $\mathcal{P} = \{P \in M(X) \mid P \ll \mu\}$, then it is not difficult to show that $T_\mu = T_\mathcal{P}$. Hence $T_\mathcal{P}$ seems to be a proper generalization of T_μ. The following fact

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about T_μ is useful in investigating properties of T_P .

PROPOSITION 2.3. *Let X be a separable metric space and μ a σ-finite measure on $(X, \mathcal{B})$. Then the convergence space T_μ is metrizable.*

Proof. Since X is a separable metric space, then Y , defined above, is also separable. Let $A = \{g_1, g_2, \ldots\}$ be a countable dense subset of Y and let R^∞ denote a countably infinite product of R with convergence structure of pointwise convergence of sequences. Define $\alpha : T_\mu \to R^\infty$ by $\alpha(\phi)(n) = \int \phi g_n d\mu$, $n \geq 1$. Then it is easy to show that α is an embedding and hence T_μ is metrizable. //

The notation $P \ll \mu$ denotes the fact that P is dominated by the σ-finite measure μ . Suppose that (ϕ_n) fails to converge to ϕ in T_P . Then there is a $P \in P$ such that $\int \phi_n dP \not\to \int \phi dP$. Let $h : T_P \to R$ be defined by $h(\psi) = \int \psi dP$. Then h is continuous and $h(\phi_n) \not\to h(\phi)$ in R . This implies that the convergent sequences of the completely regular topology on T_P generated by the continuous functions coincide with the convergent sequences in T_P . In the language of Novák [11], T_P is a *sequentially regular* convergence space. In particular, T_P and λT_P have the same convergent sequences, where λT_P is the topological space whose closed sets are precisely those closed in T_P ; that is, $\mathrm{cl}_{T_P} A = A$.

PROPOSITION 2.4. *Let X be a separable metric space. If $P \subset M(X)$ and $P \ll \mu$, then T_P is compact and metrizable.*

Proof. Since each equivalence class re μ is contained in the corresponding equivalence class re P , then let $j : T_\mu \to T_P$ denote the natural map. Since j is continuous, then by Proposition 2.2, T_P is compact. Also $j : \lambda T_\mu \to \lambda T_P$ is continuous and moreover since T_μ is compact and metrizable it follows easily that $j : \lambda T_\mu \to \lambda T_P$ is a

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topological quotient map. It is known that a Hausdorff quotient of a compact metrizable space is metrizable (for example, see Bourbaki [3, Proposition 17, p. 159]) and since T_p and λT_p agree on convergent sequences, T_p is metrizable. //

Let X, Y be metric spaces with corresponding Borel σ-fields and let $\tau : X \to Y$ be a measurable function. If $P \subset M(X)$, then let T_X denote the test space re P and let T_Y be the test space re $P\tau^{-1}$.

PROPOSITION 2.5. *Let* $\tau : X \to Y$ *be a measurable function, where* X *and* Y *are metric spaces. Let* $P \subset M(X)$ *and let* $\alpha : T_Y \to T_X$ *be defined by* $\alpha(\psi) = \psi \circ \tau$. *Then* α *is an embedding and moreover, if* τ *is a sufficient statistic for* P, *then* α *is an onto embedding.*

Proof. Suppose that $\alpha(\psi_1) = \alpha(\psi_2)$ in T_X; that is $\int \psi_1 \circ \tau dP = \int \psi_2 \circ \tau dP$ for $P \in P$. Then $\int \psi_1 dQ = \int \psi_2 dQ$ for each $Q = P\tau^{-1}$, so $\psi_1 = \psi_2$ in T_Y; that is, ψ_1 and ψ_2 belong to the same equivalence class. Hence α is one-to-one. The continuity of α and α^{-1} are easy to show and so α is an embedding. Suppose that τ is a sufficient statistic for P and let $\phi \in T_X$. Let $\psi = E(\phi|\tau)$; then $\psi \in T_Y$. Furthermore, for $Q = P\tau^{-1}$, $P \in P$,

$$\int (\psi \circ \tau) dP = \int \psi dQ = \int \phi dP$$

and hence it follows that $\psi \circ \tau = \phi$ in T_X. Thus $\alpha(\psi) = \phi$ and so α is an onto embedding. //

Let us consider a convergence structure for $P \subset M(X)$ which has properties desirable for hypothesis testing. It seems desirable to have joint continuity of the map $\omega : T_p \times P \to [0, 1]$ defined by $\omega(\phi, P) = \int \phi dP$. This, in particular, implies that the power function for each test is continuous on P. This leads us to define the following convergence structure on P: $P_n \to P$ iff for each $\phi_n \to \phi$ in T_p,

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$\int \phi_n dP_n \to \int \phi dP$. The set P equipped with this convergence structure is denoted by P_c . In fact, this is the coarsest convergence structure on P such that the map ω is jointly continuous.

Let $\tau : X \to Y$ be a measurable function between two metric spaces and let $P \subset M(X)$. Since $P\tau^{-1} \subset M(Y)$, then convergence in $P\tau^{-1}$ is defined in a similar manner as is given for $P \subset M(X)$. Let $P\tau^{-1}$ with this convergence structure be denoted by P_c^* . If $\alpha : P_c \to P_c^*$ denotes the map $\alpha(P) = P\tau^{-1}$, then it follows easily that α is continuous.

PROPOSITION 2.6. *Let* $\tau : X \to Y$ *be a sufficient statistic for* $P \subset M(X)$ *, where* X *and* Y *are metric spaces. Then* $\alpha : P_c \to P_c^*$ *, defined above, is an onto embedding.*

Proof. Let $P_1, P_2 \in P$ such that $Q_1 = \alpha(P_1) = \alpha(P_2) = Q_2$. If $A \in B$, then $\phi = 1_A \in T_X$ and so $\psi = E(\phi|\tau) \in T_Y$. Hence

$$P_1(A) = \int \phi dP_1 = \int \psi dQ_1 = \int \psi dQ_2 = \int \phi dP_2 = P_2(A) ,$$

so $P_1 = P_2$ and α is one-to-one. The continuity of α holds as mentioned above. Suppose that $Q_n = P_n\tau^{-1} \to P\tau^{-1} = Q$ in P_c^* and let $\phi_n \to \phi$ in T_X . If $\psi_n = E(\phi_n|\tau)$ and $\psi = E(\phi|\tau)$, then it follows easily that $\psi_n \to \psi$ in T_Y and hence $\int \psi_n dQ_n \to \int \psi dQ$, or $\int \phi_n dP_n \to \int \phi dP$. Hence $P_n \to P$ in P_c and so α is an embedding. //

Let $\tau : X \to Y$ be a measurable function and $\alpha : P_c \to P_c^*$ defined as above. If $\psi \in T_Y$ and is δ-similar on the boundary for testing $P_0\tau^{-1}$ vs $P_1\tau^{-1}$, where $P_0 \cup P_1 = P$ and $P_0 \cap P_1 = \phi$, then $\phi = \psi \circ \tau$ is also δ-similar on the boundary for testing P_0 vs P_1 . The boundary for testing P_0 vs P_1 is defined to be the set $\mathrm{cl}\, P_0 \cap \mathrm{cl}\, P_1$ in the convergence space P_c . The above follows from the continuity of the map

$\alpha : P_c \to P_c^*$. Moreover, since each power function $\beta_\phi : P_c \to [0, 1]$ is continuous, then from Lehmann [9, Lemma 1, p. 126] a UMP δ-similar on the boundary test of size δ is UMP unibased of size δ .

The next result follows from Proposition 2.6.

COROLLARY 2.7. *Let* $\tau : X \to Y$ *be a sufficient statistic for* $P \subset M(X)$ *, where* X *and* Y *are metric spaces. Let* ψ *be a UMP* δ*-similar on the boundary test of size* δ *for testing* $P_0\tau^{-1}$ vs $P_1\tau^{-1}$. *Then* $\psi \circ \tau$ *is a UMP* δ*-similar on the boundary test of size* δ *for testing* P_0 vs P_1 .

3. The space P_c

Let X be a metric space, B the corresponding Borel σ-field, and $P \subset M(X)$. Then for $P, Q \in P$, $d(P, Q) = \sup_{A \in B} |P(A)-Q(A)|$ defines a metric on P . Let P equipped with the convergence structure determined by the metric be denoted by P_d . Note that $P_n \to P$ in P_d iff $P_n(A) \to P(A)$ uniformly in $A \in B$. Lehmann [9, p. 352] shows that whenever $P << \mu$, $P_n \to P$ in P_d iff $\int |p_n - p| d\mu \to 0$, where p_n, p are the probability densities re μ . Let P_ω denote the subspace inherited from $M_\omega X$, where ω denotes weak convergence of probability measures.

PROPOSITION 3.1. *Let* X *be a metric space and* $P \subset M(X)$. *Then* $P_d \geq P_c \geq P_\omega$ *and when* T_p *compact,* $P_d = P_c$ *; in particular,* $P_d = P_c$ *whenever* $P << \mu$ *and* X *is separable.*

Proof. Suppose that $P_n \to P$ in P_d and $\phi_n \to \phi$ in T_p . Let μ be a σ-finite measure which dominates P_n, P , all $n \geq 1$. If p_n, p are densities re μ , then by the above remark $\int |p_n - p| d\mu \to 0$. It follows easily by using the triangle inequality that $\int \phi_n dP_n \to \int \phi dP$, so $P_n \to P$ in P_c and $P_d \geq P_c$.

Let $P_n \to P$ in P_c and $f \in C^*(X)$, $|f| \leq M$. Then $\int f dP_n \to \int f dP$

follows by decomposing f into f^+ and f^-, so $P_n \to P$ in P_ω and $P_c \geq P_\omega$.

Suppose that T_p is compact and $P_n \to P$ in P_c. If there exists an $\varepsilon > 0$ and a sequence (A_n) in $\mathcal{B}$ such that $|P_n(A_n)-P(A_n)| \geq \varepsilon$ for each $n \geq 1$, then let $\phi_n = 1_{A_n}$, so $\phi_{n_k} \to \phi$ in T_p. Hence $\left|P_{n_k}(A_{n_k}) - \int \phi dP\right| < \varepsilon/2$ and $\left|P(A_{n_k}) - \int \phi dP\right| < \varepsilon/2$ for k sufficiently large. Thus, for k sufficiently large, $|P_{n_k}(A_{n_k})-P(A_{n_k})| < \varepsilon$, which contradicts the above. This argument shows that $P_n \to P$ in P_d and so $P_d = P_c$. The last part of the proposition follows from Proposition 2.4.//

PROPOSITION 3.2. *Let* X *be a separable metric space and let* $P \subset M(X)$. *Then the following are equivalent:*

(1) T_p *is separable and metrizable;*

(2) P_c *is separable;*

(3) $P \ll \mu$;

(4) T_p *is compact and metrizable;*

(5) λT_p *is a second countable topological space;*

(6) P_c *is separable and metrizable.*

Proof. (1) ⇒ (2). It is straightforward to verify that the map $\alpha : P_c \to C_c(T_p)$, defined by $\alpha(P)(\phi) = \int \phi dP$, is an embedding. Hence from Feldman [5, Theorems 3 and 5], it follows that P_c is separable.

(2) ⇒ (3). The argument given in Lehmann [9, p. 353] applies here.

(3) ⇒ (4). This follows from Proposition 2.4.

(4) ⇒ (5). It follows since λT_p is compact, metrizable and hence second countable.

(5) ⇒ (6). Since the map α above is an embedding, then from Feldman

[5, Theorems 3 and 5], P_c is separable. This implies from the above that $P << \mu$ and so by Proposition 3.1, $P_c = P_d$ is metrizable.

(6) ⇒ (1). Since $P << \mu$, then from Proposition 2.4, T_p is compact, metrizable and hence separable. //

Let X be a metric space, $P \subset M(X)$ and consider the problem of teating P_0 vs P_1 at level α, where $P_0 \cup P_1 = P$ and $P_0 \cap P_1 = \phi$. Lehmann [9, p. 340] defines the *envelope power function*, β^*, on P by $\beta^*(P) = \sup_{\phi \in T_\alpha} \int \phi dP$, where $T_\alpha = \{\phi \in T_p \mid \phi \text{ is a level } \alpha \text{ test}\}$. Nöte that T_α is a closed subspace of T_p and hence is compact whenever the latter is. The existence of many optimal tests in the dominated case is based on the fact that T_μ is compact, where $P << \mu$. Similar results may be expected to hold in the general case whenever T_p is assumed to be compact.

Let $\gamma : T_p \to C_c(P_c)$ be defined by $\gamma(\phi)(P) = \int \phi dP$. Then it follows easily that γ is an embedding. Hence T_α equicontinuous means that $\gamma(T_\alpha)$ is an equicontinuous subset of $C_c(P_c)$. Similarly, P_1 equicontinuous means that $\alpha(P_1)$ is an equicontinuous subset of $C_c(T_p)$.

PROPOSITION 3.3. *Let X be a metric space and $P \subset M(X)$. If T_α is equicontinuous, then β^* is a continuous function on P_c.*

Proof. Let $P_n \to P$ in P_c. Since

$$\sup_{\phi \in T_\alpha} \int \phi dP_n \leq \sup_{\phi \in T_\alpha} \left| \int \phi dP_n - \int \phi dP \right| + \sup_{\phi \in T_\alpha} \int \phi dP ,$$

then by symmetry, it follows that

$$|\beta^*(P_n) - \beta^*(P)| \leq \sup_{\phi \in T_\alpha} \left| \int \phi dP_n - \int \phi dP \right| .$$

Since T_α is equicontinuous, then it follows that $\beta^*(P_n) \to \beta^*(P)$. //

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Define the function $\delta : T_\alpha \to [0, 1]$ for the above testing problem by $\delta(\phi) = \sup_{P \in P_1} \left\{\beta^*(P) - \int \phi dP\right\}$. A test making δ a minimum is called *most stringent of level* α (for example, see Lehmann [9, p. 340]).

PROPOSITION 3.4. *Let* X *be a metric space and* $P \subset M(X)$. *If* P_1 *is equicontinuous, then* δ *is a continuous function on* T_α .

Proof. Let $\phi_n \to \phi$ in T_α . Since

$$\sup_{P \in P_1} \left\{\beta^*(P) - \int \phi dP\right\} \le \sup_{P \in P_1} \left\{\beta^*(P) - \int \phi_n dP\right\} + \sup_{P \in P_1} \left|\int \phi_n dP - \int \phi dP\right| ,$$

then by symmetry,

$$|\delta(\phi)-\delta(\phi_n)| \le \sup_{P \in P_1} \left|\int \phi_n dP - \int \phi dP\right| .$$

Since P_1 is equicontinuous, then it follows that $\delta(\phi_n) \to \delta(\phi)$. //

COROLLARY 3.5. *Let* X *be a metric space and* $P \subset M(X)$. *If* T_α *is compact and* P_1 *is equicontinuous, then there exists a most stringent test of level* α .

Moreover, it is not difficult to prove the following more general result.

PROPOSITION 3.6. *Let* X *be a metric space and* $P \subset M(X)$. *Suppose that* $P_1 = \text{cl}\, P_1^*$, *where* $P_1^* = \bigcup_{n=1}^{\infty} P^n$, $P^n \subset P^{n+1}$ *and* P^n *is equicontinuous, all* $n \ge 1$. *If* T_α *is compact, then there exists a most stringent test of level* α .

If $P \ll \mu$ and X is a separable metric space, then T_p is compact and P_c is separable and so Proposition 3.6 implies that a most stringent test of level α exists for this case.

4. Weakly uniformly integrable functions

The key to the proof of Proposition 1.4 is the fact that if $P_n \to P$ in $M_\omega X$ and g is a continuous real-valued function on X which is uniformly integrable re (P_n), then $\int g dP_n \to \int g dP$. This section is devoted to giving a converse of this result, which is sometimes useful in showing that a particular convergence structure on $M(X)$ coincides with weak convergence.

PROPOSITION 4.1. *Let X be a metric space and suppose that $P_n \to P$ in $M_\omega X$ and g is a continuous real-valued function on X which is P_n, P integrable for each $n \geq 1$. Then $\int g dP_n \to \int g dP$ iff*

$$\lim_{c\to\infty} \overline{\lim_{n\to\infty}} \left| \int_{[|g|\geq c]} g dP_n \right| = 0 .$$

Proof. Suppose that $\int g dP_n \to \int g dP$. Since g is P-integrable, for $\varepsilon > 0$, choose $c_0 > 0$ such that $\int_{[|g|\geq c_0]} |g| dP < \varepsilon$. Let $c \geq c_0$ and $I = [-c, c)$, $J = (-c, c]$.

Since $g \cdot 1_{g^{-1}(I)}$ is lower semicontinuous on X, then by Ash [1, Theorem 4.5.1, p. 196],

$$\underline{\lim}_n \int_{g^{-1}(I)} g dP_n \geq \int_{g^{-1}(I)} g dP .$$

Hence

$$\underline{\lim}_n \int_{g^{-1}(I)} g dP_n + \overline{\lim}_n \int_{X-g^{-1}(I)} g dP_n \leq \overline{\lim}_n \int g dP_n = \int g dP$$
$$= \int_{g^{-1}(I)} g dP + \int_{X-g^{-1}(I)} g dP \leq \underline{\lim}_n \int_{g^{-1}(I)} g dP_n + \int_{X-g^{-1}(I)} g dP$$

and so it follows that

$$\overline{\lim}_n \int_{X-g^{-1}(I)} g dP_n \leq \int_{X-g^{-1}(I)} g dP < \varepsilon .$$

Since

$$\int_{g^{-1}(-c)} g\,dP_n \le 0 ,$$

then it follows that

$$\overline{\lim_n} \int_{[|g| \ge c]} g\,dP_n < \varepsilon .$$

A similar argument using J shows that

$$\underline{\lim_n} \int_{[|g| \ge c]} g\,dP_n > -\varepsilon ;$$

hence

$$\overline{\lim_n} \left| \int_{[|g| \ge c]} g\,dP_n \right| < \varepsilon$$

for each $c \ge c_0$ and so

$$\lim_{c \to \infty} \overline{\lim_n} \left| \int_{[|g| \ge c]} g\,dP_n \right| = 0 .$$

The converse follows easily by using the transformation theorem in conjunction with the Helley-Bray lemma. //

Let us call a sequence (f_n) of continuous real-valued functions on a metric space X *weakly uniformly integrable* re a sequence of probability measures (P_n) on $M(X)$ if each f_n is P_n integrable, $n \ge 1$, and

$$\lim_{c \to \infty} \overline{\lim_{n \to \infty}} \left| \int_{[|f_n| \ge c]} f_n\,dP_n \right| = 0 .$$

From Proposition 4.1, along with the transformation theorem, the following result is obtained.

COROLLARY 4.2. *Let* X *be a metric space and let* (f_n) *be a sequence of continuous real-valued functions on* X . *Suppose that* $P_n \to P$ *in* $M_\omega X$ *and each* f_n *is* P_n*-integrable for each* $n \ge 1$. *If* $f_n \to f$ *in* $C_c(X)$, *then* $\int f_n\,dP_n \to \int f\,dP$ *iff* (f_n) *is weakly uniformly integrable*

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re (P_n) .

Proposition 1.4 could have been proved by using Corollary 4.2 since if $f_n \to f$ in $C_c^*(X)$, then (f_n) is uniformly bounded and hence uniformly integrable re (P_n) . One can easily give examples of weakly uniformly integrable sequences which are not uniformly integrable with respect to a sequence of probability measures.

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