Compositio Mathematica **127:** 109–116, 2001.

Torsion Packets on Curves

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(Received: 12 January 2000; accepted: 8 May 2000)

Abstract. Let X be a curve of genus $g \geqslant 2$ over a field of characteristic zero. Then X has at most finitely many torsion packets of size greater than 2. Moreover, X has infinitely many torsion packets of size 2 if and only if either $g = 2$, or $g = 3$ and X is both hyperelliptic and bielliptic.

Mathematics Subject Classifications (2000). Primary 14H40.

Key words. torsion point, torsion packet, Manin–Mumford conjecture, Abelian variety, curve, Jacobian

1. Raynaud's Theorem

Let X be an algebraic curve defined over an algebraically closed field K of characteristic zero. When we say that X is a curve, we require it to be smooth, proper, and irreducible over K. We will denote by J the Jacobian variety of X, and by g the genus of X.

We define an equivalence relation on $X(K)$ by defining $P \sim Q$ if and only if the divisor $m(P) - m(Q)$ on X is principal for some positive integer m. We call an equivalence class under $\sim$ a *torsion packet* on X. Clearly the torsion packet containing $P \in X(K)$ is the set of points of $X(K)$ which map to torsion points of J via the Albanese map from X to J sending $Q \in X(K)$ to the class of the divisor $(Q) - (P)$. A torsion packet is said to be *trivial* if it has only one element.

Recall that the Manin–Mumford conjecture (proved by M. Raynaud in [10]) states that if $g \geqslant 2$, then every torsion packet on X is finite. In fact, Raynaud [11] proved the following more general result about torsion points on subvarieties of Abelian varieties:

THEOREM 1. *Suppose A is an Abelian variety over the algebraically closed field K of characteristic zero, and that V is an irreducible closed subvariety of A. If the torsion*

⋆The first author was supported by a Sloan Dissertation Fellowship and an NSF Postdoctoral Research Fellowship. The second author was supported by NSF grant DMS-9801104, a Sloan Fellowship, and a Packard Fellowship.

points of A contained in V are Zariski-dense in V, then V is the translate of an Abelian subvariety of A by a torsion point.

Remark. Raynaud's theorem is equivalent to the following assertion: Let A and K be as in Theorem 1, and let V be an irreducible closed subvariety of A. Then the set of torsion points of A lying in V is contained in a finite union $\cup Z_j$, where each Z_j is contained in V and is a translate of a (possibly zero-dimensional) Abelian subvariety of A by a torsion point.

Our main result, Theorem 2, is concerned with the number and size of torsion packets on a curve X. In Section 2 we state the theorem, and in Section 3 we use Theorem 1 to prove it. In the final section, we show that the bounds must depend on the genus of X, and we describe a generalization along the lines of the Mordell–Lang and Bogomolov conjectures.

2. Bounds for Torsion Packets

We keep the notation of the previous section: in particular, X is an algebraic curve of genus g defined over the algebraically closed field K of characteristic zero.

If $g = 2$, then X has infinitely many nontrivial torsion packets, since the Riemann–Roch theorem implies that the subtraction map $\alpha_2: X \times X \to J$ given by $(P, Q) \mapsto [(P) - (Q)]$ is surjective.

In [4, Example (iv)], one finds the statement that if $g \geqslant 3$, then there are only finitely many nontrivial torsion packets on X. This statement is false, however: we will show later that if X has genus 3 and is both hyperelliptic and bielliptic, then there are infinitely many nontrivial torsion packets on X. The smooth projective model of $y^2 = x^8 + 1$ is an example of such a curve. (Recall that X is *hyperelliptic* if it admits a degree 2 map to $\mathbf{P}^1$, and is *bielliptic* if it admits a degree 2 map to an elliptic curve.)

Motivated by these examples, we will use Theorem 1 to prove the following result:

THEOREM 2. *Suppose $g \geqslant 2$.*

(1) *There are at most finitely many torsion packets of size greater than 2 on X.*
(2) *There are infinitely many nontrivial torsion packets on X if and only if either $g = 2$, or $g = 3$ and X is both hyperelliptic and bielliptic.*

Together with the Manin–Mumford conjecture, Theorem 2 implies:

COROLLARY 3. *There is a constant M (depending on X) such that every torsion packet on X has size at most M.*

Remark. The result of Corollary 3 is known to the experts, but we could not find in the literature an explicit statement with proof. It is stated without proof in [12], and

Raynaud [11, Proposition 9.1] gave a proof for $g > 2$. It follows from [3, Theorem 1], if one generalizes that theorem according the note 'added in proof' [3, p. 782] about the work on the Bogomolov conjecture by Szpiro, Ullmo, and Zhang [13–15].

Open question: Does the constant M in the statement of Corollary 3 depend only on the genus g of X? Mazur [7, p. 234] has asked the following more general question: can $\#(X(K) \cap \Gamma)$ be bounded in terms of g and r only, when X ranges over curves of genus g each embedded in its Jacobian J, and Γ ranges over subgroups of $J(K)$ with $\dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q}) \leqslant r$?

Another open question: Fix $g \geqslant 2$ and $s \geqslant 3$. Does there exist $M_{g,s} > 0$ such that for all curves X of genus g, the number of torsion packets on X of size at least s is bounded by $M_{g,s}$? There is not a single pair (g, s) for which we know the answer.

3. Proof of Theorem 2

Consider the family of proper maps $\alpha_n: X^n \to J^{n-1}$, $n \geqslant 2$, given by

$$(P_1, P_2, \dots, P_n) \mapsto (P_1 - P_2, P_2 - P_3, \dots, P_{n-1} - P_n).$$

These maps also played a prominent role in the proof of the generalized Bogomolov conjecture [15]. Let $V_n = \alpha_n(X^n)$, which is a closed subvariety of J^{n-1}. Let Δ_n denote the 'big diagonal' in X^n, i.e., the closed subscheme whose closed points are the n-tuples $(P_1, P_2, \dots, P_n)$ with $P_i = P_j$ for some $i \neq j$. Let $\Delta'_n = \alpha_n(\Delta_n) \subseteq V_n \subseteq J^{n-1}$. We could also characterize the closed points of Δ'_n as the closed points $(Q_1, \dots, Q_{n-1}) \in V_n$ such that $Q_i + Q_{i+1} + \dots + Q_j = 0$ for some $i \leqslant j$, so $\alpha_n^{-1}(\Delta'_n) = \Delta_n$.

The following proposition concerns the geometry of the maps α_n:

PROPOSITION 4. *Suppose $g \geqslant 2$.*

(1) *Suppose that either $n \geqslant 3$ or that $n = 2$ and X is not hyperelliptic. Then the restriction of α_n to $X^n - \Delta_n$ is an isomorphism onto its image.*
(2) *If $n = 2$ and X is hyperelliptic, then the restriction $\alpha_2: X^2 - \Delta_2 \to V_2 - \Delta'_2$ is finite of degree 2.*

Proof. We begin with the proof of part (1). By the 'inverse function theorem for varieties' (see Lemma 14.8 and Theorem 14.9 of [5]), it suffices to show that the restriction of α_n to $U := X^n - \Delta_n$ is injective on closed points and gives rise to an injection of tangent spaces at all closed points in the domain.

To see that the restriction of α_n is injective, let $P = (P_1, \dots, P_n)$ and $P' = (P'_1, \dots, P'_n)$ be distinct points of U, and note that if $\alpha_n(P) = \alpha_n(P')$ in J^{n-1}, then $(P_i) - (P_{i+1}) \sim (P'_i) - (P'_{i+1})$ for all $1 \leqslant i \leqslant n-1$, where $\sim$ denotes linear equivalence of divisors on X.

https://doi.org/10.1023/A:1017557230785 Published online by Cambridge University Press

It follows that $(P_i) - (P'_i) \sim (P_j) - (P'_j)$ for all $1 \leqslant i, j \leqslant n$. Since P is not in Δ_n and no two distinct points of X are linearly equivalent as divisors, we conclude that X is hyperelliptic, and that $P'_i = hP_j$ for all $1 \leqslant i \neq j \leqslant n$, where h is the hyperelliptic involution on X. Since we are assuming that P' is not in Δ_n, this forces n to equal 2, for otherwise we have $P'_3 = hP_2 = P'_1$, a contradiction.

It remains to show that $(d\alpha_n)_P$ is injective for all points $P \in U(K)$. It is enough to show that the map $(d\alpha_n)_P \colon T_P(X^n) \to T_0(J^{n-1})$ (by which we mean the natural map $T_P(X^n) \to T_{\alpha_n(P)}(J^{n-1})$ followed by a translation $\tau_* \colon T_{\alpha_n(P)}(J^{n-1}) \to T_0(J^{n-1})$) is injective.

Given a point $P = (P_1, \ldots, P_n)$ in $U(K)$, let $v = (v_1, \ldots, v_n)$ be a nonzero tangent vector to X^n at P, where each v_i is a tangent vector to X at P_i. Also, let t_i denote the image of v_i in $T_0(J)$ under $(d\iota)_P$ followed by a translation, where $\iota \colon X \to J$ is the Albanese map associated to an arbitrary base point $P_0 \in X(K)$. Then it is not hard to check that the image in $T_0(J^{n-1})$ of the tangent vector v is $(t_1 - t_2, \ldots, t_{n-1} - t_n)$.

The map $(d\alpha_n)_P$ fails to be injective if and only if $(d\alpha_n)_P(v) = 0$ for some nonzero tangent vector v to X^n at P, which happens if and only if there is some choice of tangent vectors v_i to X at P_i, not all zero, such that $t_1 = t_2 = \cdots = t_n$. In particular, if $(d\alpha_n)_P(v) = 0$ for some nonzero v, then the corresponding t_i are all nonzero and have the same image in the projectivization $\mathbf{P}W$ of $T_0(J) \cong H^0(X, \Omega^1_X)^\vee$. According to [6, Proposition 11.1.4], the map $\psi \colon X \to \mathbf{P}W$ which sends a point Q in $X(K)$ to the projectivized image under $(d\iota)_Q$ of a nonzero tangent vector v at Q is just the canonical map ψ from X to $\mathbf{P}W$. It follows that $(d\alpha_n)_P(v) = 0$ if and only if $\psi(P_1) = \psi(P_2) = \cdots = \psi(P_n)$. Since we are assuming that the P_i are pairwise distinct, this can happen only if X is hyperelliptic and $n = 2$, which proves part (1).

To prove part (2) of the proposition, suppose that $n = 2$ and X is hyperelliptic, and let h be the hyperelliptic involution on X. The above calculations show that the map $\alpha_2 \colon X^2 - \Delta_2 \to V_2 - \Delta'_2$ is quasifinite. It is also proper, since it is the base extension of $\alpha_2 \colon X^2 \to V_2$ by $V_2 - \Delta'_2 \hookrightarrow V_2$, so it is finite. Moreover, the tangent space calculation shows that if $\widetilde{\Delta}_2$ denotes the subvariety of X^2 whose closed points are pairs (P, hP), and if $\widetilde{\Delta}'_2$ is its image under α_2, then $\alpha_2 \colon X^2 - (\Delta_2 \cup \widetilde{\Delta}_2) \to V_2 - (\Delta'_2 \cup \widetilde{\Delta}'_2)$ is finite étale of degree 2. So the degree of the finite map $\alpha_2 \colon X^2 - \Delta_2 \to V_2 - \Delta'_2$ must be 2. □

The following lemmas will be used in the proof of Proposition 7.

LEMMA 5. *Let X be a curve of genus at least 2, and let B be an Abelian variety. Then there are no dominant rational maps from B to X.*

Proof. Let f be a dominant rational map from B to X. Choose an embedding $\iota \colon X \hookrightarrow J$ of X into its Jacobian, and let h be the composite rational map from B to J. According to [8, Theorem 3.1], the rational map h extends to a morphism from B to J. Since f is dominant, it follows that f extends to a surjective morphism (which we still denote by f) from B to X. Now let $P_0 = f(0)$, and let ι' be the Albanese map from X to J corresponding to the base point P_0. Let h' be the composite morphism from B to J. Then $h'(0) = 0$, so [8, Corollary 2.2] shows that h' is a homomorphism with image X. This forces X to be a subgroup of J, which is absurd, since X generates J (which has dimension at least 2) as a group. □

Remark. One can give also a complex-analytic proof of Lemma 5 using the hyperbolicity of curves of genus $g \geqslant 2$.

LEMMA 6. *If $f: X \to Y$ is a surjective map of curves with no nontrivial unramified subcover of Y, then the induced map $f^*: J(Y) \to J(X)$ on Jacobians is a closed immersion.*

Proof. If not, then since K is algebraically closed of characteristic zero, there exists a point $[D] \in \ker f^*$, where D is a nonprincipal divisor on Y. We have $f^*D = (g)$ for some $g \in K(X)$, and pushing the divisors forward to Y yields $nD = (h)$ where $n = \deg f$ and $h =_{K(X)/K(Y)} (g)$. Since h (viewed as function on X by composing with f) and g^n both have divisor $n(f^*D)$ on X, their ratio is constant. Hence, $K(X)$ contains an nth root of h. The subextension $K(Y)(h^{1/n})$ of $K(X)/K(Y)$ corresponds to a subcover of $X \to Y$. It is unramified over Y since the divisor (h) on Y is divisible by n. It is nontrivial since D is nonprincipal. This contradicts the hypothesis. □

We define a subvariety Y of an Abelian variety A to be a *torsion subvariety* if it is a translate of an Abelian subvariety of A by a torsion point.

PROPOSITION 7. *Let $n \geqslant 2$ be an integer, and let $V_n = \alpha_n(X^n) \subseteq J^{n-1}$. Let $\Delta_n' = \alpha_n(\Delta_n)$, where Δ_n is the big diagonal. Then the following are equivalent:*

(1) *V_n contains a translate of an Abelian subvariety of J^{n-1} of positive dimension which is not contained in Δ_n'.*
(2) *V_n contains a torsion subvariety of J^{n-1} of positive dimension which is not contained in Δ_n'.*
(3) *$n = 2$; and either $g = 2$, or $g = 3$ and X is both hyperelliptic and bielliptic.*

Remark. An induction argument combined with Proposition 7 could be used to classify completely the translates of Abelian subvarieties contained in V_n.

Proof. Clearly (2) ⇒ (1). We show next that (3) ⇒ (2).

If $n = 2$ and $g = 2$ then $V_2 = J^2$ is an Abelian surface. Now suppose $n = 2$ and X is both hyperelliptic and bielliptic. Let i be the elliptic involution, let h be the hyperelliptic involution, and let $f: X \to E$ be the degree 2 map corresponding to the quotient of X by i. We claim that V_2 contains the genus 1 curve E as a torsion subvariety.

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To see this, let S be a Weierstrass point of X; then i maps S to a Weierstrass point S', since the set of Weierstrass points is preserved by automorphisms. Without loss of generality, S maps to 0 in E. Then $f^*(0) = (S) + (S')$. Furthermore, i has fixed points by the Riemann–Hurwitz formula, so the Picard map $\psi: E \to J$, which on closed points is the homomorphism given by $\psi(P) = [f^*(P) - (S) - (S')]$, is a closed immersion by Lemma 6. Let η be the map ψ followed with translation by the 2-torsion point $[(S') - (S)]$, which on closed points satisfies $\eta(P) = [f^*(P) - 2(S)]$. This map from E to J is also a closed immersion. It suffices to show, therefore, that the image of η is contained in V_2. To do this, choose a closed point $P \in E$, and write $f^*(P) = (Q) + (Q')$. Then $\eta(P) = [f^*(P) - 2(S)] = [(Q) + (Q') - (Q') - (hQ')] = [(Q) - (hQ')]$, so $\eta(E) \subseteq \alpha_2(X \times X)$ as claimed.

We now prove that (1) $\Rightarrow$ (3). Suppose that V_n contains a translate B of an Abelian subvariety of J^{n-1}, and B is not contained in Δ'_n. Then $U := B - (B \cap \Delta'_n)$ is a dense open subscheme of B. If $n \geqslant 3$ or $n = 2$ and X is not hyperelliptic, then by Proposition 4, α_n is a birational map, and moreover the restriction of α_n to $X^n - \Delta_n$ is an isomorphism onto its image $V_n - \Delta'_n$. The composition

$$\beta: U \hookrightarrow V_n - \Delta'_n \overset{\alpha_n^{-1}}{\cong} X^n - \Delta_n$$

has infinite image, so some projection $\beta_i: U \to X$ also has infinite image. In other words, β_i is a dominant rational map from B to X. But Lemma 5 shows there are no such maps.

Therefore we may assume that $n = 2$ and X is hyperelliptic with hyperelliptic involution h. Embed X in J using a Weierstrass point S as basepoint. Then -1 on J restricts to h on X, so $\alpha_2(X^2) = W_2(X)$, the image of the symmetric square map $X^2 \to J$. If $\dim B \geqslant 2$, then $B \subseteq \alpha_2(X) = W_2(X)$ forces $B = W_2(X)$, but $W_2(X)$ generates J, and the translate B of an Abelian subvariety can generate J only if $B = J$, so $W_2(X) = J$ and $g = 2$. On the other hand, if $\dim B = 1$, then $W_2(X)$ contains a genus 1 curve B, so X is bielliptic by [1, Theorem 3], and a bielliptic hyperelliptic curve has genus at most 3, by the inequality of Castelnuovo and Severi (see [2, Exer. VIII.C–1]). □

Proof of Theorem 2. Let $n \geqslant 2$ be an integer, and as before let $V_n \subseteq J^{n-1}$ be the image of α_n. Given a closed point $P := (P_1, P_2, \ldots, P_n)$ in $X^n - \Delta_n$, $\alpha_n(P)$ is a torsion point of J^{n-1} if and only if the points $P_1, P_2, \ldots, P_n$ all lie in a common torsion packet on X. According to Proposition 4, the restriction of α_n to $X^n - \Delta_n$ is an isomorphism onto its image $V_n - \Delta'_n$, unless $n = 2$ and X is hyperelliptic, in which case the fibers of this map all have degree 2. Since each torsion packet on X is finite, it follows in all cases that there are infinitely many torsion packets of size at least n on X if and only if $V_n - \Delta'_n$ contains infinitely many torsion points of J^{n-1}. By Theorem 1, this occurs if and only if V_n contains a torsion subvariety of positive dimension not contained in Δ'_n. Parts (1) and (2) of Theorem 2 now follow by taking $n = 3$ and $n = 2$, respectively, in the statement of Proposition 7. □

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4. Generalizations and Nongeneralizations

Can one uniformly bound the number or size of torsion packets on a curve if the genus is unrestricted? Certainly not! Instead we have the following:

PROPOSITION 8. *For every $n \geqslant 1$, there exists a curve X of some genus $g \geqslant 2$ such that X has at least n torsion packets each of size at least n.*

Proof. Let E be an elliptic curve over K. Since K is algebraically closed of characteristic zero, E has infinitely many torsion packets, each of which is infinite. Let $S_1, \ldots, S_n$ denote subsets of size n contained in distinct torsion packets. Choose a divisor D on E of degree n^2 with support disjoint from $S_1 \cup \ldots \cup S_n$ such that $S_1 + \ldots + S_n - D$ is the divisor of a function f on E, i.e., sums to zero in the group law on E. Let $\pi: X \to E$ denote the double branched cover corresponding to the function field extension $K(X) = K(E)(\sqrt{f})$. For $1 \leqslant i \leqslant n$, let $T_i = \pi^{-1}(S_i)$, which is of size n since π ramifies above S_i.

If $t, t' \in T_i$, then for some $m \geqslant 1$, $m(\pi(t) - \pi(t'))$ equals the divisor of a function f on E, and $2m(t - t')$ is the divisor of $f \circ \pi$ on X. Hence T_i is contained in a torsion packet. On the other hand, if $t \in T_i$ and $t' \in T_j$ for $i \neq j$, and if $m(t - t')$ were the divisor of $g \in K(X)^*$ for some $m \geqslant 1$, then $m(\pi(t) - \pi(t'))$ would be the divisor of ${}_{K(X)/K(E)}(g) \in K(E)^*$, contradicting the fact that S_i and S_j are contained in distinct torsion packets. Thus $T_1, \ldots, T_n$ are contained in distinct torsion packets each of size at least n. □

If we repeat the proof of Theorem 2 but replace Raynaud's Theorem by a known generalization such as the 'Mordell–Lang conjecture,' the 'generalized Bogomolov conjecture,' or the combined theorem of [9], we immediately obtain a generalization of Theorem 2:

THEOREM 9. *Let X be a curve of genus $g \geqslant 2$ over $\bar{\mathbb{Q}}$. Embed X in its Jacobian J using some basepoint. Let Γ be a finite rank subgroup of $J(\bar{\mathbb{Q}})$ (e.g., the division group of some finitely generated subgroup). Let $h: J(\bar{\mathbb{Q}}) \to \mathbb{R}_{\geqslant 0}$ be the canonical height function associated to the theta divisor $\overline{W_{g-1}}(X) \subset J$, and for $\varepsilon > 0$, define $B_\varepsilon := \{a \in J(\bar{\mathbb{Q}}): h(a) < \varepsilon\}$. For each $x \in X(\bar{\mathbb{Q}})$, let*

$$T_x = T_{x,\Gamma,\varepsilon} := X(\bar{\mathbb{Q}}) \cap (x + \Gamma + B_\varepsilon)$$

be the set of points on X which are 'near x mod Γ.' Then there exists $\varepsilon > 0$ (depending on X and Γ) such that

(1) *There are only finitely many $x \in X(\bar{\mathbb{Q}})$ with $\# T_x > 2$.*
(2) *There are infinitely many $x \in X(\bar{\mathbb{Q}})$ with $\# T_x > 1$ if and only if either $g = 2$, or $g = 3$ and X is both hyperelliptic and bielliptic.*

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