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ON A PROBLEM OF RUBEL CONCERNING THE SET OF FUNCTIONS SATISFYING ALL THE ALGEBRAIC DIFFERENTIAL EQUATIONS SATISFIED BY A GIVEN FUNCTION

JOHN SHACKELL

ABSTRACT. For two functions f and g, define $g \ll f$ to mean that g satisfies every algebraic differential equation over the constants satisfied by f. The order $\ll$ was introduced in one of a set of problems on algebraic differential equations given by the late Lee Rubel. Here we characterise the set of g such that $g \ll f$, when f is a given Liouvillian function.

1. **Introduction.** One tiny part of the legacy to Mathematics of the late Lee Rubel is the following problem, which appears as part of Problem 22 in [2].

> *For two functions g and f, we define $g \ll f$ to mean that g satisfies every algebraic differential equation (over $\mathbb{C}$) which f satisfies. Discuss the order $\ll$; in particular, do this for the case when f is an exponential polynomial, $\sum_{k=1}^{n} a_k e^{\lambda_k x}$.*

In order to discuss the order $\ll$ when more general functions are involved, it is necessary to say something about the domains of definition of the functions to be considered. It is generally too restrictive to require g to have the same domain of definition as f. On the other hand, one would at least want a non-empty open subset of $\mathbb{C}$ on which both functions are defined. In fact for the functions we shall be considering, we shall generally be able to take that subset to be dense. We shall also want to use a topology on various sets of functions. Since the functions concerned will be Liouvillian, most natural choices of topology are likely to give the same answers. We shall use uniform convergence of the functions and their derivatives on compact subsets.

Although a description of the order $\ll$ is of interest for its own sake, there are also applications to asymptotics. If one searches for a series solution to a non-linear ordinary differential equation in terms of base functions $\{e_n(x), n = 1, 2, \ldots\}$, where e_n denote the n-times iterated exponential function, one would like to bound the possible n that might occur. Suppose we have a solution of the form $f(x) + \big(e_n(x)\big)^{-j} g(x)$ where f is a polynomial in $e_1(x), \ldots, e_{n-1}(x)$, $j > 0$ and g gives the tail of the series. If we do not have $f(x) + \big(e_n(x)\big)^{-j} g(x) \ll f$ there is a differential polynomial P which vanishes at f but not at $f(x) + \big(e_n(x)\big)^{-j} g(x)$. Then $P\langle f(x) + \big(e_n(x)\big)^{-j} g(x)\rangle$ tends to zero approximately as a fixed negative power of $e_n(x)$. If n is too large compared with the order of the differential

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equation, one can prove that this is impossible. Under suitable conditions, one can also prove that $f(x) + \big(e_n(x)\big)^{-j} g(x) \ll f$ is impossible, thereby bounding possible values of n. This has been done within the context of nested expansions in [3].

We begin Section 2 with a purely elementary consideration of the special case in which f is an exponential polynomial. Here we are able to give a completely explicit characterisation of the set of g for which $g \ll f$. Then we re-frame our results in terminology which is closer to differential algebra. This leads into Section 3, where we consider the case when f is an arbitrary Liouvillian function. Here we characterise the set of g for which $g \ll f$ as the closure of a set of explicitly given transformations of f. Our argument in this section has some similarities with that of Section 5 of [3].

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2. **Exponential polynomials.** The first thing to be said is that if $f = \sum_{k=1}^{n} a_k e^{\lambda_k x}$ and $g \ll f$, then $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$ for some $A_1, \ldots, A_n \in \mathbb{C}$. This is because f satisfies the linear differential equation $L\langle y\rangle = \{\prod_{k=1}^{n}(d/dx - \lambda_k)\} y = 0$. Here we have used the notation $L\langle y\rangle$ to indicate that L is a polynomial in y and its derivatives, where y is an indeterminate. However in many cases, it is not sufficient that g be of the form $\sum_{k=1}^{n} A_k e^{\lambda_k x}$.

We suppose that the λ_is are all different and that no a_i is zero. We write $\mathbb{Q}[\lambda_1, \ldots, \lambda_n]$ for the $\mathbb{Q}$-linear space generated by $\lambda_1, \ldots, \lambda_n$, and let d be its dimension. Now d is also equal to the degree of transcendence of $\mathbb{C}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ over $\mathbb{C}$, and hence f cannot satisfy an algebraic differential equation of order less than d.

Let g be of the form $\sum_{k=1}^{n} A_k e^{\lambda_k x}$, and suppose first that $d = n$. Then we have a linear differential equation of order d satisfied by f, namely $L\langle y\rangle = 0$. If f satisfies another algebraic differential equation of order n, say $P\langle y\rangle = 0$, then L must divide P. For otherwise the resultant, $\mathrm{res}_{y^{(n)}}(P, L)$, would be a non-zero differential polynomial of order less than n annulled by f. Hence g satisfies every differential equation of order n satisfied by f. Let $m \geq n$ and suppose inductively that g satisfies every differential equation of order m satisfied by f. Let $Q\langle y\rangle = 0$ be a differential equation of order $m+1$ satisfied by f. On differentiating $m-n+1$ times the equation $L\langle f\rangle = 0$, we obtain a linear expression for $f^{(m+1)}$ in terms of $f, \ldots, f^{(m)}$, say $f^{(m+1)} = X(f, \ldots, f^{(m)})$; note that g also satisfies this equation. On substituting $X(y, \ldots, y^{(m)})$ for $y^{(m+1)}$ in $Q\langle y\rangle = 0$, we get an equation of order m satisfied by f, and therefore by g. When we replace $X(g, \ldots, g^{(m)})$ where it occurs in this last equation by $g^{(m+1)}$, we see that also $Q\langle g\rangle = 0$. Thus g satisfies every equation of order $m+1$ which is satisfied by f, and by induction this holds for all m. Hence when $d = n$, we have that $g \ll f$ if and only if $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$ with $A_1, \ldots, A_n \in \mathbb{C}$. It may be noted that we have in effect used the order on differential polynomials given in [1].

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Now we consider the case when $d < n$. By rearranging as necessary, we may suppose that $\lambda_1, \dots, \lambda_d$ are linearly independent, and that $\lambda_{d+1}, \dots, \lambda_n$ are $\mathbb{Q}$-linear combinations of $\lambda_1, \dots, \lambda_d$. Then for $i = d+1, \dots, n$ we have

$$\lambda_i = \sum_{j=1}^{d} c_i^j \lambda_j, \tag{1}$$

with $c_i^1, \dots, c_i^d \in \mathbb{Q}$. On differentiating the equation

$$f = \sum_{k=1}^{n} a_k e^{\lambda_k x}$$

$n-1$ times, we obtain n linear equations for $a_1 e^{\lambda_1 x}, \dots, a_n e^{\lambda_n x}$. The determinant of the system is $\prod_{i<j}(\lambda_j - \lambda_i)$, which is non-zero. Hence we may obtain linear expressions for each $a_k e^{\lambda_k x}$ in terms of $f, \dots, f^{(n-1)}$; say

$$a_k e^{\lambda_k x} = R_k(f, \dots, f^{(n-1)}), \quad k = 1, \dots, n. \tag{2}$$

On combining these with (1) and taking suitable powers to remove roots, we obtain, for $j = 1, \dots, n-d$,

$$a_{d+j}^{b_j} \prod_{i=1}^{d} a_i^{-\gamma_i^j} = R_{d+j}^{b_j} \prod_{i=1}^{d} R_i^{-\gamma_i^j} = S_j(f, \dots, f^{(n-1)}), \tag{3}$$

where each $b_j \in \mathbb{N}$, each γ_i^j is an integer equal to $c_i^j b_j$, and S_j is a rational function over $\mathbb{C}$. If $g \ll f$ then the equations (3) must also be satisfied by g. However if $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$ with all the A_ks non-zero, then applying to g the same process of elimination that was applied to f yields

$$A_k e^{\lambda_k x} = R_k(g, \dots, g^{(n-1)}).$$

On substituting into (3), we obtain

$$a_{d+j}^{b_j} \prod_{i=1}^{d} a_i^{-\gamma_i^j} = A_{d+j}^{b_j} \prod_{i=1}^{d} A_i^{-\gamma_i^j}, \tag{4}$$

for $j = 1, \dots, n-d$. So the A_k must satisfy these equations in order that $g \ll f$. It is easy to see that this holds even when some of the A_ks are zero provided that negative powers of A_ks are removed from (4) by cross multiplication. In fact these conditions are also sufficient.

THEOREM 1. *Let $f = \sum_{k=1}^{n} a_k e^{\lambda_k x}$. Suppose that $\lambda_1, \dots, \lambda_d$ are linearly independent over $\mathbb{Q}$ and that $\lambda_{d+1}, \dots, \lambda_n$ satisfy (1) with $c_i^1, \dots, c_i^d \in \mathbb{Q}$. Then $g \ll f$ if and only if $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$, with $A_1, \dots, A_n$ satisfying the relations obtained from (4) by cross multiplying to remove negative powers.*

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PROOF OF THEOREM 1. We have already established the necessity of the conditions, so suppose that $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$, with $A_1, \ldots, A_n$ all non-zero and satisfying (4). From (1), we have

$$f = \sum_{k=1}^{d} a_k e^{\lambda_k x} + \sum_{i=d+1}^{n} a_i \prod_{j=1}^{d} (e^{\lambda_j x})^{c_i^j}.$$

On differentiating this equation d times with respect to x, we obtain equations

$$f^{(\mu)} = \sum_{k=1}^{d} a_k \lambda_k^{\mu} e^{\lambda_k x} + \sum_{i=d+1}^{n} \Big\{ a_i \prod_{j=1}^{d} a_j^{-c_i^j} (c_i^1 \lambda_1 + \cdots + c_i^d \lambda_d)^{\mu} \cdot \prod_{j=1}^{d} (a_j e^{\lambda_j x})^{c_i^j} \Big\}, \tag{5}$$

for $\mu = 0, \ldots, d$. We can then use the following theorem of elimination theory to successively eliminate $W_1 = a_1 e^{\lambda_1 x}, \ldots, W_d = a_d e^{\lambda_d x}$ between these equations; see [4].

THEOREM 2. *Let $P_1, \ldots, P_N$ be polynomials in a single variable of given degree with indeterminate coefficients. Then there exists a system, $R_1, \ldots, R_r$ of integral polynomials in these coefficients with the property that if those coefficients are assigned values from a field, K, the conditions $R_1 = 0, \ldots, R_r = 0$ are necessary and sufficient in order that either the equations $P_1 = 0, \ldots, P_N = 0$ have a solution in a suitable extension field or that the formal leading coefficients of all the polynomials $P_1, \ldots, P_N$ vanish.*

We obtain differential polynomials $R_1, \ldots, R_r$ over $\mathbb{C}$ such that $R_j(f, f', \ldots, f^{(d)}) = 0$, for $j = 1, \ldots, r$. If we replace f by g in (5), the corresponding R_js of Theorem 2 will be $R_j(g, g', \ldots, g^{(d)})$, for $j = 1, \ldots, r$. However the equations

$$g^{(\mu)} = \sum_{k=1}^{d} \lambda_k^{\mu} W_k + \sum_{i=d+1}^{n} \Big\{ a_i \prod_{j=1}^{d} a_j^{-c_i^j} (c_i^1 \lambda_1 + \cdots + c_i^d \lambda_d)^{\mu} \cdot \prod_{j=1}^{d} W_j^{c_i^j} \Big\},$$

for $\mu = 0, \ldots, d$, also have a solution for $W_1, \ldots, W_d$, namely $W_k = A_k e^{\lambda_k x}/a_k$. Hence $R_j(g, g', \ldots, g^{(d)}) = 0$ for $j = 1, \ldots, r$. Now g, like f, cannot annul two differential polynomials of order d unless they have a common factor, since otherwise their resultant with respect to $y^{(d)}$ would be a (non-zero) differential polynomial of order $d-1$ annulled by g, and this is impossible since g has transcendence degree d over $\mathbb{C}$. So we have $R_j = Q^{s_j} E_j$ for $j = 1, \ldots, r$, where $s_j \in \mathbb{N}^+$, the differential polynomial Q is irreducible of order d and is annulled by g, and $E_j\langle g\rangle$ is non-zero for each j. Since the R_js are independent of the particular values of $A_1, \ldots, A_n$, we see that Q is independent of the particular g chosen. Thus Q is the unique non-zero irreducible polynomial of order d annulled by f, and the same is true if f is replaced by g.

An immediate consequence is that neither $\partial Q/\partial y^{(d)}\langle f\rangle$ nor $\partial Q/\partial y^{(d)}\langle g\rangle$ can be zero. Also if P is any differential polynomial of degree d annulled by f, then Q must divide P, and hence $P\langle g\rangle = 0$ also. Now by differentiating r times the equation $Q\langle f\rangle = 0$ and eliminating $f^{(d+1)}, \ldots, f^{(d+r-1)}$, we get a rational expression for $f^{(d+r)}$ in terms of $f, \ldots, f^{(d)}$ which has a non-vanishing denominator. Clearly the same relations hold with f replaced by g, and again the denominator does not vanish. Thus if P_1 is a differential polynomial of order $d+r$ with $P_1\langle f\rangle = 0$, we may substitute for $f^{(d+1)}, \ldots, f^{(d+r)}$ to obtain

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a differential equation of order d satisfied by f. This must then be satisfied by g, and by reversing the substitutions, we see that g also satisfies $P_1\langle g\rangle = 0$. Hence $g \ll f$, in the case when the equations (4) hold and the A_ks are all non-zero.

The case when one or more of the A_ks is zero may be handled by multiplying (4) through to eliminate any negative powers and allowing the appropriate A_ks to tend to zero. We have therefore proved Theorem 1.

Here is another way of looking at the situation. We have a tower of function rings

$$\mathbb{C} = T_0 \subset T_1 \subset \cdots \subset T_n, \tag{6}$$

with $T_k = T_{k-1}[e^{\lambda_k x}]$ for $k = 1, \ldots, n$; the function f belongs to T_n. We write $\hat{T}_k$ for the quotient field of the integral domain T_k. Then the extensions $\hat{T}_k : \hat{T}_{k-1}$ are transcendental for $k = 1, \ldots, d$ and algebraic for $k = d+1, \ldots, n$, and the minimal polynomials for the latter are

$$(e^{\lambda_k x})^{b_k} - \prod_{i=1}^{d} (e^{\lambda_i x})^{\gamma^i_{k-d}}. \tag{7}$$

Consider a transformation, $T(C_1, \ldots, C_n)$ of T_n given by $e^{\lambda_k x} \longrightarrow C_k e^{\lambda_k x}$ for $k = 1, \ldots, n$. Provided the C_k are non-zero, such a transformation preserves the differential structure of the tower (6). As above, let $f = \sum_{k=1}^n a_k e^{\lambda_k x}$ and $g = \sum_{k=1}^n A_k e^{\lambda_k x}$, and suppose that $g = T(C_1, \ldots, C_n)(f)$. Then for $k = 1, \ldots, n$,

$$A_k = C_k a_k. \tag{8}$$

The transformed minimal polynomials are, for $k = d+1, \ldots, n$,

$$(C_k e^{\lambda_k x})^{b_k} - \prod_{i=1}^{d} (C_i e^{\lambda_i x})^{\gamma^i_{k-d}},$$

and these will be the same as (7), modulo a multiplying constant for each polynomial, if and only if

$$C_k = \prod_{i=1}^{d} C_i^{c^i_{k-d}}. \tag{9}$$

On substituting from (8) into (9), we obtain once again (4). So these are the conditions that the minimal polynomials should be the same. However the latter are also the conditions for the transformation $T(C_1, \ldots, C_n)$ to be a differential isomorphism, and this turns out to be the key to the more general case as treated in the next section. If some of the C_ks are zero, then as before, the equations (9) and (4) need to be multiplied through to clear negative powers before inserting the values of the C_ks and A_ks.

3. **Liouvillian functions.** Consider a tower of function rings $\mathbb{C} = T_0 \subset T_1 \subset \cdots \subset T_n$, as in (6), where $T_k = T_{k-1}[z_k]$ for $k = 1, \ldots, n$, and now z_k satisfies one of the following three conditions:

i. z_k is algebraic over T_{k-1} with minimal polynomial m_k;
ii. $z_k = \exp(w_{k-1})$ with $w_{k-1} \in \hat{T}_{k-1}$.

iii. $z_k = \int w_{k-1}$ with $w_{k-1} \in \hat{T}_{k-1}$.

We assume that the path of integration is arranged so that (iii) makes sense, and that a determination of the constant of integration is standardised in some way. An element of a field $\hat{T}_k$ is called a *Liouvillian function*. Clearly any elementary function is Liouvillian. Also any Liouvillian function is analytic on $\mathbb{C}$ except perhaps at a countable number of singularities.

We wish to use induction on $k = 0, \ldots, n$ to define a set, $\mathbf{G}_k$, of transformations, each taking T_k to some ring of Liouvillian functions. We shall use the following notation. For any differential polynomial, P, over T_k and any $\rho \in \mathbf{G}_k$, we write $\tilde{\rho}(P)$ for the differential polynomial obtained by applying ρ to all the coefficients of P.

We take $\mathbf{G}_0$ to consist of the identity map, I, on $\mathbb{C}$. Suppose that $\mathbf{G}_{k-1}$ has already been defined. We begin with the case when z_k is algebraic over T_{k-1}. Let ρ be any element of $\mathbf{G}_{k-1}$. For s any root of $\tilde{\rho}(m_k)$, we define $T_i(\rho, s)$ to be the homomorphism on T_k which reduces to ρ on T_{k-1} and takes z_k to s. We then define

$$\mathbf{G}_k = \{T_i(\rho, s) : \rho \in \mathbf{G}_{k-1} \text{ and } s \text{ is a root of } \tilde{\rho}(m_k)\}. \tag{10}$$

The case when z_k is transcendental over $\hat{T}_{k-1}$ is similar. Suppose that $z_k = \int w_{k-1}$, where $w_{k-1} \in \hat{T}_{k-1}$. Let $w_{k-1} = \xi_{k-1}/\eta_{k-1}$ where ξ_{k-1} and η_{k-1} belong to T_{k-1} and have no common factor. For any $K \in \mathbb{C}$ and any $\rho \in \mathbf{G}_{k-1}$ such that $\rho(\eta_{k-1}) \not\equiv 0$, we define $T_{ii}(\rho, K)$ be the homomorphism on T_k which reduces to ρ on T_{k-1} and takes z_k to $K + \int \rho(w_{k-1})$. Then we set

$$\mathbf{G}_k = \{T_{ii}(\rho, K) : K \in \mathbb{C}, \rho \in \mathbf{G}_{k-1} \text{ and } \rho(\eta_{k-1}) \not\equiv 0\}. \tag{11}$$

Finally suppose that z_k is transcendental over $\hat{T}_{k-1}$ and $z_k = \exp(w_{k-1})$ with $w_{k-1} \in \hat{T}_{k-1}$; let $w_{k-1} = \xi_{k-1}/\eta_{k-1}$ as before. Then for $K \in \mathbb{C}$ and $\rho \in \mathbf{G}_{k-1}$ with $\rho(\eta_{k-1}) \not\equiv 0$, we define $T_{iii}(\rho, K)$ to be the homomorphism on T_k which reduces to ρ on T_{k-1} and takes z_k to $K \exp \rho(w_{k-1})$. We put

$$\mathbf{G}_k = \{T_{iii}(\rho, K) : K \in \mathbb{C}, \rho \in \mathbf{G}_{k-1} \text{ and } \rho(\eta_{k-1}) \not\equiv 0\}. \tag{12}$$

Note that K can be zero. It would be possible to exclude this case and make $\mathbf{G}_k$ the differential Galois group of a suitably chosen field of functions. However it seems more natural here to allow the case $K = 0$. We extend the maps $\rho \in \mathbf{G}_k$ to part of $\hat{T}_k$ by setting $\rho(h_1/h_2) = \rho(h_1)/\rho(h_2)$ provided $\rho(h_2) \not\equiv 0$.

The following is a consequence of the constructions above, and is analogous to Proposition 2 in [3].

PROPOSITION 1. *Let f be a Liouvillian function, defined by a tower (6) with $f \in \hat{T}_n$. For each $k = 0, \ldots, n$, there is a set $\mathbf{G}_k$ with the following properties:*

(1) Each $\rho \in \mathbf{G}_k$ is a differential homomorphism from T_k into a ring of Liouvillian functions.

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(2) Suppose that z_k is algebraic over T_{k-1} and let $\sigma \in \mathbf{G}_{k-1}$. Then if ν belongs to the Galois group of the polynomial $\tilde{\sigma}(m_k)$ (with variable z_k), there exists a $\rho \in \mathbf{G}_k$ which agrees with σ on T_{k-1} and maps z_k to $\nu\big(\sigma(z_k)\big)$.

(3) Now suppose that z_k is transcendental over $\hat{T}_{k-1}$. Let $\sigma \in \mathbf{G}_{k-1}$ and suppose that $\sigma(\eta_{k-1}) \not\equiv 0$. If $z_k = \int w_{k-1}$ (respectively $\exp w_{k-1}$) and $K \in \mathbb{C}$, there exists a $\rho \in \mathbf{G}_k$ which agrees with σ on T_{k-1} and maps z_k to $K + \int \sigma(w_{k-1})$ (respectively $K \exp \sigma(w_{k-1})$).

Our main result is the following.

THEOREM 3. *Let (6) and $\mathbf{G}_n$ be as above, and let $f \in \hat{T}_n$. Suppose that $f = f_1/f_2$ where f_1 and f_2 belong to T_n and have no common factor. Then $g \ll f$ if and only if there exists an open dense subset, W, of $\mathbb{C}$ such that g belongs to the closure of the set*

$$\tau(f) =_{\text{def}} \{\rho(f) : \rho \in \mathbf{G}_n \textit{ and } \rho(f_2) \not\equiv 0\}$$

in the topology of uniform, C^∞, convergence on compact subsets of W.

We shall write $\mathrm{cl}(V)$ for the closure of a subset $V \subset W$ in the above topology. The reason why we need to take the closure in Theorem 3 is to accommodate the possibility of ρ mapping both f_1 and f_2 to zero but nonetheless $\rho(f_1/f_2)$ being definable as a limit.

EXAMPLE. Let $f = \big(\exp(e^x) - 1\big)e^{-x}$. Then $f' = \exp(e^x) - \exp(e^x)e^{-x} + e^{-x}$, and so

$$f' - 1 = \big(\exp(e^x) - 1\big)(1 - e^{-x}) = (e^x - 1)f.$$

Thus f satisfies

$$\Big(\frac{f'-1}{f}\Big)' = \frac{f'-1}{f} + 1,$$

and this simplifies to

$$ff'' - f'^2 - ff' + f' + f - f^2 = 0. \tag{13}$$

Here

$$\tau(f) = \Big\{\frac{K_2\exp(K_1e^x) - 1}{K_1e^x} : K_1, K_2 \in \mathbb{C} \text{ and } K_1 \neq 0\Big\},$$

but the set $\tau(f)$ is not closed, as can be seen on taking $K_2 = 1$ and letting K_1 tend to zero. We have

$$\frac{\exp(K_1e^x) - 1}{K_1e^x} \longrightarrow 1,$$

and of course the function 1 does indeed satisfy (13). Theorem 3 shows that it also satisfies every other algebraic differential equation satisfied by f.

We now turn our attention to the proof of Theorem 3. In one direction, this is given by the following proposition.

PROPOSITION 2. *Let f and $\tau(f)$ be as above. Then $g \ll f$ for every $g \in \mathrm{cl}\big(\tau(f)\big)$.*

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PROOF OF PROPOSITION 2. Let $P\langle y\rangle = P(y, y', \ldots, y^{(N)})$ be a differential polynomial such that $P\langle f\rangle = 0$. Let ρ be an element of $\mathbf{G}_n$ such that $\rho(f_2) \neq 0$. Since ρ is a differential homomorphism on T_n, we have

$$P\langle \rho(f)\rangle = \rho(P\langle f\rangle) = 0.$$

Now let $g \in \mathrm{cl}\big(\tau(f)\big)$. Then there exists a sequence $\{\rho_i(f)\}$ such that for all i $\rho_i(f_2) \not\equiv 0$ and $\rho_i(f) \longrightarrow g$. Then for any P as above,

$$P\langle g\rangle = \lim\{P\langle \rho_i(f)\rangle\} = 0.$$

This completes the proof of Proposition 2.

The converse needs more work! The idea is as follows. Let g be a function such that g and f are both defined on some open subset of $\mathbb{C}$, and suppose that g does not belong to $\mathrm{cl}\big(\tau(f)\big)$. Starting from $k = n$, and working down to $k = 0$, we construct a differential polynomial, P_k, over T_k with the following properties:

1. $P_k\langle f\rangle = 0$.
2. g does not belong to the set

$$\Omega(\mathbf{S}_k) =_{\mathrm{def}} \Big(\{h : \exists \rho \in \mathbf{G}_k / \tilde{\rho}(P_k)\langle h\rangle = 0 \text{ and } \tilde{\rho}(P_k) \not\equiv 0\}\Big). \tag{14}$$

Note that $\tilde{\rho}(P_k) \not\equiv 0$ means that ρ does not annihilate every coefficient of P_k.

The initial case is simple enough. If $f = f_1/f_2$ with $f_1, f_2 \in T_{n-1}[z_n]$, we take P_n to be the differential polynomial of order zero, $yf_2 - f_1$. We note that this means that $\Omega(\mathbf{S}_n) = \mathrm{cl}\big(\tau(f)\big)$. It is then a matter of handling the induction step in the various cases.

PROPOSITION 3. *Suppose that* $1 \leq k \leq n$ *and that* z_k *is algebraic over* T_{k-1}. *Let* P_k *be such that* $\tau(f) \subset \Omega(P_k) \subset \mathrm{cl}\big(\tau(f)\big)$. *Then there exists a differential polynomial* P_{k-1} *over* T_{k-1} *with similar properties; i.e.,* $\tau(f) \subset \Omega(P_{k-1}) \subset \mathrm{cl}\big(\tau(f)\big)$.

PROOF OF PROPOSITION 3. We regard P_k as a polynomial in z_k with coefficients in $T_{k-1}\langle y\rangle$, and we replace z_k by an indeterminate z. Then we define

$$P_{k-1} = \mathrm{res}_z\{P\langle y\rangle, m_k\}. \tag{15}$$

We show that P_{k-1} has the required properties.

Firstly, $P_{k-1}\langle f\rangle = 0$ because $P_k\langle f\rangle(z)$ is zero at a root of m_k, namely $z = z_k$. If $g \in \tau(f)$ then $g = \rho(f)$ for some $\rho \in \mathbf{G}_n$ with $\rho(f_2) \neq 0$. Since ρ is a differential homomorphism,

$$P_{k-1}\langle \rho(f)\rangle = \rho(P_{k-1}\langle f\rangle) = \rho(0) = 0.$$

Thus $\tau(f) \subset \Omega(P_{k-1})$.

Now suppose that $g \in \Omega(\mathbf{S}_{k-1})$. Then there exists a $\sigma \in \mathbf{G}_{k-1}$ such that $\tilde{\sigma}(P_{k-1}) \not\equiv 0$ but $\tilde{\sigma}(P_{k-1})\langle g\rangle = 0$. Since σ is an algebra homomorphism and the resultant is given by the Sylvester determinant,

$$\tilde{\sigma}(P_{k-1}) = \mathrm{res}_z\{\tilde{\sigma}(P_k), \tilde{\sigma}(m_k)\} = \pm \prod \tilde{\sigma}(P_k)(\beta_j),$$

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where the β_js are the roots of $\tilde{\sigma}(m_k)$. So there exists a root β such that $\tilde{\sigma}(P_k)\langle g\rangle(\beta) = 0$. However, by Proposition 1(2), there is then a $\rho \in \mathbf{G}_k$ which agrees with σ on T_{k-1} and takes z_k to β. But then $\tilde{\rho}\big(P_k(z_k)\big)\langle g\rangle = \tilde{\sigma}(P_k)(\beta)\langle g\rangle = 0$, and it follows that $g \in \Omega(P_k)$. Hence $g \in \mathrm{cl}\big(\tau(f)\big)$ and we have established Proposition 3.

For the cases when z_k is a transcendental extension, we use what is essentially a differential version of the above. So as to allow the two cases, of an exponential and an integral extension, to be treated together, we let the differential equation satisfied by z_k be

$$z' = \Lambda_k(z), \tag{16}$$

with Λ_k a polynomial over T_{k-1}. Of course (16) is either $z' = w_{k-1}$ or $z' = zw'_{k-1}$. We introduce a derivation D^* on the ring of differential polynomials over $T_{k-1}[z]$, by defining

$$D^*\big(P(z)\big) = \tilde{D}(P) + \Lambda_k(z)\frac{\partial P}{\partial z} + \sum_{j=0} y^{(j+1)}\frac{\partial P}{\partial y^{(j)}};$$

here $\tilde{D}(P)$ denotes the differential polynomial obtained by differentiating the coefficients of P (as a polynomial in z_k and the derivatives of y). As a function of x, $D^*(P)(z_k)\langle y(x)\rangle$ is just the derivative of $P(z_k)\langle y(x)\rangle$. We shall have need of the following lemma, which is a slight adaption of Lemma 11 of [3] to the present set-up. The proof, as in [3], is a straightforward application of the fact that σ is a differential homomorphism.

LEMMA 1. *Let $Q \in T_{k-1}[z]\langle y\rangle$ and $\sigma \in \mathbf{G}_k$. Then*

$$\tilde{\sigma}\big(D^*_k(Q)(z_k)\big) = \Theta^*_{k,\sigma}\big(\tilde{\sigma}(Q)\big)\big(\sigma(z_k)\big),$$

where $\Theta^*_{k,\sigma}$ is defined analogously to D^*_k. i.e. for $S \in \tilde{\sigma}(T_{k-1}[z]\langle y\rangle)$,

$$\Theta^*_{k,\sigma}(S) = \tilde{D}(S) + \tilde{\sigma}\big(\Lambda_k(z)\big)\frac{\partial S}{\partial z} + \sum_{j=0} y^{(j+1)}\frac{\partial S}{\partial y^{(j)}}.$$

The following result is the analogue for transcendental extensions of Proposition 3.

PROPOSITION 4. *Suppose that z_k is transcendental over $\hat{T}_{k-1}$, and let P_k be a differential polynomial such that $\tau(f) \subset \Omega(P_k) \subset \mathrm{cl}\big(\tau(f)\big)$. Then there exists a differential polynomial, P_{k-1} over T_{k-1} such that $\tau(f) \subset \Omega(P_{k-1}) \subset \mathrm{cl}\big(\tau(f)\big)$.*

PROOF OF PROPOSITION 4. We need to make use of the following result, which is a slight modification of Proposition 6 of [3].

LEMMA 2. *Under the conditions of Proposition 4, there exists a differential polynomial Q_k over T_k such that $\tau(f) \subset \Omega(Q_k) \subset \mathrm{cl}\big(\tau(f)\big)$ and $\tilde{\rho}(Q_k\langle h\rangle)$ is a square-free polynomial in z_k for every $\rho \in \mathbf{G}_{k-1}$ and every $h \in \tau(f)$.*

The proof of Proposition 6 in [3] is a dimension argument, and this needs only very minor modification to take account of the dimension of $\tau(f)$.

We then take $P_{k-1} = \mathrm{res}_z\{Q_k, D^*(Q_k)\}$, where we replaced z_k by an indeterminate, z. In order to establish Proposition 4, it is a matter of showing that P_{k-1} has the required properties.

Firstly, $Q_k\langle f\rangle(z)$ and $D^*\big(Q_k\langle f\rangle(z)\big)$ have a factor in common, namely $z - z_k$. Hence $P_{k-1}\langle f\rangle = 0$ and it then follows that $\tau(f) \subset \Omega(P_{k-1})$ as in Proposition 3.

Now suppose that $g \in \Omega(P_{k-1})$. Then there is a $\sigma \in \mathbf{G}_{k-1}$ such that $\tilde{\sigma}(P_{k-1}) \not\equiv 0$ but $\tilde{\sigma}(P_{k-1})\langle g\rangle = 0$. Using Lemma 1, we get

$$\mathrm{res}_z\Big\{\tilde{\sigma}\big(Q_k\langle g\rangle(z)\big), \Theta_{k,\sigma}\Big(\tilde{\sigma}\big(Q_k\langle g\rangle(z)\big)\Big)\Big\} = \tilde{\sigma}\Big(\mathrm{res}_z\big\{\big(Q_k\langle g\rangle(z)\big), D^*\big(Q_k\langle g\rangle(z)\big)\big\}\Big) = 0.$$

So $\tilde{\sigma}\big(Q_k\langle g\rangle(z)\big)$ and $\Theta_{k,\sigma}\Big(\tilde{\sigma}\big(Q_k\langle g\rangle(z)\big)\Big)$ have a root, $z = \phi$ say, in common. Let

$$\tilde{\sigma}\big(Q_k\langle g\rangle(z)\big) = (z - \phi)H.$$

Then

$$\Theta_{k,\sigma}\Big(\tilde{\sigma}\big(Q_k\langle g\rangle(z)\big)\Big) = \big(\tilde{\sigma}(\Lambda_k)(z) - \phi'\big)H + (z - \phi)\Theta_{k,\sigma}H.$$

Since $\tilde{\sigma}(Q_k)$ is square free, $\tilde{\sigma}(\Lambda_k)(z) - \phi'$ must vanish when $z = \phi$. In other words, ϕ satisfies the differential equation for $\sigma(z_k)$. This implies that there is a $\rho \in \mathbf{G}_k$ which agrees with σ on $\mathbf{G}_{k-1}$ and sends z_k to ϕ. But then $\tilde{\rho}(Q_k)\langle g\rangle = 0$, and hence $g \in \Omega(Q_k)$. Thus $g \in \mathrm{cl}\big(\tau(f)\big)$, by the induction hypothesis. So $\Omega(P_{k-1}) \subset \mathrm{cl}\big(\tau(f)\big)$ as required. This completes the proof of Proposition 4.

CONCLUSION OF THE PROOF OF THEOREM 3. Suppose that $g \notin \mathrm{cl}\big(\tau(f)\big)$. As previously indicated, we take $\mathbf{S}_n = \{f_2 y - f_1\}$. Then $P\langle f\rangle = 0$ for every $P \in \mathbf{S}_n$, but $g \notin \Omega(\mathbf{S}_n)$. By Propositions 3 and 4 we can find a set, $\mathbf{S}_0$, of polynomials over $\mathbb{C}$ such that $P_0\langle f\rangle = 0$ for all $P_0 \in \mathbf{S}_0$ but $g \notin \Omega(\mathbf{S}_0)$. In particular, $P_0\langle g\rangle$ is not zero for every $P_0 \in \mathbf{S}_0$, and so there is a P_0 with $P_0\langle f\rangle = 0$ but $P_0\langle g\rangle \neq 0$. Thus we cannot have $g \ll f$, and we have therefore proved Theorem 3.

The same method can possibly be applied to more general situations. For example, one might allow some of the z_k to be given by other first-order, first-degree differential equations. However the eventual result may be of less interest in cases when an explicit description of the relevant differential Galois groups is not available.

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Institute of Mathematics and Statistics
The University
Canterbury
Kent CT2 7NF
England
e-mail: J.R.Shackell@ukc.ac.uk