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GEOMETRIC ρ-MIXING PROPERTY OF THE INTERARRIVAL TIMES OF A STATIONARY MARKOVIAN ARRIVAL PROCESS

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Abstract

In this note, the sequence of the interarrivals of a stationary Markovian arrival process is shown to be ρ-mixing with a geometric rate of convergence when the driving process is ρ-mixing. This provides an answer to an issue raised in the recent work of Ramirez-Cobo and Carrizosa (2012) on the geometric convergence of the autocorrelation function of the stationary Markovian arrival process.

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1. Introduction

We provide a positive answer to a question raised in [4] on the geometric convergence of the autocorrelation function associated with the interarrival times of a stationary m-state Markovian arrival process (MAP). Indeed, it is shown in [3, Proposition 3.1] that the increment sequence $\{T_n := S_n - S_{n-1}\}_{n\geq 1}$ associated with a discrete-time stationary Markov additive process $\{(X_n, S_n)\}_{n\in\mathbb{N}} \in \mathbb{X} \times \mathbb{R}^d$ is ρ-mixing with a geometric rate provided that the driving stationary Markov chain $\{X_n\}_{n\in\mathbb{N}}$ is ρ-mixing. There, $\mathbb{X}$ may be any measurable set. In the case where the increments $\{T_n\}_{n\geq 1}$ are nonnegative random variables, $\{(X_n, S_n)\}_{n\in\mathbb{N}}$ is a Markov renewal process (MRP). Therefore, we obtain the expected answer to the question in [4] since such an MRP with $\{T_n\}_{n\geq 1}$ being the interarrival times can be associated with an m-state MAP and the ρ-mixing property of $\{T_n\}_{n\geq 1}$ with geometric rate ensures the geometric convergence of the autocorrelation function of $\{T_n\}_{n\geq 1}$. We refer the reader to [1, Chapter XI] for basic properties of MAPs and Markov additive processes.

2. Geometric ρ-mixing of the sequence of interarrivals of a MAP

Let us recall the definition of the ρ-mixing property of a (strictly) stationary sequence of random variables $\{T_n\}_{n\geq 1}$ (see, e.g. [2]). The ρ-mixing coefficient with time lag $k > 0$, usually denoted by $\rho(k)$, is defined by

$$\rho(k) := \sup_{n\geq 1} \sup_{m\in\mathbb{N}} \sup\{|\mathrm{Corr}(f(T_1, \dots, T_n); h(T_{n+k}, \dots, T_{n+k+m}))|,\\ f, g \ \mathbb{R}\text{-valued functions such that } \mathbb{E}[|f(T_1, \dots, T_n)|^2]\\ \text{and } \mathbb{E}[|h(T_{n+k}, \dots, T_{n+k+m})|^2] \text{ are finite}\}, \qquad (1)$$

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where $\mathrm{Corr}(f(T_1,\ldots,T_n);h(T_{n+k},\ldots,T_{n+k+m}))$ is the correlation coefficient of the two square-integrable random variables. Note that $\{\rho(k)\}_{n\geq 1}$ is a nonincreasing sequence. Then $\{T_n\}_{n\geq 1}$ is said to be ρ-mixing if

$$\lim_{k\to+\infty}\rho(k)=0.$$

When, for any $n\in\mathbb{N}$, the random variable T_n has a moment of order 2, the autocorrelation function of $\{T_n\}_{n\geq 1}$ as studied in [4], that is, $\mathrm{Corr}(T_1;T_{k+1})$ as a function of the time lag k, clearly satisfies

$$|\mathrm{Corr}(T_1;T_{k+1})|\leq\rho(k)\quad\text{for all } k\geq 1. \tag{2}$$

Therefore, any rate of convergence of the ρ-mixing coefficients $\{\rho(k)\}_{k\geq 1}$ is a rate of convergence for the autocorrelation function.

We only outline the main steps to obtain from [3, Proposition 3.1] a geometric convergence rate of $\{\rho(k)\}_{n\geq 1}$ for the m-state MRP $\{(X_n,S_n)\}_{n\in\mathbb{N}}$ associated with an m-state MAP. In [4, Section 2], the analysis of the autocorrelation function in the two-state case is based on such an MRP (notation and background in [4] are that of [5]). Recall that an m-state MAP is a bivariate continuous-time Markov process $\{(J_t,N_t)\}_{t\geq 0}$ on $\{1,\ldots,m\}\times\mathbb{N}$, where N_t represents the number of arrivals up to time t, while the states of the driving Markov process $\{J_t\}_{t\geq 0}$ are called phases. Let S_n be the time at the nth arrival ($S_0=0$ almost surely), and let X_n be the state of the driving process just after the nth arrival. Then $\{(X_n,S_n)\}_{n\in\mathbb{N}}$ is known to be an MRP with the following semi-Markov kernel Q on $\{1,\ldots,m\}\times[0,\infty)$:

$$Q(x_1;\{x_2\}\times\,\mathrm{d}y):=(\mathrm{e}^{D_0 y}D_1)(x_1,x_2)\,\mathrm{d}y\quad\text{for all } (x_1,x_2)\in\{1,\ldots,m\}^2 \tag{3}$$

parameterized by a pair of $(m\times m)$-matrices usually denoted by D_0 and D_1. The matrix D_0+D_1 is the infinitesimal generator of the background Markov process $\{J_t\}_{t\geq 0}$ which is always assumed to be irreducible, and D_0 is stable. The process $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain with state space $\mathbb{X}:=\{1,\ldots,m\}$ and transition probability matrix P:

$$P(x_1,x_2)=Q(x_1;\{x_2\}\times[0,\infty))=((-D_0)^{-1}D_1)(x_1,x_2)\quad\text{for all } (x_1,x_2)\in\mathbb{X}^2. \tag{4}$$

$\{X_n\}_{n\in\mathbb{N}}$ has an invariant probability measure $\boldsymbol{\phi}$ (i.e. $\boldsymbol{\phi}P=\boldsymbol{\phi}$). It is well known that, for $n\geq 1$, the interarrival time $T_n:=S_n-S_{n-1}$ has a moment of order 2 (whatever the probability distribution of X_0). We refer the reader to [1] for details about the above basic facts of a MAP and its associated MRP.

Let us introduce the $(m\times m)$-matrix

$$\Phi:=\boldsymbol{e}^{\top}\boldsymbol{\phi} \tag{5}$$

when $\boldsymbol{e}$ is the m-dimensional row vector with all components equal to 1. Any $\mathbb{R}$-valued function v on $\mathbb{X}$ may be identified to an $\mathbb{R}^m$-dimensional vector. We use the subordinate matrix norm induced by the $\ell^2(\phi)$-norm $\|v\|_2:=\sqrt{\sum_{x\in\mathbb{X}}|v(x)|^2\boldsymbol{\phi}(x)}$ on $\mathbb{R}^m$:

$$\|M\|_2:=\sup_{\{v:\,\|v\|_2=1\}}\|Mv\|_2.$$

Let $\mathbb{E}_{\boldsymbol{\phi}}$ be the expectation with respect to the initial conditions $(X_0,S_0)\sim(\boldsymbol{\phi},\delta_0)$. Recall that

$$T_n:=S_n-S_{n-1}\quad\text{for } n\geq 1.$$

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When $X_0 \sim \boldsymbol{\phi}$, the following statements hold (see [3, Section 3]).

(i) If g is an $\mathbb{R}$-valued function such that $\mathbb{E}[|g(X_1, T_1, \ldots, X_n, T_n)|] < \infty$ then, for all $k \geq 0$ and all $n \geq 1$,

$$\begin{aligned}
&\mathbb{E}[g(X_{k+1}, T_{k+1}, \ldots, X_{k+n}, T_{k+n}) \mid \sigma(X_l, T_l \colon l \leq k)] \\
&\quad = \int_{(\mathbb{X}\times[0,\infty))^n} Q(X_s;\, \mathrm{d}x_1 \times \mathrm{d}z_1) \prod_{i=2}^{n} Q(x_{i-1};\, \mathrm{d}x_i \times \mathrm{d}z_i) g(x_1, z_1, \ldots, x_n, z_n) \\
&\quad = (Q^{\otimes n})(g)(X_k), \qquad (6)
\end{aligned}$$

where $Q^{\otimes n}$ denotes the n-fold kernel product $\bigotimes_{i=1}^{n} Q$ of Q defined in (3).

(ii) Let f and h be two $\mathbb{R}$-valued functions such that

$$\mathbb{E}_{\boldsymbol{\phi}}[|f(T_1, \ldots, T_n)|^2] < \infty \quad \text{and} \quad \mathbb{E}_{\boldsymbol{\phi}}[|h(T_{n+k}, \ldots, T_{n+k+m})|^2] < \infty$$

for $(k, n) \in (\mathbb{N}^*)^2$, $m \in \mathbb{N}$. From (6), with

$$g(x_1, z_1, \ldots, x_{n+k+m}, z_{n+k+m}) \equiv f(z_1, \ldots, z_n) h(z_{n+k}, \ldots, z_{n+k+m}),$$

the process $\{T_n\}_{n\geq 1}$ is stationary and the following covariance formula holds (see [3, Lemma 3.3] for details):

$$\begin{aligned}
&\operatorname{cov}(f(T_1, \ldots, T_n); h(T_{n+k}, \ldots, T_{n+k+m})) \\
&\qquad = \mathbb{E}_{\boldsymbol{\phi}}[f(T_1, \ldots, T_n)(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))(X_n)]. \qquad (7)
\end{aligned}$$

The matrices P and Φ are defined in (4) and (5).

First, note that the random variables $f(\cdot)$ and $h(\cdot)$ in (1) may be assumed to be of $\mathbb{L}^2$-norm 1. Thus we just have to deal with covariances. Second, the Cauchy–Schwarz inequality and (7) allow us to write

$$\begin{aligned}
&\operatorname{cov}(f(T_1, \ldots, T_n); h(T_{n+k}, \ldots, T_{n+k+m}))^2 \\
&\quad \leq \mathbb{E}_{\boldsymbol{\phi}}[|f(T_1, \ldots, T_n)|^2]\mathbb{E}_{\boldsymbol{\phi}}[|(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))(X_n)|^2] \\
&\quad = \mathbb{E}_{\boldsymbol{\phi}}[|(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))(X_0)|^2] \quad (\boldsymbol{\phi} \text{ is } P\text{-invariant}) \\
&\quad = \|(P^{k-1} - \Phi)(Q^{\otimes m+1}(h))\|_2^2 \\
&\quad \leq \|P^{k-1} - \Phi\|_2^2 \|Q^{\otimes m+1}(h)\|_2^2 \\
&\quad \leq \|P^{k-1} - \Phi\|_2^2 \quad (\text{since } \|Q^{\otimes m+1}(h)\|_2 \leq 1).
\end{aligned}$$

Therefore, it follows from (1) and (2) that the autocorrelation coefficient $\operatorname{Corr}(T_1; T_{k+1})$ as studied in [4] satisfies

$$|\operatorname{Corr}(T_1; T_{k+1})| \leq \rho(k) \leq \|P^{k-1} - \Phi\|_2^2 \quad \text{for all } k \geq 1. \qquad (8)$$

The convergence rate to 0 of the sequence $\{\operatorname{Corr}(T_1; T_{k+1})\}_{n\geq 1}$ is bounded from above by that of $\{\|P^{k-1} - \Phi\|_2\}_{k\geq 1}$. Under usual assumptions on the MAP, $\{X_n\}_{n\in\mathbb{N}}$ is irreducible and aperiodic, so there exists $r \in (0, 1)$ such that

$$\|P^k - \Phi\|_2 = O(r^k) \qquad (9)$$

with $r = \max(|\lambda|, \lambda$ is an eigenvalue of P such that $|\lambda| < 1)$. For a stationary Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with general state space, we know from [6, pp. 200, 207] that property (9) is equivalent to the ρ-mixing property of $\{X_n\}_{n\in\mathbb{N}}$.

3. Comments on [4]

In [4], the analysis is based on a known explicit formula of the correlation function in terms of the parameters of the m-state MRP (see [4, Formula (2.6)]). Note that this formula can be obtained using $n = 1, m = 0$ and $f(T_1) = T_1, h(T_{1+k}) = T_{1+k}$ in (7). When $m := 2$ and under standard assumptions on MAPs, matrix P is diagonalizable with two distinct real eigenvalues, 1 and $0 < \lambda < 1$, which has an explicit form in terms of the entries of P. Consequently, the authors have analyzed the correlation function with respect to the entries of matrix P [4, Equations (3.4)–(3.7)]. However, as the authors pointed out, such an analysis would be tedious and difficult with $m > 2$ due to the increasing number of parameters defining an m-state MAP. Note that inequality (8) and estimate (9) when $m := 2$ provide the same convergence rate as in [4], that is, λ the second eigenvalue of matrix P.

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