2 Monotone Operators and Base Splitting Schemes

In this chapter, we present the basic notion of monotone operators and the base splitting schemes. Throughout this book, we use this machinery to derive and analyze a wide variety of classical and modern algorithms in a unified and streamlined manner. The approach is to first pose the problem at hand as a monotone inclusion problem, then use one of the base splitting schemes to encode the solution as a fixed point of a related operator, and finally find the solution with a fixed-point iteration.

2.1 SET-VALUED OPERATORS

We say $\mathbb{T}$ is a *(set-valued) operator*, *point-to-set mapping*, *set-valued mapping*, *multi-valued function*, or *correspondence* on $\mathbb{R}^n$ if $\mathbb{T}$ maps a point in $\mathbb{R}^n$ to a (possibly empty) subset of $\mathbb{R}^n$. We denote this as $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. So, $\mathbb{T}(x) \subseteq \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. For notational simplicity, we write $\mathbb{T}x = \mathbb{T}(x)$.

If $\mathbb{T}x$ is a singleton or empty for all x, then $\mathbb{T}$ is a *function* or is *single-valued* with domain $\{x \mid \mathbb{T}(x) \neq \emptyset\}$. In this case, we mix function and operator notation and write $\mathbb{T}x = y$ (function notation) although $\mathbb{T}x = \{y\}$ (operator notation) would be strictly correct.

We define the graph of an operator as

$$\operatorname{Gra}\mathbb{T} = \{(x,u) \,|\, u \in \mathbb{T}x\} \subseteq \mathbb{R}^n \times \mathbb{R}^n.$$

An operator and its graph are mathematically equivalent. In other words, we can view $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as a point-to-set mapping and as a subset of $\mathbb{R}^n \times \mathbb{R}^n$. In this book, we will often not distinguish the operator itself and its graph; we will often write $\mathbb{T}$ when we really mean $\operatorname{Gra}\mathbb{T}$.

We extend many notions for functions to operators. For example, the domain and range of an operator $\mathbb{T}$ are defined as

$$\operatorname{dom}\mathbb{T} = \{x \mid \mathbb{T}x \neq \emptyset\}, \qquad \operatorname{range}\mathbb{T} = \{y \mid y \in \mathbb{T}x,\, x \in \mathbb{R}^n\}.$$

If $C \subseteq \mathbb{R}^n$, we write $\mathbb{T}(C) = \cup_{c\in C}\mathbb{T}(c)$ for the image of C under $\mathbb{T}$. If $\mathbb{T}$ and $\mathbb{S}$ are operators, we define the composition as

$$\mathbb{T} \circ \mathbb{S}x = \mathbb{T}\mathbb{S}x = \mathbb{T}(\mathbb{S}(x))$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

and the sum as

$$(\mathbb{T} + \mathbb{S})x = \mathbb{T}(x) + \mathbb{S}(x),$$

where $\mathbb{T}(x) + \mathbb{S}(x)$ is the Minkowski sum. Alternate equivalent definitions that use the graph are

$$\mathbb{T}\mathbb{S} = \{(x, z) \mid \exists\, y\ (x, y) \in \mathbb{S},\ (y, z) \in \mathbb{T}\},$$
$$\mathbb{T} + \mathbb{S} = \{(x, y + z) \mid (x, y) \in \mathbb{T},\ (x, z) \in \mathbb{S}\}.$$

We write $\mathbb{I}$ and $\mathbb{O}$ for the identity and zero operators

$$\mathbb{I} = \{(x, x) \mid x \in \mathbb{R}^n\}, \qquad \mathbb{O} = \{(x, 0) \mid x \in \mathbb{R}^n\}.$$

So, for any operator $\mathbb{T}$, we have $\mathbb{T} + \mathbb{O} = \mathbb{T}$, $\mathbb{T}\mathbb{I} = \mathbb{T}$, and $\mathbb{I}\mathbb{T} = \mathbb{T}$.

For an $L > 0$, we say an operator $\mathbb{T}$ is L-Lipschitz if

$$\|u - v\| \le L\|x - y\| \qquad \forall\,(x, u), (y, v) \in \mathbb{T},$$

or, more concisely, if

$$\|\mathbb{T}x - \mathbb{T}y\| \le L\|x - y\| \qquad \forall\, x, y \in \operatorname{dom} \mathbb{T}.$$

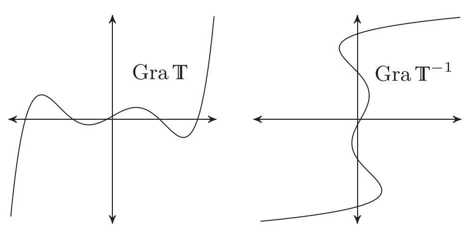
If $\mathbb{T}$ is L-Lipschitz, it is single-valued; if $\mathbb{T}x$ is not a singleton, then we have a contradiction by setting $y = x$. (This generalizes the previous definition of §1, as it allows $\operatorname{dom} \mathbb{T} \neq \mathbb{R}^n$.)

The *inverse operator* of $\mathbb{T}$ is defined as

$$\mathbb{T}^{-1} = \{(y, x) \mid (x, y) \in \mathbb{T}\}.$$

Since $\mathbb{T}^{-1}$ can be multi-valued, it is always well defined. It is easy to see that $(\mathbb{T}^{-1})^{-1} = \mathbb{T}$ and $\operatorname{dom} \mathbb{T}^{-1} = \operatorname{range} \mathbb{T}$. As a note of caution, the inverse operator is not an inverse in the usual sense, as we can have $\mathbb{T}^{-1}\mathbb{T} \neq \mathbb{I}$. The zero operator is such an example. However, we do have $\mathbb{T}^{-1}\mathbb{T}x = x$ when $\mathbb{T}^{-1}$ is single-valued and $x \in \operatorname{dom} \mathbb{T}$. See Exercise 2.1.

Example 2.1 The inverse of an operator $\mathbb{T}$ always exists since we do not require it to be single-valued.



When $0 \in \mathbb{T}(x)$, we say that x is a *zero* of $\mathbb{T}$. We write the *zero set* of an operator $\mathbb{T}$ as

$$\operatorname{Zer} \mathbb{T} = \{x \mid 0 \in \mathbb{T}x\} = \mathbb{T}^{-1}(0).$$

We will see that many interesting problems can be posed as finding zeros of an operator.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Subdifferential

Let f be a convex function on $\mathbb{R}^n$. Then ∂f is a set-valued operator, and

$$\operatorname{argmin} f = \operatorname{Zer} \partial f,$$

that is, $0 \in \partial f(x)$ if and only if x minimizes f. When f is known or assumed to be differentiable, we write ∇f instead of ∂f. As an aside, $\operatorname{dom} \partial f \subseteq \operatorname{dom} f$, and it is possible to have $\operatorname{dom} \partial f \neq \operatorname{dom} f$. Example 1.7 is one such example.

When f is CCP, we have the elegant formula

$$(\partial f)^{-1} = \partial f^*, \tag{2.1}$$

which is known as *Fenchel's identity*. This follows from

$$\begin{aligned} u \in \partial f(x) \quad &\Leftrightarrow \quad 0 \in \partial f(x) - u \\ &\Leftrightarrow \quad x \in \operatorname*{argmin}_z \{f(z) - u^\intercal z\} \\ &\Leftrightarrow \quad -f(x) + u^\intercal x = f^*(u) \\ &\Leftrightarrow \quad f(x) + f^*(u) = u^\intercal x \\ &\Leftrightarrow \quad f^{**}(x) + f^*(u) = u^\intercal x \\ &\Leftrightarrow \quad x \in \partial f^*(u), \end{aligned}$$

where the second-to-last step uses the fact that $f^{**} = f$ when f is CCP, as discussed in §1.3.8, and the last step takes the whole argument backward.

Consider $g(y) = f^*(A^\intercal y)$, where f is CCP. If $\mathcal{R}(A^\intercal) \cap \operatorname{ri} \operatorname{dom} f^* \neq \emptyset$, we have

$$\begin{aligned} u \in \partial g(y) \quad &\Leftrightarrow \quad u \in A\partial f^*(A^\intercal y) \\ &\Leftrightarrow \quad u = Ax,\ x \in \partial f^*(A^\intercal y) \\ &\Leftrightarrow \quad u = Ax,\ \partial f(x) \ni A^\intercal y \\ &\Leftrightarrow \quad u = Ax,\ 0 \in \partial f(x) - A^\intercal y \\ &\Leftrightarrow \quad u = Ax,\ x \in \operatorname*{argmin}_z \{f(z) - \langle y, Az\rangle\}. \end{aligned} \tag{2.2}$$

This means we can find an element of ∂g by solving a minimization problem.

2.2 MONOTONE OPERATORS

An operator $\mathbb{T}$ on $\mathbb{R}^n$ is said to be *monotone* if

$$\langle u - v, x - y\rangle \geq 0 \qquad \forall\, (x,u),\ (y,v) \in \mathbb{T}.$$

Equivalently and more concisely, we can express monotonicity as

$$\langle \mathbb{T}x - \mathbb{T}y, x - y\rangle \geq 0 \qquad \forall\, x, y \in \mathbb{R}^n.$$

To clarify, $\langle \mathbb{T}x - \mathbb{T}y, x - y\rangle$ is a subset of $\mathbb{R}$ and the inequality means the subset is contained in $[0, \infty)$. When $x \notin \operatorname{dom} \mathbb{T}$ or $y \notin \operatorname{dom} \mathbb{T}$, then $\langle \mathbb{T}x - \mathbb{T}y, x - y\rangle = \emptyset$ and the inequality is vacuous.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

An operator $\mathbb{T}$ is *maximal monotone* if there is no other monotone operator $\mathbb{S}$ such that $\mathrm{Gra}\,\mathbb{T} \subset \mathrm{Gra}\,\mathbb{S}$ properly. In other words, if the monotone operator $\mathbb{T}$ is not maximal, then there exists $(x,u) \notin \mathbb{T}$ such that $\mathbb{T} \cup \{(x,u)\}$ is still monotone. Maximality is a technical but fundamental detail.

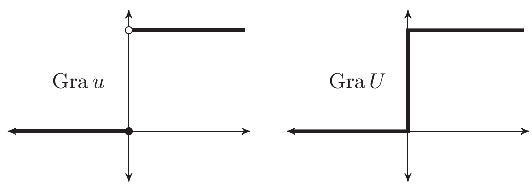
Example 2.2 The Heaviside step function $u\colon \mathbb{R} \to \mathbb{R}$ defined as

$$u(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0 \end{cases}$$

is monotone but not maximal. The operator $U\colon \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$U(x) = \begin{cases} \{0\} & \text{for } x < 0 \\ [0,1] & \text{for } x = 0 \\ \{1\} & \text{for } x > 0 \end{cases}$$

is maximal monotone.



Subdifferential

If f is convex and proper, then ∂f is a monotone operator. If f is CCP, then ∂f is maximal monotone. To prove monotonicity, add the inequalities

$$f(y) \ge f(x) + \langle \partial f(x), y - x\rangle, \qquad f(x) \ge f(y) + \langle \partial f(y), x - y\rangle,$$

which hold by the definition of subdifferentials, to get

$$\langle \partial f(x) - \partial f(y), x - y\rangle \ge 0.$$

We prove maximality later in §10. See Exercise 2.2 for an example where ∂f is not maximal.

Not all maximal monotone operators are subdifferential operators. Subdifferential operators of CCP functions form a subclass of monotone operators that enjoy certain nice properties that general maximal monotone operators do not.

2.2.1 Stronger Monotonicity Properties

An operator $\mathbb{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *μ-strongly monotone* or *μ-coercive* if $\mu > 0$ and

$$\langle u - v, x - y\rangle \ge \mu\|x - y\|^2 \qquad \forall\,(x,u),(y,v) \in \mathbb{A}.$$

We say $\mathbb{A}$ is strongly monotone if it is μ-strongly monotone for some unspecified $\mu \in (0,\infty)$. An operator $\mathbb{A}$ is *β-cocoercive* or *β-inverse strongly monotone* if $\beta > 0$ and

$$\langle u - v, x - y\rangle \ge \beta\|u - v\|^2 \qquad \forall\,(x,u),(y,v) \in \mathbb{A}.$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

We say $\mathbb{A}$ is cocoercive if it is β-cocoercive for some unspecified $\beta \in (0, \infty)$. Cocoercivity is the dual property of strong monotonicity; $\mathbb{A}$ is β-cocoercive if and only if $\mathbb{A}^{-1}$ is β-strongly monotone. Clearly, strongly monotone and cocoercive operators are monotone.

We can more concisely express μ-strong monotonicity as

$$\langle \mathbb{A}x - \mathbb{A}y, x - y \rangle \geq \mu \|x - y\|^2 \qquad \forall\, x, y \in \mathbb{R}^n,$$

and, when $\mathbb{A}$ is a priori known or assumed to be single-valued, express β-cocoercivity as

$$\langle \mathbb{A}x - \mathbb{A}y, x - y \rangle \geq \beta \|\mathbb{A}x - \mathbb{A}y\|^2 \qquad \forall\, x, y \in \mathbb{R}^n.$$

When $\mathbb{A}$ is β-cocoercive, the Cauchy–Schwartz inequality tells us

$$(1/\beta)\|x - y\| \geq \|\mathbb{A}x - \mathbb{A}y\| \qquad \forall\, x, y \in \mathbb{R}^n.$$

that is, $\mathbb{A}$ is $(1/\beta)$-Lipschitz. Therefore, cocoercive operators are single-valued. The converse is not true. The single-valued operator $\mathbb{A} \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\mathbb{A}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

is an example of an operator that is maximal monotone and Lipschitz, but not cocoercive since $\langle \mathbb{A}x - \mathbb{A}y, x - y \rangle = 0, \forall\, x, y \in \mathbb{R}^n$.

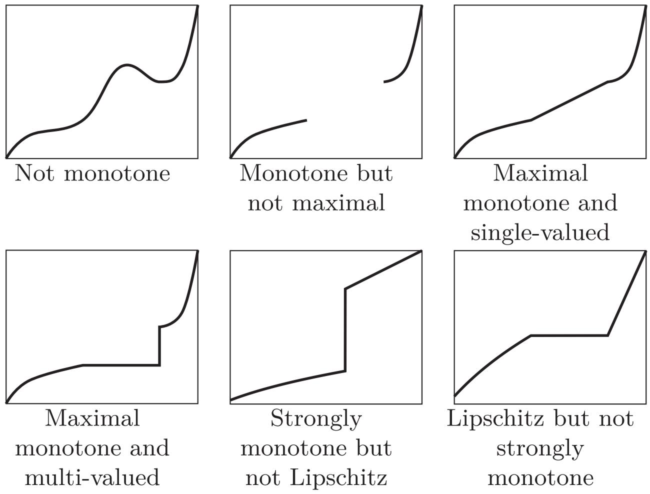
We say $\mathbb{A}$ is maximal μ-strongly monotone if there is no other μ-strongly monotone operator $\mathbb{B}$ such that $\operatorname{Gra} \mathbb{A} \subset \operatorname{Gra} \mathbb{B}$ properly. We say $\mathbb{A}$ is maximal β-cocoercive if there is no other β-cocoercive operator $\mathbb{B}$ such that $\operatorname{Gra} \mathbb{A} \subset \operatorname{Gra} \mathbb{B}$ properly. Maximal cocoercivity is the dual property of maximal strong monotonicity; $\mathbb{A}$ is maximal β-cocoercive if and only if $\mathbb{A}^{-1}$ is maximal β-strongly monotone. A β-cocoercive operator $\mathbb{A}$ is maximal if and only if $\operatorname{dom} \mathbb{A} = \mathbb{R}^n$. (We show this fact in §10.3 as Theorem 15.) Since a β-cocoercive operator is single-valued, the statement "$\mathbb{A} \colon \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoercive" is equivalent to "$\mathbb{A} \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal β-cocoercive" since the notation $\mathbb{A} \colon \mathbb{R}^n \to \mathbb{R}^n$ implicitly assumes $\operatorname{dom} \mathbb{A} = \mathbb{R}^n$. For further discussion, see §10 and Exercises 10.11 and 10.12.

Assume f is CCP. Then f is μ-strongly convex if and only if ∂f is μ-strongly monotone, and f is L-smooth if and only if ∂f is $(1/L)$-cocoercive. Since ∂f is μ-strongly monotone if and only if $(\partial f)^{-1} = \partial f^*$ is μ-cocoercive, f is μ-strongly convex if and only if f^* is $(1/\mu)$-smooth.

The notion of Lipschitz continuity and cocoercivity coincide for subdifferential operators of convex functions: ∂f is L-Lipschitz if and only if ∂f is $(1/L)$-cocoercive. This result is known as the *Baillon–Haddad theorem*.

Example 2.3 An operator on $\mathbb{R}$ is monotone if its graph is a nondecreasing curve in $\mathbb{R}^2$. If it has vertical portions, the operator is multi-valued. If it is continuous with no end points, then it is maximal. If its slope is at least μ everywhere, then it is μ-strongly monotone. If its slope is never more than L, then it is L-Lipschitz. The notion of Lipschitz continuity and cocoercivity coincide for operators on $\mathbb{R}$.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press



2.2.2 Operations Preserving (Maximal) Monotonicity

If $\mathbb{T}$ is (maximal) monotone, then $\mathbb{S}(x) = y + \alpha\mathbb{T}(x + z)$ for any $\alpha > 0$ and $y, z \in \mathbb{R}^n$ is (maximal) monotone. If $\mathbb{T}$ is (maximal) monotone, then $\mathbb{T}^{-1}$ is (maximal) monotone. If $\mathbb{T}$ and $\mathbb{S}$ are monotone, then $\mathbb{T} + \mathbb{S}$ is monotone. If $\mathbb{T}$ and $\mathbb{S}$ are maximal monotone and if $\operatorname{dom}\mathbb{T} \cap \operatorname{int}\operatorname{dom}\mathbb{S} \neq \emptyset$, then $\mathbb{T} + \mathbb{S}$ is maximal monotone. If $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is monotone and $M \in \mathbb{R}^{n\times m}$, then $M^\intercal\mathbb{T}M$ is a monotone operator on $\mathbb{R}^m$. If $\mathbb{T}$ is maximal and $\mathcal{R}(M) \cap \operatorname{int}\operatorname{dom}\mathbb{T} \neq \emptyset$, then $M^\intercal\mathbb{T}M$ is maximal. See §10 for proofs of maximality.

If $\mathbb{R}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $\mathbb{S}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, then the operator $\mathbb{T}\colon \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{n+m}$ defined by

$$\mathbb{T}(x, y) = \{(u, v) \mid u \in \mathbb{R}x,\ v \in \mathbb{S}y\}$$

is (maximal) monotone if $\mathbb{R}$ and $\mathbb{S}$ are. We call $\mathbb{T}$ the *concatenation* of $\mathbb{R}$ and $\mathbb{S}$ and use the notation

$$\mathbb{T} = \begin{bmatrix} \mathbb{R} \\ \mathbb{S} \end{bmatrix}, \qquad \mathbb{T}(x, y) = \begin{bmatrix} \mathbb{R}x \\ \mathbb{S}y \end{bmatrix}.$$

If $\mathbb{T}$ is μ-strongly monotone and $\mathbb{S}$ is monotone, then $\mathbb{T} + \mathbb{S}$ is μ-strongly monotone and $\alpha\mathbb{T}$ is $(\alpha\mu)$-strongly monotone for $\alpha > 0$. If $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is μ-strongly monotone and $M \in \mathbb{R}^{n\times m}$ has rank m (so $n \geq m$), then $M^\intercal\mathbb{T}M$ is $(\mu\sigma_{\min}^2(M))$-strongly monotone. If $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is L-Lipschitz and $M \in \mathbb{R}^{n\times m}$, then $M^\intercal\mathbb{T}M$ is $(L\sigma_{\max}^2(M))$-Lipschitz.

2.2.3 Examples

Affine Operators

An affine operator $\mathbb{T}(x) = Ax + b$ is maximal monotone if and only if $A + A^\intercal \succeq 0$. It is a subdifferential operator of a CCP function if and only if $A = A^\intercal$ and $A \succeq 0$. It is $\lambda_{\min}(A + A^\intercal)/2$-strongly monotone and $\sigma_{\max}(A)$-Lipschitz.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Continuous Operators

We say an operator $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is continuous if $\operatorname{dom}\mathbb{T} = \mathbb{R}^n$, $\mathbb{T}$ is single-valued, and $\mathbb{T}$ is continuous as a function. A continuous monotone operator $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is maximal. See Exercise 2.4 for a proof. Therefore maximality comes into question only with discontinuous or set-valued operators.

Differentiable Operators

We say anoperator is differentiable if it is single-valued, continuous, and differentiable. A differentiable operator $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is monotone if and only if $D\mathbb{T}(x) + D\mathbb{T}(x)^\intercal \succeq 0$ for all $x \in \mathbb{R}^n$, where $D\mathbb{T}(x)$ is the $n \times n$ Jacobian matrix evaluated at x. It is μ-strongly monotone if and only if $D\mathbb{T}(x)+D\mathbb{T}(x)^\intercal \succeq 2\mu I$ for all x, and it is L-Lipschitz if and only if $\sigma_{\max}(D\mathbb{T}(x)) \le L$ for all x. See Exercises 2.7 and 2.8 for proofs.

If a monotone operator $\mathbb{T}$ is differentiable with continuous $D\mathbb{T}$, then $\mathbb{T}$ is a subdifferential operator of a CCP function if and only if $D\mathbb{T}(x)$ is symmetric for all $x \in \mathbb{R}^n$. When $n = 3$, this condition is equivalent to the so-called curl-less (or irrotational) condition discussed in the context of electromagnetic potentials.

Saddle Subdifferential

Let $\mathbf{L}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ be a convex-concave saddle function, that is, $\mathbf{L}(x,u)$ is convex in x for fixed u and concave in u for fixed x. The *saddle subdifferential operator* $\partial\mathbf{L}\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is defined as

$$\partial\mathbf{L}(x,u) = \begin{bmatrix} \partial_x \mathbf{L}(x,u) \\ \partial_u(-\mathbf{L}(x,u)) \end{bmatrix}. \tag{2.3}$$

To clarify, ∂_x and ∂_u respectively denote the subgradients with respect to x and u. To clarify, $\partial\mathbf{L}(x,u)$ is nonempty if both $\partial_x\mathbf{L}(x,u)$ and $\partial_u(-\mathbf{L}(x,u))$ are nonempty. $\operatorname{Zer}\partial\mathbf{L}$ is the set of saddle points of $\mathbf{L}$, that is, $0 \in \partial\mathbf{L}(x^\star,u^\star)$ if and only if $(x^\star,u^\star)$ is a saddle point of $\mathbf{L}$.

For most well-behaved convex-concave saddle functions, their saddle subdifferentials are maximal monotone. Specifically, "closed proper" convex-concave saddle functions have maximal monotone saddle subdifferentials. (See the bibliographical notes section.) In this book, we avoid this notion, as it is usually straightforward to verify the maximality of saddle subdifferentials on a case-by-case basis.

As a technical note, we adopt the convention $+\infty - \infty = -\infty + \infty = -\infty$ in saddle functions. We do encounter $+\infty - \infty$ in certain cases such as the Lagrangians for DRS (2.17), PDHG (1.8), and Condat–Vũ (3.12). The specific value that we ascribe to $+\infty-\infty$ does not matter, but we define it for concreteness.

KKT Operator

Consider the problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1,\dots,m \\ & h_i(x) = 0, \quad i = 1,\dots,p, \end{array}$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

where $f_0,\dots,f_m$ are CCP and $h_1,\dots,h_p$ are affine. The associated Lagrangian

$$\mathbf{L}(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) - \delta_{\mathbb{R}^m_+}(\lambda),$$

where $\mathbb{R}^m_+$ denotes the nonnegative orthant, is a convex-concave saddle function, and we define the Karush–Kuhn–Tucker (KKT) operator as

$$\mathbb{T}(x,\lambda,\nu) = \begin{bmatrix} \partial_x \mathbf{L}(x,\lambda,\nu) \\ -\mathbb{F}(x) + \mathbb{N}_{\mathbb{R}^m_+}(\lambda) \\ -\mathbb{H}(x) \end{bmatrix} = \begin{bmatrix} \partial_x \mathbf{L}(x,\lambda,\nu) \\ \partial_\lambda(-\mathbf{L}(x,\lambda,\nu)) \\ \partial_\nu(-\mathbf{L}(x,\lambda,\nu)) \end{bmatrix},$$

where

$$\mathbb{F}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}, \qquad \mathbb{H}(x) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{bmatrix}.$$

$\mathbb{T}$ is monotone, since it is a special case of the saddle subdifferential. Arguments based on total duality tell us that $0 \in \mathbb{T}(x^\star,\lambda^\star,\nu^\star)$ if and only if $x^\star$ solves the primal problem, $(\lambda^\star,\nu^\star)$ solves the dual problem, and strong duality holds.

2.2.4 Monotone Inclusion Problem

Monotone inclusion problems are problems of the form

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in \mathbb{A}x,$$

where $\mathbb{A}$ is monotone. Many interesting problems can be formulated as monotone inclusion problems. For example, minimizing a convex function f is equivalent to finding a zero of ∂f.

2.3 NONEXPANSIVE AND AVERAGED OPERATORS

Nonexpansive and Contractive Operators

We say an operator $\mathbb{T}$ is nonexpansive if

$$\|\mathbb{T}x - \mathbb{T}y\| \le \|x - y\| \qquad \forall\, x,y \in \operatorname{dom} \mathbb{T}.$$

In other words, $\mathbb{T}$ is nonexpansive if it is 1-Lipschitz. We say $\mathbb{T}$ is a contraction if it is L-Lipschitz with $L < 1$. Mapping a pair of points by a contraction reduces the distance between them; mapping them by a nonexpansive operator does not increase the distance between them.

If $\mathbb{T}$ and $\mathbb{S}$ are nonexpansive, then $\mathbb{TS}$ is nonexpansive. If $\mathbb{T}$ or $\mathbb{S}$ is furthermore contractive, then $\mathbb{TS}$ is contractive. If $\mathbb{T}$ and $\mathbb{S}$ are nonexpansive, then $\theta\mathbb{T} + (1-\theta)\mathbb{S}$ with $\theta \in [0,1]$, a convex combination, is nonexpansive. If $\mathbb{T}$ is furthermore contractive and $\theta \in (0,1]$, then $\theta\mathbb{T} + (1-\theta)\mathbb{S}$ is contractive.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

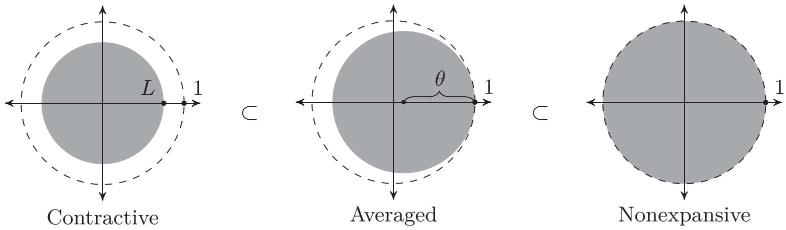


Figure 2.1 Illustration of classes of contractive, averaged, and nonexpansive operators. The figure illustrates the relationship: contractive ⊂ averaged ⊂ nonexpansive. The precise meaning of these figures will be defined in §13.

Averaged Operators

For $\theta \in (0,1)$, we say an operator $\mathbb{T}$ is *θ-averaged* if $\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$ for some nonexpansive operator $\mathbb{S}$. We say an operator is averaged if it is θ-averaged for some unspecified $\theta \in (0,1)$. In other words, taking a weighted average (convex combination) of $\mathbb{I}$ and a nonexpansive operator gives an averaged operator. We say an operator is *firmly nonexpansive* if it is $(1/2)$-averaged. See Figure 2.1. When operators $\mathbb{T}$ and $\mathbb{S}$ are averaged, the composition $\mathbb{T}\mathbb{S}$ is averaged. We prove this as Theorem 27 later in §13.

Averagedness is the central notion in establishing convergence for many splitting methods. In fact, Theorems 1, 2, and 3, the main convergence theorems of Part I, are based on the notion of averagedness.

2.4 FIXED-POINT ITERATION

We now discuss the first meta-algorithm of this book: the fixed-point iteration. Using the fixed-point iteration involves two steps. First, find a suitable operator whose fixed points are solutions to a monotone inclusion problem of interest. Second, show that the iteration converges to a fixed point.

2.4.1 Fixed Points

We say x is a *fixed point* of $\mathbb{T}$ if $x = \mathbb{T}x$, and write

$$\operatorname{Fix}\mathbb{T} = \{x \mid x = \mathbb{T}x\} = (\mathbb{I} - \mathbb{T})^{-1}(0)$$

for the set of fixed points of $\mathbb{T}$. If $\mathbb{T}$ is nonexpansive and $\operatorname{dom}\mathbb{T} = \mathbb{R}^n$, then its set of fixed points is closed and convex. Certainly, $\operatorname{Fix}\mathbb{T}$ can be empty (for example, $\mathbb{T}x = x + 1$ on $\mathbb{R}$) or contain many points (for example, $\mathbb{T}x = |x|$ on $\mathbb{R}$).

Let us show $\operatorname{Fix}\mathbb{T}$ is closed and convex when $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive. That $\operatorname{Fix}\mathbb{T}$ is closed follows from the fact that $\mathbb{T} - \mathbb{I}$ is a continuous function. Now suppose $x, y \in \operatorname{Fix}\mathbb{T}$ and $\theta \in [0,1]$. We will show that $z = \theta x + (1-\theta)y \in \operatorname{Fix}\mathbb{T}$. Since $\mathbb{T}$ is nonexpansive, we have

$$\|\mathbb{T}z - x\| \le \|z - x\| = (1-\theta)\|y - x\|,$$

and similarly, we have

$$\|\mathbb{T}z - y\| \leq \theta\|y - x\|.$$

So, the triangle inequality

$$\|x - y\| \leq \|\mathbb{T}z - x\| + \|\mathbb{T}z - y\|$$

holds with equality, which means the previous inequalities hold with equality and $\mathbb{T}z$ is on the line segment between x and y. From $\|\mathbb{T}z - y\| = \theta\|y - x\|$, we conclude that $\mathbb{T}z = \theta x + (1-\theta)y = z$. Thus $z \in \operatorname{Fix}\mathbb{T}$.

2.4.2 Fixed-Point Iteration

The algorithm *fixed-point iteration* (FPI), also called the *Picard iteration*, is

$$x^{k+1} = \mathbb{T}x^k$$

for $k = 0, 1, \ldots$, where $x^0 \in \mathbb{R}^n$ is some starting point and $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is single-valued. The FPI is used to find a fixed point of $\mathbb{T}$. Clearly, the algorithm stays at a fixed point if it starts at a fixed point. For the sake of brevity, we will usually omit stating that $x^0 \in \mathbb{R}^n$ is some starting point and that $k = 0, 1, \ldots$ when we write an FPI.

In general, an FPI need not converge, even if we assume $\mathbb{T}$ is nonexpansive. For example, this is the case when $\mathbb{T}$ is a rotation about some line or a reflection through a plane. We provide two conditions that guarantee convergence, although these are not the only possible approaches.

Contractive Operators

Suppose that $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a contraction with Lipschitz constant $L < 1$. In this setting, FPI is also called the *contraction mapping algorithm*. For $x^\star \in \operatorname{Fix}\mathbb{T}$, we have

$$\|x^k - x^\star\| \leq L\|x^{k-1} - x^\star\| \leq \cdots \leq L^k\|x^0 - x^\star\|.$$

This is the basis of the classic Banach fixed-point theorem; see Exercise 2.14.

So, when $\mathbb{T}$ is a contraction, the convergence analysis is very simple. In many optimization setups, however, a contraction is too much to ask for, and we need an approach to establish convergence under weaker assumptions.

Averaged Operators

Suppose $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is averaged. In this setting, FPI is also called the *averaged* or *Krasnosel'skiĭ–Mann* iteration, and it converges to a solution if one exists.

Theorem 1 Assume $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is θ-averaged with $\theta \in (0,1)$ and $\operatorname{Fix}\mathbb{T} \neq \emptyset$. Then $x^{k+1} = \mathbb{T}x^k$ with any starting point $x^0 \in \mathbb{R}^n$ converges to one fixed point, that is,

$$x^k \to x^\star$$

for some $x^\star \in \operatorname{Fix}\mathbb{T}$. The quantities $\operatorname{dist}(x^k, \operatorname{Fix}\mathbb{T})$, $\|x^{k+1} - x^k\|$, and $\|x^k - x^\star\|$ for any $x^\star \in \operatorname{Fix}\mathbb{T}$ are monotonically nonincreasing with k. Finally, we have

$$\operatorname{dist}(x^k, \operatorname{Fix}\mathbb{T}) \to 0$$

and

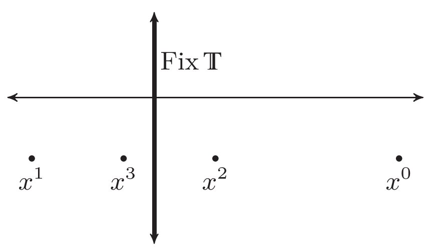
$$\|x^{k+1} - x^k\|^2 \leq \frac{\theta}{(k+1)(1-\theta)} \operatorname{dist}^2(x^0, \operatorname{Fix}\mathbb{T}).$$

To find a fixed point of a nonexpansive operator $\mathbb{T}$ that is not necessarily averaged, we can perform FPI on the averaged operator $(1-\theta)\mathbb{I} + \theta\mathbb{T}$ with $\theta \in (0,1)$. $\mathbb{T}$ and $(1-\theta)\mathbb{I} + \theta\mathbb{T}$ share the same set of fixed points, that is, $\operatorname{Fix}\mathbb{T} = \operatorname{Fix}((1-\theta)\mathbb{I} + \theta\mathbb{T})$. This ensures the iteration converges, with essentially no additional computational cost.

Example 2.4 Consider $\mathbb{T}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\mathbb{T}x = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix} x = \left(\frac{3}{4} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) x.$$

This is a $(3/4)$-averaged operator with $\operatorname{Fix}\mathbb{T} = \{(0,z) \mid z \in \mathbb{R}\}$.



We can see that FPI with respect to $\mathbb{T}$ converges to one of the fixed points and that the limit depends on the starting point x^0.

Proof of Theorem 1. Before we begin the proof in earnest, we summarize the core idea of the proof. Assume we have *nonnegative* scalar sequences $V^0, V^1, \ldots$ and $S^0, S^1, \ldots$. (To clarify, the superscripts denote iteration count, not exponents.) Say we establish the inequality

$$V^{k+1} \leq V^k - S^k$$

for $k = 0, 1, 2, \ldots$. Such an inequality has two useful consequences. The first is that V^k is monotonically nonincreasing, although there is no guarantee that V^k decreases to 0.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

The second is that $S^k \to 0$. To see why, sum both sides from 0 to k to get

$$\sum_{i=0}^{k} S^i \le V^0 - V^{k+1} \le V^0.$$

Taking $k \to \infty$ gives us

$$\sum_{i=0}^{\infty} S^i \le V^0 < \infty,$$

and we say the sequence $S^0, S^1, \dots$ is *summable*. Nonnegative summable sequences converge to 0, so $S^k \to 0$. If, furthermore, we can show that $S^0, S^1, \dots$ is nonincreasing, then

$$(k+1)S^k \le \sum_{i=0}^{k} S^i \le V^0,$$

and hence $S^k \le \frac{1}{k+1} V^0$. As an aside, we call V^k the *Lyapunov function* and S^k the *summable term*.

The proof technique of showing that a Lyapunov function produces a summable term, which converges to zero, is called the *summability argument*.

Stage 1 Note

$$\|(1-\theta)x + \theta y\|^2 = (1-\theta)\|x\|^2 + \theta\|y\|^2 - \theta(1-\theta)\|x-y\|^2,$$

for all $\theta \in \mathbb{R}$, $x, y \in \mathbb{R}^n$. Verifying the identity is a matter of expanding both sides.

Write $\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$, where $\mathbb{S}$ is nonexpansive. Write the FPI as

$$x^{k+1} = \mathbb{T}x^k = (1-\theta)x^k + \theta\mathbb{S}x^k.$$

For any $x^\star \in \operatorname{Fix}\mathbb{T}$, we use the previous identity to get

$$\begin{aligned} \|x^{k+1} - x^\star\|^2 &= (1-\theta)\|x^k - x^\star\|^2 + \theta\|\mathbb{S}(x^k) - x^\star\|^2 - \theta(1-\theta)\|\mathbb{S}(x^k) - x^k\|^2 \\ &\le (1-\theta)\|x^k - x^\star\|^2 + \theta\|x^k - x^\star\|^2 - \theta(1-\theta)\|\mathbb{S}(x^k) - x^k\|^2 \\ &= \underbrace{\|x^k - x^\star\|^2}_{=V^k} - \underbrace{\theta(1-\theta)\|\mathbb{S}(x^k) - x^k\|^2}_{=S^k}, \end{aligned} \tag{2.4}$$

where we used nonexpansiveness of $\mathbb{S}$ in the second line.

We now establish the monotonic decreases. The core inequality (2.4) tells us

$$\|x^{k+1} - x^\star\| \le \|x^k - x^\star\|$$

for any $x^\star \in \operatorname{Fix}\mathbb{T}$, that is, the distance of the iterates to any fixed point is monotonically nonincreasing. Minimizing both sides with respect to $x^\star \in \operatorname{Fix}\mathbb{T}$ gives us

$$\operatorname{dist}(x^{k+1}, \operatorname{Fix}\mathbb{T}) \le \operatorname{dist}(x^k, \operatorname{Fix}\mathbb{T}),$$

that is, the distance of the iterates to the set of fixed points is monotonically nonincreasing. As another aside, an algorithm is said to be *Fejér monotone* if the distance of the iterates to the solution set is monotonically nonincreasing.

We call $\mathbb{T}(x^k) - x^k = x^{k+1} - x^k$ the *fixed-point residual*. If $\mathbb{T}(x^k) - x^k = 0$, the FPI is at a fixed point, and the iteration stops, so one can use $\|\mathbb{T}(x^k) - x^k\|$ as a measure of progress of the FPI. Since $\mathbb{T}$ is nonexpansive, we have

$$\|x^{k+1} - x^k\| = \|\mathbb{T}x^k - \mathbb{T}x^{k-1}\| \le \|x^k - x^{k-1}\|,$$

that is, the magnitude of the fixed-point residual is monotonically nonincreasing.

Using the monotonic decrease of $\|x^{k+1} - x^k\|$, we obtain a rate of convergence for $\|x^{k+1} - x^k\| \to 0$. Summing the inequality (2.4) from 0 to k gives us

$$\|x^{k+1} - x^\star\|^2 \le \|x^0 - x^\star\|^2 - \frac{1-\theta}{\theta}\sum_{j=0}^{k}\|\mathbb{T}x^j - x^j\|^2.$$

Reorganizing, we get

$$\sum_{j=0}^{k}\|\mathbb{T}x^j - x^j\|^2 \le \frac{\theta}{1-\theta}\|x^0 - x^\star\|^2 - \frac{\theta}{1-\theta}\|x^{k+1} - x^\star\|^2.$$

With the monotonic decrease of $\|x^{k+1} - x^k\|$ we get

$$(k+1)\|x^{k+1} - x^k\|^2 \le \sum_{j=0}^{k}\|x^{j+1} - x^j\|^2 \le \frac{\theta}{1-\theta}\|x^0 - x^\star\|^2,$$

and we conclude that

$$\|x^{k+1} - x^k\|^2 \le \frac{\theta}{(k+1)(1-\theta)}\|x^0 - x^\star\|^2.$$

Minimizing the right-hand side with respect to $x^\star \in \operatorname{Fix}\mathbb{T}$, we get

$$\|x^{k+1} - x^k\|^2 \le \frac{\theta}{(k+1)(1-\theta)}\operatorname{dist}^2(x^0, \operatorname{Fix}\mathbb{T}).$$

Stage 2 We now show $x^k \to x^\star$ for some $x^\star \in \operatorname{Fix}\mathbb{T}$. Consider any $\tilde{x}^\star \in \operatorname{Fix}\mathbb{T}$. Then (2.4) tells us that $x^0, x^1, \ldots$ lie within the compact set $\{x \mid \|x - \tilde{x}^\star\| \le \|x^0 - \tilde{x}^\star\|\}$, and $x^0, x^1, \ldots$ has an accumulation point $x^\star$. Let x^{k_j} be a subsequence such that $x^{k_j} \to x^\star$. Then $(\mathbb{T} - \mathbb{I})(x^k) \to 0$ implies $(\mathbb{T} - \mathbb{I})(x^{k_j}) \to 0$. Since $\mathbb{T} - \mathbb{I}$ is continuous, $x^{k_j} \to x^\star$ and $(\mathbb{T} - \mathbb{I})(x^{k_j}) \to 0$ implies $(\mathbb{T} - \mathbb{I})(x^\star) = 0$. In other words, $x^\star \in \operatorname{Fix}\mathbb{T}$. Finally, applying (2.4) to this accumulation point $x^\star \in \operatorname{Fix}\mathbb{T}$, we conclude that $\|x^k - x^\star\|$ monotonically decreases to 0, that is, the entire sequence converges to $x^\star$. □

Termination Criterion

Although we avoid the discussion of termination criterion throughout this book for the sake of simplicity, detecting when an iterate is a sufficiently accurate approximation of the solution is essential for a practical iterative method. We simply point out that $\|x^{k+1} - x^k\| < \varepsilon$ for some small $\varepsilon > 0$ can generally be used as a termination criterion. Specific setups may have other termination criteria that better capture the particular goals of the setup.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

2.4.3 Methods

Gradient Descent

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x).$$

Assume f is CCP and differentiable. Then x is a solution if and only if

$$0 = \nabla f(x) \quad \Leftrightarrow \quad x = (\mathbb{I} - \alpha\nabla f)(x)$$

for any nonzero $\alpha \in \mathbb{R}$. In other words, x is a solution if and only if it is a fixed point of the operator $\mathbb{I} - \alpha\nabla f$.

The FPI for this setup is

$$x^{k+1} = x^k - \alpha\nabla f(x^k).$$

This algorithm is called the *gradient method* or *gradient descent*, and α is called the *stepsize*.

Now assume f is L-smooth. By the cocoercivity inequality,

$$\begin{aligned}
&\|(\mathbb{I} - (2/L)\nabla f)x - (\mathbb{I} - (2/L)\nabla f)y\|^2 \\
&= \|x-y\|^2 - \frac{4}{L}\left(\langle x-y, \nabla f(x) - \nabla f(y)\rangle - \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2\right) \\
&\le \|x-y\|^2.
\end{aligned}$$

Therefore, $\mathbb{I} - \alpha\nabla f$ is averaged for $\alpha \in (0, 2/L)$ since

$$\mathbb{I} - \alpha\nabla f = (1-\theta)\mathbb{I} + \theta(\mathbb{I} - (2/L)\nabla f),$$

where $\theta = \alpha L/2 < 1$. Consequently, $x^k \to x^\star$ for some solution $x^\star$, if one exists, with rate

$$\|\nabla f(x^k)\|^2 = O(1/k),$$

for any

$$\alpha \in (0, 2/L). \tag{2.5}$$

If we furthermore assume f is strongly convex, we can show the iteration is a contraction.

Forward Step Method

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 = \mathbb{F}(x),$$

where $\mathbb{F}\colon \mathbb{R}^n \to \mathbb{R}^n$.

By the same argument as for gradient descent, x is a solution if and only if it is a fixed point of $\mathbb{I} - \alpha\mathbb{F}$ for any nonzero $\alpha \in \mathbb{R}$. The FPI for this setup is

$$x^{k+1} = x^k - \alpha\mathbb{F}x^k,$$

which we call the *forward step method*.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

The forward step method converges if $\mathbb{F}$ is β-cocoercive and $\alpha \in (0, 2\beta)$. The forward step iteration is a contraction for small enough $\alpha > 0$ if $\mathbb{F}$ is strongly monotone and Lipschitz.

However, the method does not necessarily converge if $\mathbb{F}$ is merely monotone and Lipschitz. The operator

$$\mathbb{F}(x, y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is such an example, since the 2×2 matrix representing $\mathbb{I} - \alpha\mathbb{F}$ has singular values strictly greater than 1 for any $\alpha \neq 0$. (This operator arises as, say, the KKT operator of the problem of minimizing x subject to $x = 0$.) The scaled relative graphs of §13 will provide the geometric intuition of this counterexample.

Dual Ascent

Consider the primal-dual problem pair (1.6) and (1.7),

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b, \end{array} \qquad \underset{u \in \mathbb{R}^m}{\text{maximize}} \quad -f^*(-A^\intercal u) - b^\intercal u,$$

generated by the Lagrangian (1.5)

$$\mathbf{L}(x, u) = f(x) + \langle u, Ax - b \rangle.$$

Define $g(u) = f^*(-A^\intercal u) + b^\intercal u$. By the discussion of §1.3.8, if f is μ-strongly convex, then f^* is differentiable and ∇f^* is $(1/\mu)$-Lipschitz. By the discussion of §2.2.2,

$$\nabla g(u) = -A \nabla f^*(-A^\intercal u) + b$$

is Lipschitz with parameters $\sigma_{\max}^2(A)/\mu$.

Using (2.2), write the gradient method applied to g, the FPI on $\mathbb{I} - \alpha\nabla g$, as

$$\begin{aligned} x^{k+1} &= \underset{x}{\operatorname{argmin}}\, \mathbf{L}(x, u^k) \\ u^{k+1} &= u^k + \alpha(Ax^{k+1} - b). \end{aligned}$$

The first step is minimizing the Lagrangian, and the second is a multiplier update. This method is called the *Uzawa method* or *dual ascent*. If f is μ-strongly convex, total duality holds, and $0 < \alpha < 2\mu/\sigma_{\max}^2(A)$, then $x^k \to x^\star$ and $u^k \to u^\star$. See Exercise 2.17.

2.5 RESOLVENTS

The *resolvent* of an operator $\mathbb{A}$ is defined as

$$\mathbb{J}_{\mathbb{A}} = (\mathbb{I} + \mathbb{A})^{-1}.$$

The *reflected resolvent*, also called the *Cayley operator* or the *reflection operator*, of $\mathbb{A}$ is defined as

$$\mathbb{R}_{\mathbb{A}} = 2\mathbb{J}_{\mathbb{A}} - \mathbb{I}.$$

Often, we will use $\mathbb{J}_{\alpha\mathbb{A}}$ and $\mathbb{R}_{\alpha\mathbb{A}}$ with $\alpha > 0$. If $\mathbb{A}$ is maximal monotone, $\mathbb{R}_{\mathbb{A}}$ is a nonexpansive (single-valued) with $\operatorname{dom} \mathbb{R}_{\mathbb{A}} = \mathbb{R}^n$, and $\mathbb{J}_{\mathbb{A}}$ is a (1/2)-averaged with $\operatorname{dom} \mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Let us prove nonexpansiveness. Assume we have $(x,u),(y,v) \in \mathbb{J}_{\mathbb{A}}$. By definition of resolvents, we have

$$x \in u + \mathbb{A}u, \qquad y \in v + \mathbb{A}v.$$

By monotonicity of $\mathbb{A}$,

$$\langle (x-u) - (y-v), u - v\rangle \geq 0$$

and

$$\begin{aligned}\|(2u-x)-(2v-y)\|^2 &= \|x-y\|^2 - 4\langle (x-u)-(y-v), u-v\rangle \\ &\leq \|x-y\|^2.\end{aligned}$$

This proves $\mathbb{R}_{\mathbb{A}}$ is nonexpansive and therefore single-valued, and $\mathbb{J}_{\mathbb{A}} = (1/2)\mathbb{I} + (1/2)\mathbb{R}_{\mathbb{A}}$ is $(1/2)$-averaged.

The *Minty surjectivity theorem* states that $\operatorname{dom} \mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$ when $\mathbb{A}$ is maximal monotone. This result is easy to intuitively see in 1D but is nontrivial in higher dimensions. We prove this in §10.

Zero Set of a Maximal Monotone Operator

Using resolvents, we can quickly show $\operatorname{Zer} \mathbb{A}$ is a closed convex set when $\mathbb{A}$ is maximal monotone. Since

$$0 \in \mathbb{A}x \quad \Leftrightarrow \quad x \in x + \mathbb{A}x \quad \Leftrightarrow \quad \mathbb{J}_{\mathbb{A}}x = x,$$

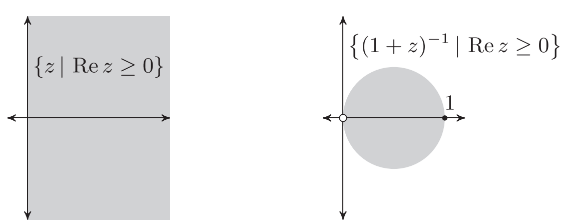
we have $\operatorname{Zer} \mathbb{A} = \operatorname{Fix} \mathbb{J}_{\mathbb{A}}$. Since $\mathbb{J}_{\mathbb{A}}$ is nonexpansive, $\operatorname{Fix} \mathbb{J}_{\mathbb{A}}$ is a closed convex set. Note that this proof relies on maximality through the condition $\operatorname{dom} \mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$.

Example 2.5 When $\mathbb{A}$ is a monotone linear operator represented by a symmetric matrix, it is easier to see why $\mathbb{J}_{\mathbb{A}}$ and $\mathbb{R}_{\mathbb{A}}$ are nonexpansive. In this case, $\mathbb{A}$ has eigenvalues in $[0,\infty)$ and $\mathbb{J}_{\mathbb{A}} = (\mathbb{I} + \mathbb{A})^{-1}$ has eigenvalues in $(0,1]$. The reflected resolvent,

$$\mathbb{R}_{\mathbb{A}} = 2\mathbb{J}_{\mathbb{A}} - \mathbb{I} = (\mathbb{I} - \mathbb{A})(\mathbb{I} + \mathbb{A})^{-1} = (\mathbb{I} + \mathbb{A})^{-1}(\mathbb{I} - \mathbb{A}),$$

also called the Cayley transform of $\mathbb{A}$, has eigenvalues in $(-1,1]$.

Example 2.6 Let $z \in \mathbb{C}$ be a complex number. We can identify z with a linear operator from $\mathbb{C}$ to $\mathbb{C}$ defined by multiplication, that is, we can view z as the operator that maps $x \mapsto zx$ for any $x \in \mathbb{C}$. We equip the set of complex numbers with the inner product $\langle x,y\rangle = \operatorname{Re} x\overline{y}$ for any $x,y \in \mathbb{C}$, where $\overline{y}$ is the complex conjugate of y. Then $z \in \mathbb{C}$ is a monotone operator if and only if $\operatorname{Re} z \geq 0$.



So a monotone z is a complex number on the right half-plane, and its resolvent $(1+z)^{-1}$ is a complex number within the disk with center $1/2$ and radius $1/2$ except for the origin.

2.5.1 Examples

Subdifferential

When f is CCP and $\alpha > 0$, we have

$$\mathbb{J}_{\alpha\partial f} = \mathrm{Prox}_{\alpha f}.$$

This follows from

$$\begin{aligned} z = (I + \alpha\partial f)^{-1}(x) \quad &\Leftrightarrow \quad z + \alpha\partial f(z) \ni x \\ &\Leftrightarrow \quad 0 \in \partial_z \left(\alpha f(z) + \frac{1}{2}\|z - x\|^2\right) \\ &\Leftrightarrow \quad z = \operatorname*{argmin}_z \left\{\alpha f(z) + \frac{1}{2}\|z - x\|^2\right\} \\ &\Leftrightarrow \quad z = \mathrm{Prox}_{\alpha f}(x). \end{aligned}$$

Subdifferential of Conjugate

Let $g(u) = f^*(A^\intercal u)$, and assume f is CCP and $\mathrm{ri}\,\mathrm{dom}\,f^* \cap \mathcal{R}(A^\intercal) \neq \emptyset$. Then

$$v = \mathrm{Prox}_{\alpha g}(u) \quad \Leftrightarrow \quad \begin{array}{l} x \in \mathrm{argmin}_x \left\{f(x) - \langle u, Ax\rangle + \frac{\alpha}{2}\|Ax\|^2\right\} \\ v = u - \alpha Ax. \end{array} \tag{2.6}$$

This follows from

$$\begin{aligned} v = (I + \alpha\partial g)^{-1}(u) \quad &\Leftrightarrow \quad v + \alpha A\partial f^*(A^\intercal v) \ni u \\ &\Leftrightarrow \quad v + \alpha Ax = u,\ x \in \partial f^*(A^\intercal v) \\ &\Leftrightarrow \quad v = u - \alpha Ax,\ \partial f(x) \ni A^\intercal v \\ &\Leftrightarrow \quad v = u - \alpha Ax,\ \partial f(x) \ni A^\intercal(u - \alpha Ax) \\ &\Leftrightarrow \quad v = u - \alpha Ax,\ x \in \operatorname*{argmin}_x \left\{f(x) - \langle u, Ax\rangle + \frac{\alpha}{2}\|Ax\|^2\right\}. \end{aligned}$$

Projection

Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set. Remember from §1 that δ_C is the indicator function of C, $\mathbb{N}_C$ is the normal cone operator of C, and Π_C is the projection onto C. These satisfy the following properties: $\delta_C = \alpha\delta_C$ and $\mathbb{N}_C = \alpha\mathbb{N}_C$ for any $\alpha > 0$; $\partial\delta_C = \mathbb{N}_C$; and $\mathbb{J}_{\mathbb{N}_C} = \mathrm{Prox}_{\delta_C} = \Pi_C$.

KKT Operator for Linearly Constrained Problems

Consider the Lagrangian

$$\mathbf{L}(x, u) = f(x) + \langle u, Ax - b\rangle,$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

which generates the primal problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b. \end{array}$$

We can compute its resolvent with

$$\mathbb{J}_{\alpha\partial\mathbf{L}}(x,u) = (y,v) \quad \Leftrightarrow \quad \begin{array}{l} y = \text{argmin}_z \left\{\mathbf{L}_\alpha(z,u) + \frac{1}{2\alpha}\|z-x\|^2\right\} \\ v = u + \alpha(Ay-b), \end{array} \tag{2.7}$$

where $\mathbf{L}_\alpha = f(x) + \langle u, Ax-b\rangle + \frac{\alpha}{2}\|Ax-b\|^2$ is the augmented Lagrangian of (1.11).

Let us show this. For any $\alpha > 0$, we have

$$\begin{aligned} \mathbb{J}_{\alpha\partial\mathbf{L}}(x,u) = (y,v) \quad &\Leftrightarrow \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} y \\ v \end{bmatrix} + \alpha \begin{bmatrix} \partial f(y) + A^\intercal v \\ b - Ay \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \alpha \begin{bmatrix} \partial f(y) \\ b \end{bmatrix} + \begin{bmatrix} I & \alpha A^\intercal \\ -\alpha A & I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}. \end{aligned}$$

We left-multiply the invertible matrix

$$\begin{bmatrix} I & -\alpha A^\intercal \\ 0 & I \end{bmatrix}$$

to get

$$\Leftrightarrow \quad \begin{bmatrix} x - \alpha A^\intercal u \\ u \end{bmatrix} \in \alpha \begin{bmatrix} \partial f(y) - \alpha A^\intercal b \\ b \end{bmatrix} + \begin{bmatrix} I + \alpha^2 A^\intercal A & 0 \\ -\alpha A & I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.$$

We call this the *Gaussian elimination technique* and discuss it in more detail in §3.4. Now that the first line of the inclusion is independent of v, we can compute y first and then compute v. Reorganizing, we get

$$\begin{aligned} 0 \in \partial f(y) + A^\intercal u + \alpha A^\intercal(Ay-b) + (1/\alpha)(y-x) \\ v = u + \alpha(Ay-b), \end{aligned}$$

and we have the formula

$$\begin{aligned} &y = \underset{z}{\text{argmin}} \left\{ f(z) + \langle u, Az-b\rangle + \frac{\alpha}{2}\|Az-b\|^2 + \frac{1}{2\alpha}\|z-x\|^2 \right\} \\ &v = u + \alpha(Ay-b). \end{aligned}$$

2.5.2 Basic Identities

Resolvent Identities

If $\mathbb{A}$ is maximal monotone, $\alpha > 0$, and $\mathbb{B}(x) = \mathbb{A}(x) + t$, then

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = \mathbb{J}_{\alpha\mathbb{A}}(u - \alpha t). \tag{2.8}$$

This follows from

$$\begin{aligned} \mathbb{J}_{\alpha\mathbb{B}} u = v \quad &\Leftrightarrow \quad u \in v + \alpha\mathbb{B}v \\ &\Leftrightarrow \quad u - \alpha t \in v + \alpha\mathbb{A}v \\ &\Leftrightarrow \quad v = \mathbb{J}_{\alpha\mathbb{A}}(u - \alpha t). \end{aligned}$$

With similar calculations, one can show that if $\mathbb{A}$ is maximal monotone, $\alpha > 0$, and $\mathbb{B}(x) = \mathbb{A}(x - t)$, then

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = \mathbb{J}_{\alpha\mathbb{A}}(u - t) + t, \tag{2.9}$$

and if $\mathbb{A}$ is maximal monotone, $\alpha > 0$, and $\mathbb{B}(x) = -\mathbb{A}(t - x)$, then

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = t - \mathbb{J}_{\alpha\mathbb{A}}(t - u). \tag{2.10}$$

The *inverse resolvent identity* states

$$\mathbb{J}_{\alpha^{-1}\mathbb{A}}(x) + \alpha^{-1}\mathbb{J}_{\alpha\mathbb{A}^{-1}}(\alpha x) = x, \tag{2.11}$$

for maximal monotone $\mathbb{A}$ and $\alpha > 0$. This follows from

$$\begin{aligned}
x - \mathbb{J}_{\alpha^{-1}\mathbb{A}}x = y \quad &\Leftrightarrow \quad x \in x - y + \alpha^{-1}\mathbb{A}(x - y)\\
&\Leftrightarrow \quad \alpha y \in \mathbb{A}(x - y)\\
&\Leftrightarrow \quad \mathbb{A}^{-1}(\alpha y) \ni x - y\\
&\Leftrightarrow \quad (\mathbb{I} + \alpha\mathbb{A}^{-1})(\alpha y) \ni \alpha x\\
&\Leftrightarrow \quad y = (1/\alpha)\mathbb{J}_{\alpha\mathbb{A}^{-1}}(\alpha x).
\end{aligned}$$

When $\alpha = 1$, we get the further elegant formula

$$\mathbb{J}_{\mathbb{A}} + \mathbb{J}_{\mathbb{A}^{-1}} = \mathbb{I}.$$

The *Moreau identity*, a special case, states that for any CCP f,

$$\mathrm{Prox}_f + \mathrm{Prox}_{f^*} = \mathbb{I},$$

or more generally,

$$\mathrm{Prox}_{\alpha^{-1}f}(x) + \alpha^{-1}\mathrm{Prox}_{\alpha f^*}(\alpha x) = x. \tag{2.12}$$

An important practical consequence of the Moreau identity is that $\mathrm{Prox}_{\alpha f}$ and $\mathrm{Prox}_{\alpha f^*}$ require essentially the same computational cost. In other words, if you can compute $\mathrm{Prox}_{\alpha f}$, then you can compute $\mathrm{Prox}_{\alpha f^*}$, and vice versa.

Reflected Resolvent Identities

If $\mathbb{A}$ is maximal monotone and single-valued and $\alpha > 0$, we have

$$\mathbb{R}_{\alpha\mathbb{A}} = (\mathbb{I} - \alpha\mathbb{A})(\mathbb{I} + \alpha\mathbb{A})^{-1}.$$

This follows from

$$\begin{aligned}
\mathbb{R}_{\alpha\mathbb{A}} &= 2(\mathbb{I} + \alpha\mathbb{A})^{-1} - \mathbb{I}\\
&= 2(\mathbb{I} + \alpha\mathbb{A})^{-1} - (\mathbb{I} + \alpha\mathbb{A})(\mathbb{I} + \alpha\mathbb{A})^{-1}\\
&= (\mathbb{I} - \alpha\mathbb{A})(\mathbb{I} + \alpha\mathbb{A})^{-1},
\end{aligned}$$

where we used the result of Exercise 2.1 in the second equality.

If $\mathbb{A}$ is maximal monotone (but not necessarily single-valued) and $\alpha > 0$, we have

$$\mathbb{R}_{\alpha\mathbb{A}}(\mathbb{I} + \alpha\mathbb{A}) = \mathbb{I} - \alpha\mathbb{A}. \tag{2.13}$$

Let us prove this. Since $(\mathbb{I} + \alpha\mathbb{A})^{-1}$ is single-valued, for any $x \in \mathrm{dom}\,\mathbb{A}$ we have

$$\mathbb{R}_{\alpha\mathbb{A}}(\mathbb{I} + \alpha\mathbb{A})(x) = 2(\mathbb{I} + \alpha\mathbb{A})^{-1}(\mathbb{I} + \alpha\mathbb{A})(x) - (\mathbb{I} + \alpha\mathbb{A})(x)$$

$$\begin{aligned} &= 2\mathbb{I}(x) - (\mathbb{I} + \alpha\mathbb{A})(x) \\ &= (\mathbb{I} - \alpha\mathbb{A})(x), \end{aligned}$$

where we used the result of Exercise 2.1 in the second equality. For any $x \notin \operatorname{dom} \mathbb{A}$, both sides are empty sets.

2.6 PROXIMAL POINT METHOD

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in \mathbb{A}x,$$

where $\mathbb{A}$ is maximal monotone. This problem is equivalent to finding a fixed point of $\mathbb{J}_{\alpha\mathbb{A}}$, since $\operatorname{Zer}\mathbb{A} = \operatorname{Fix}\mathbb{J}_{\alpha\mathbb{A}}$ for any $\alpha > 0$. The FPI

$$x^{k+1} = \mathbb{J}_{\alpha\mathbb{A}}(x^k),$$

called the *proximal point method* (PPM) or *proximal minimization*, converges to a solution if one exists, since $\mathbb{J}_{\alpha\mathbb{A}}$ is averaged.

2.6.1 Methods of Multipliers

Consider the primal-dual problem pair,

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & Ax = b, \end{array} \qquad \underset{u\in\mathbb{R}^m}{\text{maximize}} \quad -f^*(-A^\intercal u) - b^\intercal u,$$

of (1.6) and (1.7) generated by the Lagrangian $\mathbf{L}(x,u) = f(x) + \langle u, Ax - b\rangle$. The associated augmented Lagrangian discussed in Example 1.11 is

$$\mathbf{L}_\alpha(x,u) = f(x) + \langle u, Ax - b\rangle + \frac{\alpha}{2}\|Ax - b\|^2.$$

Method of Multipliers

Assume $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq \emptyset$. Write $g(u) = f^*(-A^\intercal u) + b^\intercal u$ for the dual function. Using (2.6) and (2.8), we can write the FPI $u^{k+1} = \mathbb{J}_{\alpha\partial g}(u^k)$ with $\alpha > 0$ as

$$\begin{aligned} x^{k+1} &\in \operatorname*{argmin}_x \mathbf{L}_\alpha(x, u^k) \\ u^{k+1} &= u^k + \alpha(Ax^{k+1} - b), \end{aligned}$$

which is called the *method of multipliers*, also known as the *augmented Lagrangian method* or *ALM*. The first step is minimizing the augmented Lagrangian, and the second is a multiplier update.

If a dual solution exists and $\alpha > 0$, then $u^k \to u^\star$. If we further assume f is strictly convex, we can show $x^k \to x^\star$. See Exercises 2.18 and 10.4.

Proximal Method of Multipliers

Using (2.7), we can write the FPI $(x^{k+1}, u^{k+1}) = \mathbb{J}_{\alpha\partial\mathbf{L}}(x^k, u^k)$ with $\alpha > 0$ as

$$\begin{aligned} x^{k+1} &= \operatorname*{argmin}_x \left\{ \mathbf{L}_\alpha(x, u^k) + \frac{1}{2\alpha}\|x - x^k\|^2 \right\} \\ u^{k+1} &= u^k + \alpha(Ax^{k+1} - b), \end{aligned}$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

which is called the *proximal method of multipliers*, also *the proximal augmented Lagrangian method*. The first step is minimizing the augmented Lagrangian with an additional proximal term, and the second is a multiplier update. If total duality holds and $\alpha > 0$, then $x^k \to x^\star$ and $u^k \to u^\star$.

The proximal method of multipliers becomes useful when it is combined with the linearization technique. We discuss this in §3.5.

2.7 OPERATOR SPLITTING

Consider the monotone inclusion problems of finding an $x \in \operatorname{Zer}(\mathbb{A}+\mathbb{B})$ or $x \in \operatorname{Zer}(\mathbb{A}+\mathbb{B}+\mathbb{C})$, where $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are maximal monotone. In this section, we present a few *base splitting schemes*, which transform these monotone inclusion problems into fixed-point equations with averaged operators constructed from $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, and their resolvents.

The key technique is to formulate a given optimization problem as a monotone inclusion problem, apply one of the base splitting schemes, and use the fixed-point iteration discussed in §2.4.2, or the randomized coordinate or asynchronous variants of §5 and §6. The main message of Part I of this book is that a wide range of methods can be derived and analyzed through this unified approach.

2.7.1 Base Splitting Schemes

Forward-Backward and Backward-Forward Splitting

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A}+\mathbb{B})x,$$

where $\mathbb{A}$ and $\mathbb{B}$ are maximal monotone and $\mathbb{A}$ is single-valued. Then for any $\alpha > 0$, we have

$$\begin{aligned} 0 \in (\mathbb{A}+\mathbb{B})x \quad &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{B})x - (\mathbb{I}-\alpha\mathbb{A})x \\ &\Leftrightarrow \quad (\mathbb{I}+\alpha\mathbb{B})x \ni (\mathbb{I}-\alpha\mathbb{A})x \\ &\Leftrightarrow \quad x = \mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I}-\alpha\mathbb{A})x. \end{aligned}$$

So, x is a solution if and only if it is a fixed point of $\mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I}-\alpha\mathbb{A})$. This splitting is called *forward-backward splitting* (FBS).

Assume $\mathbb{A}$ is β-cocoercive and $\alpha \in (0,2\beta)$. Then the *forward step* $\mathbb{I}-\alpha\mathbb{A}$ and the *backward step* $(\mathbb{I}+\alpha\mathbb{B})^{-1}$ are averaged. So, the composition $\mathbb{J}_{\alpha\mathbb{B}}(\mathbb{I}-\alpha\mathbb{A})$ is an averaged operator.

The FPI with FBS

$$x^{k+1} = \mathbb{J}_{\alpha\mathbb{B}}(x^k - \alpha\mathbb{A}x^k)$$

converges if $\alpha \in (0,2\beta)$ and $\operatorname{Zer}(\mathbb{A}+\mathbb{B}) \neq \emptyset$.

We can also consider a similar splitting with a permuted order:

$$\begin{aligned} 0 \in (\mathbb{A}+\mathbb{B})x \quad &\Leftrightarrow \quad (\mathbb{I}+\alpha\mathbb{B})x \ni (\mathbb{I}-\alpha\mathbb{A})x \\ &\Leftrightarrow \quad z = (\mathbb{I}-\alpha\mathbb{A})x,\ z \in (\mathbb{I}+\alpha\mathbb{B})x \\ &\Leftrightarrow \quad z = (\mathbb{I}-\alpha\mathbb{A})x,\ \mathbb{J}_{\alpha\mathbb{B}}z = x \\ &\Leftrightarrow \quad z = (\mathbb{I}-\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}z,\ \mathbb{J}_{\alpha\mathbb{B}}z = x. \end{aligned}$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

So, x is a solution if and only if there is a $z \in \operatorname{Fix}(\mathbb{I}-\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}$ and $x = \mathbb{J}_{\alpha\mathbb{B}}z$. This splitting is called *backward-forward splitting* (BFS).

The FPI with BFS

$$\begin{aligned} x^{k+1} &= \mathbb{J}_{\alpha\mathbb{B}} z^k \\ z^{k+1} &= x^{k+1} - \alpha\mathbb{A}x^{k+1} \end{aligned}$$

converges if $\alpha \in (0, 2\beta)$ and $\operatorname{Zer}(\mathbb{A}+\mathbb{B}) \neq \emptyset$.

Since BFS is FBS with the order permuted, BFS may seem like an unnecessary complication. In fact, the FPIs with FBS and BFS have the same iterates if the starting points x^0 for FBS and z^0 for BFS are matched in the sense that $x^0 = \mathbb{J}_{\alpha\mathbb{B}}z^0$. However, we will later see that BFS can be more natural to work with when using the randomized or asynchronous coordinate fixed-point iterations of §5 and §6.

Peaceman–Rachford and Douglas–Rachford Splitting

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A}+\mathbb{B})x,$$

where $\mathbb{A}$ and $\mathbb{B}$ are maximal monotone.

For any $\alpha > 0$, we have

$$\begin{aligned} 0 \in (\mathbb{A}+\mathbb{B})x \quad &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - (\mathbb{I}-\alpha\mathbb{B})x \\ &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - \mathbb{R}_{\alpha\mathbb{B}}(\mathbb{I}+\alpha\mathbb{B})x \\ &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - \mathbb{R}_{\alpha\mathbb{B}}z,\ z \in (\mathbb{I}+\alpha\mathbb{B})x \\ &\Leftrightarrow \quad \mathbb{R}_{\alpha\mathbb{B}}z \in (\mathbb{I}+\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}z,\ x = \mathbb{J}_{\alpha\mathbb{B}}z \\ &\Leftrightarrow \quad \mathbb{J}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}z = \mathbb{J}_{\alpha\mathbb{B}}z,\ x = \mathbb{J}_{\alpha\mathbb{B}}z \\ &\Leftrightarrow \quad \mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}z = z,\ x = \mathbb{J}_{\alpha\mathbb{B}}z, \end{aligned}$$

where we have used (2.13). So x is a solution if and only if there is a $z \in \operatorname{Fix}\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}$ and $x = \mathbb{J}_{\alpha\mathbb{B}}z$. This splitting is called *Peaceman–Rachford splitting* (PRS).

Since the operator $\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}$ is merely nonexpansive, the FPI with PRS

$$z^{k+1} = \mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}(z^k) \tag{2.14}$$

is not guaranteed to converge. See Exercise 2.27.

To ensure convergence, we average. For any $\alpha > 0$, we have

$$0 \in (\mathbb{A}+\mathbb{B})x \quad \Leftrightarrow \quad \left(\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}\right)(z) = z,\ x = \mathbb{J}_{\alpha\mathbb{B}}(z).$$

This splitting is called *Douglas–Rachford splitting* (DRS).

The FPI with DRS

$$\begin{aligned} x^{k+1/2} &= \mathbb{J}_{\alpha\mathbb{B}}(z^k) \\ x^{k+1} &= \mathbb{J}_{\alpha\mathbb{A}}(2x^{k+1/2} - z^k) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2} \end{aligned}$$

converges for any $\alpha > 0$ if $\operatorname{Zer}(\mathbb{A}+\mathbb{B}) \neq \emptyset$. See Exercise 2.26.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

We can think of the $x^{k+1/2}$- and x^{k+1}-iterates as estimates of a solution with different properties. For example, if $\mathbb{J}_{\alpha\mathbb{B}}$ is a projection onto a constraint set, $x^{k+1/2}$-iterates satisfy these constraints exactly.

Davis–Yin Splitting

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A}+\mathbb{B}+\mathbb{C})x,$$

where $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are maximal monotone, and $\mathbb{C}$ is single-valued.

Then for any $\alpha > 0$, we have

$$\begin{aligned}
0 \in (\mathbb{A}+\mathbb{B}+\mathbb{C})x \quad &\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - (\mathbb{I}-\alpha\mathbb{B})x + \alpha\mathbb{C}x \\
&\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - \mathbb{R}_{\alpha\mathbb{B}}(\mathbb{I}+\alpha\mathbb{B})x + \alpha\mathbb{C}x \\
&\Leftrightarrow \quad 0 \in (\mathbb{I}+\alpha\mathbb{A})x - \mathbb{R}_{\alpha\mathbb{B}}z + \alpha\mathbb{C}x,\ z \in (\mathbb{I}+\alpha\mathbb{B})x \\
&\Leftrightarrow \quad (\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}})z \in (\mathbb{I}+\alpha\mathbb{A})\mathbb{J}_{\alpha\mathbb{B}}z, \\
&\qquad\quad x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad \mathbb{J}_{\alpha\mathbb{A}}(\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}})z = \mathbb{J}_{\alpha\mathbb{B}}z,\ x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad (\mathbb{R}_{\alpha\mathbb{A}}(\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}}) - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}})z = z, \\
&\qquad\quad x = \mathbb{J}_{\alpha\mathbb{B}}z \\
&\Leftrightarrow \quad ((1/2)\mathbb{I} + (1/2)\mathbb{T})z = z,\ x = \mathbb{J}_{\alpha\mathbb{B}}z, \\
&\qquad\quad \mathbb{T} = \mathbb{R}_{\alpha\mathbb{A}}(\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}}) - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}}.
\end{aligned}$$

So, x is a solution if and only if there is a $z \in \operatorname{Fix}((1/2)\mathbb{I} + (1/2)\mathbb{T})$ and $x = \mathbb{J}_{\alpha\mathbb{B}}z$. This splitting is called *Davis–Yin splitting* (DYS). We can also write

$$(1/2)\mathbb{I} + (1/2)\mathbb{T} = \mathbb{I} - \mathbb{J}_{\alpha\mathbb{B}} + \mathbb{J}_{\alpha\mathbb{A}}(\mathbb{R}_{\alpha\mathbb{B}} - \alpha\mathbb{C}\mathbb{J}_{\alpha\mathbb{B}}).$$

Assume $\mathbb{C}$ is β-cocoercive and $\alpha \in (0, 2\beta)$, then $(1/2)\mathbb{I} + (1/2)\mathbb{T}$ is averaged. We prove this in §13 as Theorem 28. $\mathbb{T}$ itself may not be nonexpansive. The FPI with DYS

$$\begin{aligned}
x^{k+1/2} &= \mathbb{J}_{\alpha\mathbb{B}}(z^k) \\
x^{k+1} &= \mathbb{J}_{\alpha\mathbb{A}}(2x^{k+1/2} - z^k - \alpha\mathbb{C}x^{k+1/2}) \\
z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}
\end{aligned}$$

converges for $\alpha \in (0, 2\beta)$ if $\operatorname{Zer}(\mathbb{A}+\mathbb{B}+\mathbb{C}) \neq \emptyset$. Note that DYS reduces to BFS when $\mathbb{A} = 0$, to FBS when $\mathbb{B} = 0$, and to DRS when $\mathbb{C} = 0$.

2.7.2 Splitting for Convex Optimization and Total Duality

In §3, we combine the base splittings with various techniques to derive a wide range of methods. In this section, we directly apply the base splittings to convex optimization problems as is.

Proximal Gradient Method

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x),$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

where f and g are CCP functions on $\mathbb{R}^n$ and f is differentiable. Then x is a solution if and only if $x \in \operatorname{Zer}(\nabla f + \partial g)$.

The FPI with FBS is

$$x^{k+1} = \operatorname{Prox}_{\alpha g}(x^k - \alpha \nabla f(x^k)),$$

which is also called the *proximal gradient method*. Assume a primal solution exists, f is L-smooth, and $\alpha \in (0, 2/L)$. Then $x^k \to x^\star$.

We can write the proximal gradient method equivalently as

$$x^{k+1} = \operatorname*{argmin}_x \left\{ f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + g(x) + \frac{1}{2\alpha} \|x - x^k\|_2^2 \right\}.$$

So, the proximal gradient method uses a first-order approximation of f about x^k.

When $g = \delta_C$ for some nonempty convex set C, the proximal gradient method reduces to the *projected gradient method*:

$$x^{k+1} = \Pi_C(x^k - \alpha \nabla f(x^k)).$$

DRS for Convex Optimization and Total Duality

Consider the primal-dual problem pair

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x) \tag{2.15}$$

and

$$\underset{u \in \mathbb{R}^n}{\text{maximize}} \quad -f^*(-u) - g^*(u) \tag{2.16}$$

generated by the Lagrangian

$$\mathbf{L}(x, u) = f(x) + \langle x, u \rangle - g^*(u), \tag{2.17}$$

where f and g are CCP functions on $\mathbb{R}^n$.

As we soon prove, the primal problem is equivalent to

$$\underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 \in (\partial f + \partial g)x$$

when total duality holds. The FPI with DRS is

$$\begin{aligned} x^{k+1/2} &= \operatorname{Prox}_{\alpha g}(z^k) \\ x^{k+1} &= \operatorname{Prox}_{\alpha f}(2x^{k+1/2} - z^k) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}. \end{aligned} \tag{2.18}$$

Assume total duality holds and $\alpha > 0$. Then $x^k \to x^\star$ and $x^{k+1/2} \to x^\star$. In §9, we furthermore show that fixed points are of the form $z^\star = x^\star + \alpha u^\star$. So, $z^k \to x^\star + \alpha u^\star$.

The FPI with DRS requires f and g to be CCP, and the method converges for all $\alpha > 0$. In contrast, the proximal gradient method furthermore requires f to be L-smooth, and the parameter α must lie within a specific range. DRS is useful when evaluating $\operatorname{Prox}_{\alpha f}$ and $\operatorname{Prox}_{\alpha g}$ is easy but evaluating $\operatorname{Prox}_{\alpha(f+g)}$ is not. The proximal gradient method is useful when evaluating ∇f and $\operatorname{Prox}_{\alpha g}$ is easy. The proximal point method is useful when evaluating $\operatorname{Prox}_{\alpha(f+g)}$ is easy.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Note that although the primal problem (2.15) is symmetric in f and g, the dual problem (2.16) is not. Swapping the roles of f and g changes the sign of the dual variable. The algorithm (2.18) is also not symmetric in f and g, and swapping the roles of f and g changes the sign of the dual variable in $z^k \to x^\star + \alpha u^\star$.

DYS for Convex Optimization and Total Duality

Consider the primal-dual problem pair

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x) + h(x)$$

and

$$\underset{u \in \mathbb{R}^n}{\text{maximize}} \quad -(f+h)^*(-u) - g^*(u)$$

generated by the Lagrangian

$$\mathbf{L}(x,u) = f(x) + h(x) + \langle x, u \rangle - g^*(u).$$

The FPI with DYS is

$$\begin{aligned} x^{k+1/2} &= \text{Prox}_{\alpha g}(z^k) \\ x^{k+1} &= \text{Prox}_{\alpha f}(2x^{k+1/2} - z^k - \alpha \nabla h(x^{k+1/2})) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}. \end{aligned}$$

Assume total duality holds, h is L-smooth, and $\alpha \in (0, 2/L)$. Then $x^k \to x^\star$ and $x^{k+1/2} \to x^\star$. In §9, we furthermore show that fixed points are of the form $z^\star = x^\star + \alpha u^\star$. So, $z^k \to x^\star + \alpha u^\star$.

Necessity and Sufficiency of Total Duality

The following equivalence summarizes the role of total duality in splitting methods:

$$\text{argmin}(f+g) = \text{Zer}(\partial f + \partial g) \neq \emptyset \quad \Leftrightarrow \quad \text{total duality holds between (2.15) and (2.16).}$$

Therefore, we can write

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) + g(x) \qquad \Leftrightarrow \qquad \underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 \in (\partial f + \partial g)(x)$$

when total duality holds. This fact explains why total duality is required for the convergence of so many operator splitting methods.

Let us see why. First, assume that total duality holds. Then $x^\star \in \text{argmin}(f+g)$ if and only if $(x^\star, u^\star)$ is a saddle point of

$$\mathbf{L}(x,u) = f(x) + \langle x, u \rangle - g^*(u)$$

for some $u^\star \in \mathbb{R}^n$, and

$$\begin{aligned} (x^\star, u^\star) \text{ is a saddle point of } \mathbf{L} \quad &\Leftrightarrow \quad 0 \in \partial \mathbf{L}(x^\star, u^\star) \\ &\Leftrightarrow \quad 0 \in \partial_x \mathbf{L}(x^\star, u^\star),\ 0 \in \partial_u(-\mathbf{L})(x^\star, u^\star) \\ &\Leftrightarrow \quad -u^\star \in \partial f(x^\star),\ u^\star \in \partial g(x^\star) \\ &\Leftrightarrow \quad 0 \in (\partial f + \partial g)(x^\star). \end{aligned}$$

We conclude that $\text{argmin}(f+g) = \text{Zer}(\partial f + \partial g) \neq \emptyset$.

Next, assume $\operatorname{argmin}(f+g) = \operatorname{Zer}(\partial f + \partial g) \neq \emptyset$. Then any $x^\star \in \operatorname{argmin}(f+g)$ satisfies $0 \in (\partial f + \partial g)(x^\star)$. By a similar chain of arguments, $(x^\star, u^\star)$ is a saddle point of $\mathbf{L}$ for some $u^\star \in \mathbb{R}^n$, and we conclude that total duality holds.

2.7.3 Discussion

Fixed-Point Encoding

A *fixed-point encoding* establishes a correspondence between solutions of a monotone inclusion problem and fixed points of a related operator. The splittings we discussed are fixed-point encodings.

Upon reading §2.7.1, one may ask why there is no "forward-forward" splitting. A "forward-forward splitting" of the form $\mathbb{I} - \alpha(\mathbb{A} + \mathbb{B})$ is an instance of the forward-step method. A "forward-forward splitting" of the form $(\mathbb{I} - \alpha\mathbb{A})(\mathbb{I} - \beta\mathbb{B})$ would not be a valid fixed-point encoding; that is, we cannot recover a zero of $A + B$ from a fixed point of $(\mathbb{I} - \alpha\mathbb{A})(\mathbb{I} - \beta\mathbb{B})$. Likewise, a "backward-backward splitting" of the form $\mathbb{J}_{\alpha\mathbb{A}}\mathbb{J}_{\alpha\mathbb{B}}$ is not a valid fixed-point encoding. See Exercise 2.28.

Why Use the Resolvent?

The splittings we discuss use resolvents or direct evaluations of single-valued operators. Why do we not use other operators such as $(\mathbb{I} - \alpha\mathbb{A})^{-1}$? One reason is computational convenience. The resolvent is often easy to evaluate for many interesting operators, while evaluating something like $(\mathbb{I} - \alpha\partial f)^{-1}$ is often difficult.

Another reason is that only single-valued operators are, in a sense, algorithmically actionable. On a computer, we can compute and store a vector in $\mathbb{R}^n$, but we cannot store a subset of $\mathbb{R}^n$ in most cases. While multi-valued operators are a useful mathematical concept, single-valued operators, such as resolvents, are more algorithmically useful.

The Role of Maximality

An FPI $x^{k+1} = \mathbb{T}x^k$ becomes undefined if its iterates ever escape the domain of $\mathbb{T}$. In §2.4.2, we implicitly assumed $\operatorname{dom}\mathbb{T} = \mathbb{R}^n$ through stating $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$. When the operators are maximal monotone, FPIs defined with resolvents do not run into this issue.

So, we assume maximality out of theoretical necessity, but in practice the non-maximal monotone operators, such as the gradient operator of a nonconvex function, are usually ones we cannot efficiently compute the resolvent for anyway. In other words, there is little need to consider resolvents of non-maximal monotone operators, theoretically or practically.

Computational Efficiency

These base splitting methods are useful when the operators used in the splitting are efficient to compute. For example, although the convergence of DRS iteration

$$z^{k+1} = \left(\frac{1}{2}\mathbb{I} + \frac{1}{2}\mathbb{R}_{\alpha\mathbb{A}}\mathbb{R}_{\alpha\mathbb{B}}\right) z^k$$

does not depend on the value of α, it is most useful when $\mathbb{R}_{\alpha\mathbb{A}}$ and $\mathbb{R}_{\alpha\mathbb{B}}$ can be computed efficiently.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

For a given optimization problem, there is often more than one applicable method. The trick is to find a method using computationally efficient split components.

2.7.4 Methods

LASSO and ISTA

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \frac{1}{2}\|Ax-b\|^2 + \lambda\|x\|_1,$$

for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $\lambda > 0$. This particular optimization problem is called LASSO. Let $S(x;\kappa)$ be the soft-thresholding operator of Example 1.12.

The FPI with DRS

$$\begin{aligned} x^{k+1/2} &= (I+\alpha A^\intercal A)^{-1}(z^k + \alpha A^\intercal b) \\ x^{k+1} &= S(2x^{k+1/2} - z^k; \alpha\lambda) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2} \end{aligned}$$

converges for any $\alpha > 0$.

The FPI with FBS

$$x^{k+1} = S(x^k - \alpha A^\intercal(Ax^k - b); \alpha\lambda)$$

converges for $0 < \alpha < 2/\lambda_{\max}(A^\intercal A)$. This particular instance of the proximal gradient method is called the Iterative Shrinkage-Thresholding Algorithm (ISTA).

Note that DRS uses the matrix inverse $(I+\alpha A^\intercal A)^{-1}$, while FBS does not. When m and n are large, computing the matrix inverse can be prohibitively expensive. Therefore, FBS is the more computationally effective splitting for large-scale LASSO problems.

Consensus Technique

Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^m g_i(x),$$

where $g_1,\dots,g_m$ are CCP functions on $\mathbb{R}^n$. This problem is equivalent to

$$\begin{array}{ll} \underset{\mathbf{x}\in\mathbb{R}^{nm}}{\text{minimize}} & \sum_{i=1}^m g_i(x_i) \\ \text{subject to} & \mathbf{x} \in C, \end{array}$$

where $\mathbf{x} = (x_1,\dots,x_m)$ and

$$C = \{(x_1,\dots,x_m) \in \mathbb{R}^{nm} \mid x_1 = \cdots = x_m\} \tag{2.19}$$

is the *consensus set*. In turn, this problem is equivalent to

$$\underset{\mathbf{x}\in\mathbb{R}^{nm}}{\text{find}} \quad 0 \in \begin{bmatrix} \partial g_1(x_1) \\ \vdots \\ \partial g_m(x_m) \end{bmatrix} + \mathbb{N}_C(\mathbf{x}),$$

assuming $\bigcap_{i=1}^m \operatorname{int}\operatorname{dom} g_i \neq \emptyset$.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

The projection onto the consensus set is simple averaging:

$$\Pi_C \mathbf{x} = \overline{\mathbf{x}} = (\overline{x}, \overline{x}, \dots, \overline{x}), \qquad \overline{x} = \frac{1}{m}\sum_{i=1}^{m} x_i.$$

Define $\overline{\mathbf{z}}^k = \Pi_C \mathbf{z}^k$. The FPI with DRS for this setup

$$\begin{aligned} x_i^{k+1} &= \mathrm{Prox}_{\alpha g_i}(2\overline{z}^k - z_i^k - \alpha \nabla f_i(\overline{z}^k)) \qquad \text{for } i = 1, \dots, m, \\ \mathbf{z}^{k+1} &= \mathbf{z}^k + \mathbf{x}^{k+1} - \overline{\mathbf{z}}^k, \end{aligned}$$

converges for any $\alpha > 0$, if $\bigcap_{i=1}^m \mathrm{int}\,\mathrm{dom}\, g_i \neq \emptyset$ and a solution exists. Since $\mathrm{Prox}_{\alpha g_i}$ for $i = 1, \dots, m$ can be evaluated independently, this method is well-suited for parallel and distributed computing, which we discuss in §4.2.1 and §11.1.

The use of the consensus set (2.19) is called the *consensus technique* and it can more generally solve

$$\underset{x \in \mathbb{R}^n}{\text{find}} \quad 0 \in \sum_{i=1}^{m} \mathbb{A}_i x,$$

where $\mathbb{A}_1, \dots, \mathbb{A}_m$ are maximal monotone. See Exercise 2.36.

Forward-Douglas–Rachford

Consider the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^{m} (f_i(x) + g_i(x)),$$

where $g_1, \dots, g_m$ are CCP and $f_1, \dots, f_m$ are L-smooth. With the consensus technique, we can recast the problem into

$$\begin{array}{ll} \underset{\mathbf{x} \in \mathbb{R}^{nm}}{\text{minimize}} & \sum_{i=1}^{m} f_i(x_i) + \sum_{i=1}^{m} g_i(x_i) \\ \text{subject to} & \mathbf{x} \in C, \end{array}$$

where we use the same notation as we did for consensus optimization.

The FPI with DYS for this setup

$$\begin{aligned} x_i^{k+1} &= \mathrm{Prox}_{\alpha g_i}(2\overline{z}^k - z_i^k) \qquad \text{for } i = 1, \dots, m, \\ \mathbf{z}^{k+1} &= \mathbf{z}^k + \mathbf{x}^{k+1} - \overline{\mathbf{z}}^k \end{aligned}$$

is called *generalized forward-backward* or *forward-Douglas–Rachford*. This method converges if total duality holds, $\bigcap_{i=1}^m \mathrm{int}\,\mathrm{dom}\, g_i \neq \emptyset$, and $\alpha \in (0, 2/L)$.

2.8 VARIABLE METRIC METHODS

In the theory we have developed so far, the Euclidean norm plays a special role. In the definition of the proximal operator

$$\mathrm{Prox}_f(x) = \underset{z}{\mathrm{argmin}} \left\{ f(z) + \frac{1}{2}\|z - x\|^2 \right\},$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

the $(1/2)\|z - x\|^2$ term, called the *proximal term*, is defined with the Euclidean norm. Theorem 1 is stated in terms of the Euclidean norm. *Variable metric* methods generalize many of the notions we have discussed so far with the M-norm.

One reason to consider this generalization is preconditioning. A good choice of the norm $\|\cdot\|_M$ can reduce the number of iterations needed for convergence. Variable metric methods are also useful when an operator $\mathbb{A}$ has structure and a well-chosen M cancels certain terms to make $(M+\mathbb{A})^{-1}$ easy to evaluate. We explore this technique thoroughly in §3.3.

Despite the name variable *metric* methods, the generalization works only with M-norms since they are the norms induced by the inner product $\langle x, y\rangle_M = x^\intercal My$. The analysis of this section does not extend to other metrics, such as the ℓ^1-norm.

Variable Metric Proximal Point Method

Let $\mathbb{A}$ be maximal monotone and $M \succ 0$. Then $M^{-1/2}\mathbb{A}M^{-1/2}$ is maximal monotone and the proximal point method

$$y^{k+1} = (\mathbb{I} + M^{-1/2}\mathbb{A}M^{-1/2})^{-1}y^k$$

converges.

With the change of variables $x^k = M^{-1/2}y^k$, we get

$$\begin{aligned}(\mathbb{I} + M^{-1/2}\mathbb{A}M^{-1/2})y^{k+1} &\ni y^k\\ (\mathbb{I} + M^{-1}\mathbb{A})x^{k+1} &\ni x^k.\end{aligned}$$

This gives us

$$\begin{aligned}x^{k+1} &= \mathbb{J}_{M^{-1}\mathbb{A}}x^k\\ &= (M + \mathbb{A})^{-1}Mx^k.\end{aligned}$$

We call this the *variable metric PPM*. The iterates x^k inherit the convergence properties of y^k. For example, the fact that $\|y^k - y^\star\|$ is monotonically nonincreasing translates to the fact that $\|x^k - x^\star\|_M$ is monotonically nonincreasing. Likewise, $\|x^{k+1} - x^k\|_M \to 0$ monotonically at rate $O(1/k)$.

When $\mathbb{A} = \partial f$, then

$$\mathbb{J}_{M^{-1}\partial f}(x) = \underset{z\in\mathbb{R}^d}{\operatorname{argmin}}\left\{f(z) + \frac{1}{2}\|z - x\|_M^2\right\}.$$

We can interpret the variable metric PPM as PPM performed with the norm $\|\cdot\|_M$ instead of the Euclidean norm.

Variable Metric Forward-Backward Splitting

Let $\mathbb{A}$ and $\mathbb{B}$ be maximal monotone and let $\mathbb{A}$ be single-valued. Then with the same reasoning, we can use a change of variables to write the FBS FPI with respect to $M^{-1/2}\mathbb{A}M^{-1/2}$ and $M^{-1/2}\mathbb{B}M^{-1/2}$ as

$$\begin{aligned}x^{k+1} &= (M + \mathbb{B})^{-1}(M - \mathbb{A})x^k\\ &= \mathbb{J}_{M^{-1}\mathbb{B}}(\mathbb{I} - M^{-1}\mathbb{A})x^k.\end{aligned}$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

We call this splitting *variable metric FBS*. This method converges if $\mathbb{I} - M^{-1/2}\mathbb{A}M^{-1/2}$ is averaged.

When $\mathbb{A} = \nabla f$ and $\mathbb{B} = \partial g$, then

$$\mathbb{J}_{M^{-1}\partial g}(\mathbb{I} - M^{-1}\nabla f)x = \operatorname*{argmin}_{z\in\mathbb{R}^d}\left\{g(z) + \langle \nabla f(x), z\rangle + \frac{1}{2}\|z-x\|_M^2\right\}.$$

We can interpret the variable metric FBS as the proximal gradient method performed with the norm $\|\cdot\|_M$ instead of the Euclidean norm.

If $\mathbb{A}$ is β-cocoercive, then $M^{-1/2}\mathbb{A}M^{-1/2}$ is $(\beta/\|M^{-1}\|)$-cocoercive. See Exercise 2.9. Therefore, the FPI with variable metric FBS converges if $\|M^{-1}\| < 2\beta$.

Averagedness with Respect to $\|\cdot\|_M$

Assume $M \succ 0$. We say $\mathbb{T}$ is nonexpansive in $\|\cdot\|_M$ if

$$\|\mathbb{T}x - \mathbb{T}y\|_M \le \|x-y\|_M \qquad \forall x,y \in \operatorname{dom}\mathbb{T}.$$

For $\theta \in (0,1)$, we say $\mathbb{T}$ is θ-averaged in $\|\cdot\|_M$ if $\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$ for some $\mathbb{S}$ that is nonexpansive in $\|\cdot\|_M$. We say $\mathbb{T}$ is averaged in $\|\cdot\|_M$ if it is θ-averaged in $\|\cdot\|_M$ for some unspecified $\theta \in (0,1)$.

The operator $M^{-1/2}\mathbb{T}M^{-1/2}$ is nonexpansive (in $\|\cdot\|$) if and only if $M^{-1}\mathbb{T}$ is nonexpansive in $\|\cdot\|_M$. This is easy to verify since

$$\|M^{-1/2}\mathbb{T}M^{-1/2}x - M^{-1/2}\mathbb{T}M^{-1/2}y\|^2 \le \|x-y\|^2$$

is equivalent to

$$\|M^{-1}\mathbb{T}\tilde{x} - M^{-1}\mathbb{T}\tilde{y}\|_M^2 \le \|\tilde{x}-\tilde{y}\|_M^2$$

with the change of variables $M^{-1/2}x = \tilde{x}$ and $M^{-1/2}y = \tilde{y}$.

2.9 COMMONLY USED FORMULAS

For later convenience, we list a few commonly used formulas derived in this section.

- If $g(y) = f^*(A^\intercal y)$, where f is CCP and $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq \emptyset$, then

$$u \in \partial g(y) \quad \Leftrightarrow \quad \begin{aligned} &x \in \operatorname{argmin}_z \{f(z) - \langle y, Az\rangle\} \\ &u = Ax. \end{aligned} \tag{2.2}$$

- If $g(y) = f^*(A^\intercal y)$, where f is CCP and $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq \emptyset$, then

$$v = \operatorname{Prox}_{\alpha g}(u) \quad \Leftrightarrow \quad \begin{aligned} &x \in \operatorname{argmin}_x \left\{f(x) - \langle u, Ax\rangle + \tfrac{\alpha}{2}\|Ax\|^2\right\} \\ &v = u - \alpha Ax. \end{aligned} \tag{2.6}$$

- Let $\mathbf{L}(x,u) = f(x) + \langle u, Ax-b\rangle$ and let $\mathbf{L}_\alpha$ be the augmented Lagrangian of (1.11). Then

$$\mathbb{J}_{\alpha\partial\mathbf{L}}(x,u) = (y,v) \quad \Leftrightarrow \quad \begin{aligned} &y = \operatorname{argmin}_z \left\{\mathbf{L}_\alpha(z,u) + \tfrac{1}{2\alpha}\|z-x\|^2\right\} \\ &v = u + \alpha(Ay-b). \end{aligned} \tag{2.7}$$

- If $\mathbb{B}(x) = \mathbb{A}(x) + t$, where $\mathbb{A}$ is maximal monotone and $\alpha > 0$, then

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = \mathbb{J}_{\alpha\mathbb{A}}(u - \alpha t). \tag{2.8}$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

- If $\mathbb{B}(x) = \mathbb{A}(x - t)$, where $\mathbb{A}$ is maximal monotone and $\alpha > 0$, then

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = \mathbb{J}_{\alpha\mathbb{A}}(u - t) + t. \tag{2.9}$$

- If $\mathbb{B}(x) = -\mathbb{A}(t - x)$, where $\mathbb{A}$ is maximal monotone and $\alpha > 0$, then

$$\mathbb{J}_{\alpha\mathbb{B}}(u) = t - \mathbb{J}_{\alpha\mathbb{A}}(t - u). \tag{2.10}$$

- Inverse resolvent identity: If $\mathbb{A}$ is maximal monotone and $\alpha > 0$, then

$$\mathbb{J}_{\alpha^{-1}\mathbb{A}}(x) + \alpha^{-1}\mathbb{J}_{\alpha\mathbb{A}^{-1}}(\alpha x) = x. \tag{2.11}$$

- Moreau identity: If f is CCP and $\alpha > 0$, then

$$\mathrm{Prox}_{\alpha^{-1}f}(x) + \alpha^{-1}\mathrm{Prox}_{\alpha f^*}(\alpha x) = x. \tag{2.12}$$

BIBLIOGRAPHICAL NOTES

There are many classical and recent review papers based on the core insight that monotone operators serve as an elegant and unifying abstraction in the analysis of optimization algorithms: Lemaire and Penot in 1989 [LP89], Iusem in 1999 [Ius99], Combettes in 2004 [Com04], Combettes and Wajs in 2005 [CW05], Combettes and Pesquet in 2011[CP11b], Combettes, Condat, Pesquet, and Vũ in 2014 [CCPV14], Komodakis and Pesquet in 2015 [KP15], Clason and Valkonen in 2020 [CV20], and Condat, Kitahara, Contreras, and Hirabayashi in 2020 [CKCH22]. This book is largely influenced by these prior treatments.

Early Development: Basic Notions The notion of monotonicity was first formalized by Zarantonello in 1960 [Zar60]. The fact that derivatives of convex functions on $\mathbb{R}$ are nondecreasing was established by Jensen in 1906 [Jen06], and this monotonicity property was extended to gradients of convex functions on higher-dimensional spaces by Kačurovskiĭ in 1960 [Kac60] and Minty in 1962 [Min62]. The notion of *maximal* monotonicity was first established by Minty in 1962 [Min62]. Maximal monotonicity of subdifferentials of CCP functions on Hilbert spaces (and thus on $\mathbb{R}^n$) was established by Minty in 1964 [Min64] and Moreau in 1965 [Mor65]. This maximality result was generalized to convex functions on Banach spaces by Rockafellar [Roc66, Roc70b].

Fenchel's identity (2.1) was first presented by Fenchel in 1951 in his lectures [Fen53, Section 5]. The proximal operator was first introduced by Moreau in 1962 [Mor62, Mor65], and the Moreau identity was introduced in 1965 [Mor65]. The proof of $\mathrm{dom}\,\mathbb{J}_{\mathbb{A}} = \mathbb{R}^n$ when $\mathbb{A}$ is maximal monotone, the Minty surjectivity theorem, was established by Minty in 1962 [Min62]. The (1/2)-averagedness of resolvents was first discussed by Browder and Petryshyn in 1967 [BP67].

The study of convex-concave saddle functions and their saddle subdifferentials was pioneered by Rockafellar. His work started in the 1960s [Roc64, Roc68], and the maximal monotonicity of "closed proper" saddle subdifferentials was established in 1970 [Roc70a].

The augmented Lagrangian was used in [Hes69, Pow69] and later studied by Rockafellar in the late 1970s [Roc76b, Roc78].

Early Development: Methods Gradient descent dates back to Cauchy in 1847 [Cau47]. Fixed-point iterations date back to Picard, Lindelöf, and Banach in the late 1800s and early 1900s [Pic90, Lin94, Ban22]. The proximal point method was first studied in the 1970s [Mar70, Mar72b, Roc76b, BL78], and its convergence rate in terms of function values was later studied by Güler in 1991 [Gül91]. The method of multipliers was first presented in 1969 by Hestenes and Powell [Hes69, Pow69] and was interpreted as an instance of PPM by Rockafellar in 1973 [Roc73]. Dual ascent was first presented by Uzawa in 1972 [AHU58] and was later further studied by Tseng, Bertsekas, and Tsitsiklis [TB87, Tse90a]. The projected gradient method was first presented in the 1960s by Goldstein, Levitin, and Polyak [Gol64, LP66]. The forward step method is due to Bruck in 1977 [Bru77] and forward-backward splitting in its operator theoretic form was first presented in the 70s by Bruck and Passty [Bru77, Pas79]. In modern literature, FBS applied to the sum of two convex functions has been referred to as the proximal-gradient method [CW05].

Peaceman–Rachford and Douglas–Rachford splitting methods were first presented as splitting methods to solve the heat equation in 1955 and 1956 [PR55, DR56]. In 1979, Lions and Mercier generalized the technique to a sum of two maximal monotone operators [LM79]. The effort of combining Douglas–Rachford and Forward–Backward splitting schemes was initiated by Raguet, Fadili, and Peyré [RFP13, Rag19], extended by Briceño-Arias [Bri15], and completed by Davis and Yin [DY17b] as they proved averagedness in the general case with two maximal monotone operators and one cocoercive operator. This splitting method, which we refer to as Davis–Yin splitting, is also called the Forward-Douglas–Rachford splitting.

As we explore further in §3, many of the splitting methods are intimately connected. Since the DRS operator is firmly nonexpansive, it is a resolvent of a maximal monotone operator, and this was first pointed out by Lawrence and Spingarn in 1987 [LS87] and later by Eckstein and Bertsekas in 1992 [EB92]. That the gradient update can be viewed as the proximal operator of the function's first-order approximation, as discussed in §2.7.2, was first identified by Polyak in 1987 [Pol87].

Fixed-Point Iteration The FPI analyzed in Theorem 1 is also called the Krasnosel'skiĭ–Mann iteration. In 1953, Mann showed that the FPI converges when $n = 1$, $C \subset \mathbb{R}$ is a compact interval, and $T\colon C \to C$ is 1/2-averaged. In 1955, Krasnosel'skiĭ established convergence when $C \subset \mathbb{R}^n$ is compact and $T\colon C \to C$ is 1/2-averaged [Kra55]. In 1957, Schaefer extended Krasnosel'skiĭ's result to θ-averaged operators with $\theta \in (0, 1)$ [Sch57]. The general convergence result of Theorem 1 (without any compactness assumption) is due to Martinet's 1972 work [Mar72a, Théorème 5.5.2]. A key component of our (and Martinet's) proof is the subsequence convergence argument of Stage 2, which is due to Opial's 1967 work [Opi67]. In fact, Theorem 1 of [Opi67] captures this subsequence argument and is known as Opial's lemma. The notion of averaged operators was first formally defined in 1978 by Baillon, Bruck, and Reich [BBR78].

Infinite-Dimensional Analysis Although we focus on finite-dimensional spaces in this book, much of the monotone operator theory is developed in the infinite-dimensional setup, where a new set of interesting challenges arise. For example, the convergence $x^k \to x^\star$ of Theorem 1 becomes weak when the underlying space is an infinite-dimensional Hilbert space instead of $\mathbb{R}^n$. Bauschke and Combette's textbook [BC17a] provides a thorough treatment for operators on Hilbert spaces. Works on other setups include Reich and Shoikhet's [Rei79, RS98] work studying averaged operators in Banach spaces and Goebel and Reich's work [GR84, Rei85] studying averaged operators on the Hilbert ball with the hyperbolic metric.

Forward and Backward Nomenclature and Gradient-Flow The operators $\mathbb{I} - \alpha\mathbb{A}$ and $(\mathbb{I} + \alpha\mathbb{A})^{-1}$ are respectively called forward and backward steps in analogy to the forward and backward Euler discretizations of $\dot{x}(t) = -\mathbb{A}x(t)$, a continuous-time differential equation defined for single-valued $\mathbb{A}$. This interpretation is due to Lamaire and Penot [LP89, Lem92] and Eckstein [Eck89, §3.2.2] in 1989. However, the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ for functions f itself was studied earlier by Bruck in 1975 [Bru75a] and Botsaris and Jacobson in 1976 [BJ76].

Consensus Technique The first use of the consensus technique, also called the *product space trick*, seems to be due to Pierra in 1984 [Pie84] and Spingarn in 1983 through the "method of partial inverses" [Spi83, Spi85]. The use of the technique for distributed optimization and machine learning was popularized through the works of Boyd, Parikh, Chu, Peleato, and Eckstein [BPC+11, PB14b, PB14a].

Variable Metric Methods The variable metric proximal point method can be thought of as a special case of the Bregman proximal point method, which was first presented by Censor and Zenios for minimizing convex functions [CZ92] and Burachik and Iusem for monotone inclusions [BI98]. Other early work includes that of Chen and Teboulle [CT93], Bonnans, Gilbert, Lemaréchal, and Sagastizábal [BGLS95], Parente, Lotito, and Solodov [PLS08], and He and Yuan [HY12b]. Variable metric forward-backward splitting was first formalized by Combettes and Vũ [CV14]. A block coordinate extension was given by Chouzenoux, Pesquet, and Repetti [CPR16]. Liu and Yin [LY19] used variable metrics to analyze the Davis–Yin splitting for smooth nonconvex problems. Vũ [Vũ13b] proposed variable metric extensions of Tseng's forward-backward-forward splitting. Briceño-Arias and Davis [BD18] proposed variable metric extensions of their forward-backward-half forward splitting. A different approach to apply variable metrics was introduced by Burke and Qian [BQ99].

LASSO Application LASSO (least absolute shrinkage and selection operator) first introduced in geophysics literature in 1986 [SS86]. It was later independently rediscovered, popularized, and named LASSO by the statistician Tibshirani in 1996 [Tib96]. LASSO is one of the main models of compressed sensing [Don06, CT05, CT06] when the sensing is corrupted by noise or the signal to sense is approximately sparse.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Early work regarding the computation of LASSO includes [EHJT04, FNW07, HYZ08, YOGD08]. The Nesterov acceleration to the iterative soft thresholding algorithm was introduced in [BT09].

EXERCISES

2.1 *When $\mathbb{T}^{-1}$ is a left-inverse of* $\mathbb{T}$. Show that if $x \in \operatorname{dom} \mathbb{T}$ and $\mathbb{T}^{-1}$ is single-valued, then $\mathbb{T}^{-1}\mathbb{T}x = x$.

2.2 *Non-maximal subdifferential.* Consider the function f on $\mathbb{R}$ defined as

$$f(x) = \begin{cases} \infty & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0. \end{cases}$$

Show that f is convex and proper but not closed. Show that ∂f is not maximal.

2.3 *Monotonicity of saddle subdifferential.* Assume $\mathbf{L}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathbf{L}(x,u)$ is convex-concave. Recall $\partial \mathbf{L}$ is defined in (2.3). Show that $\partial \mathbf{L}$ is monotone.

Hint. Add the four subgradient inequalities that lower bound

- $\mathbf{L}(x_2,u_1)$ with a subgradient of $\mathbf{L}(\cdot,u_1)$ at x_1
- $-\mathbf{L}(x_1,u_2)$ with a subgradient of $-\mathbf{L}(x_1,\cdot)$ at u_1
- $\mathbf{L}(x_1,u_2)$ with a subgradient of $\mathbf{L}(\cdot,u_2)$ at x_2
- $-\mathbf{L}(x_2,u_1)$ with a subgradient of $-\mathbf{L}(x_2,\cdot)$ at u_2

to show

$$\langle \partial_x \mathbf{L}(x_1,u_1) - \partial_x \mathbf{L}(x_2,u_2), x_1 - x_2\rangle + \langle \partial_u(-\mathbf{L}(x_1,u_1)) - \partial_u(-\mathbf{L}(x_2,u_2)), u_1 - u_2\rangle \geq 0.$$

2.4 *Maximality of continuous monotone operators.* Show that if $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone, then $\mathbb{T}$ is maximal.

Hint. Assume for contradiction that there is a pair $(y,v) \notin \mathbb{T}$ such that

$$0 \leq \langle v - \mathbb{T}x, y - x\rangle$$

for all $x \in \mathbb{R}^n$. Plug in $x = y - \delta$ and use continuity of $\mathbb{T}$ to argue

$$0 \leq \langle v - \mathbb{T}(y-\delta), \delta\rangle = \langle v - \mathbb{T}y, \delta\rangle + o(\|\delta\|)$$

as $\delta \to 0$. Argue that $v = \mathbb{T}y$ and draw a contradiction.

2.5 Show that if f is a strictly convex CCP function, then (i) ∂f^* is single-valued and (ii) f^* is differentiable on $\operatorname{int}\operatorname{dom} f^*$.

Remark. Since f^* is CCP, f^* is subdifferentiable on $\operatorname{int}\operatorname{dom} f^*$ and $\partial f^*(u)$ is a singleton if and only if f^* is differentiable at u.

2.6 *Recovering a primal solution from a dual solution.* Let f be a strictly convex CCP function on $\mathbb{R}^n$, g a CCP function on $\mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$. Consider the primal problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) + g(Ax)$$

and dual problem

$$\underset{u\in\mathbb{R}^m}{\text{maximize}} \quad -f^*(-A^\intercal u) - g^*(u)$$

generated by the Lagrangian

$$\mathbf{L}(x,u) = f(x) + \langle Ax, u\rangle - g^*(u).$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Assume total duality holds. Show that $\nabla f^*(-A^\intercal u^\star)$ is a primal solution.
Hint. Use Exercise 2.5.
Remark. Without the strict convexity, this statement is not true. The setting $n = 1$, $m = 1$, $f(x) = 0$, $A = 1$, $g(x) = \delta_{\{0\}}(x)$, and $\mathbf{L}(x,u) = xu$ is a counterexample: $x^\star = 0$ and $u^\star = 0$ are the unique primal and dual solutions, but $\partial f^*(-u^\star) = \mathbb{R}$.

2.7 *Differentiable monotone operators.* Show that a differentiable operator $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is monotone if and only if $D\mathbb{T}(x) + D\mathbb{T}(x)^\intercal \succeq 0$ for all x.
Hint. Assume $\mathbb{T}$ is monotone, and use

$$D\mathbb{T}(x)v = \lim_{h\to 0} \frac{1}{h}(\mathbb{T}(x+hv) - \mathbb{T}(x))$$

to show $v^\intercal D\mathbb{T}(x)v \geq 0$ for all $v \in \mathbb{R}^n$. For the other direction, assume $D\mathbb{T}(x)+D\mathbb{T}(x)^\intercal \succeq 0$ for all x, define $g(t) = \langle x-y, \mathbb{T}(tx+(1-t)y)\rangle$, and use the mean value theorem to show

$$\langle x-y, \mathbb{T}x - \mathbb{T}y\rangle = g(1) - g(0) = g'(\xi)$$

for some $\xi \in [0,1]$.

2.8 *Differentiable Lipschitz operators.* Show that a differentiable operator $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is L-Lipschitz if and only if $\sigma_{\max}(D\mathbb{T}(x)) \leq L$ for all x.
Hint. Assume $\sigma_{\max}(D\mathbb{T}(x)) \leq L$, define $g(t) = \mathbb{T}(tx + (1-t)y)$, and use the mean value theorem and the Cauchy–Schwartz inequality to get

$$\|\mathbb{T}x - \mathbb{T}y\|^2 = \langle \mathbb{T}x - \mathbb{T}y, g(1) - g(0)\rangle = \langle \mathbb{T}x - \mathbb{T}y, g'(\xi)\rangle \leq \|\mathbb{T}x - \mathbb{T}y\|\|g'(\xi)\|.$$

For the other direction, assume $\mathbb{T}$ has Lipschitz parameter L and use

$$\|D\mathbb{T}(x)v\| = \lim_{h\to 0} \frac{1}{h}\|\mathbb{T}(x+hv) - \mathbb{T}(x)\|.$$

2.9 Show that if $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoercive and $M \in \mathbb{R}^{n\times n}$ is symmetric positive definite, then $M^{-1/2}\mathbb{T}M^{-1/2}$ is $(\beta/\|M^{-1}\|)$-cocoercive.

2.10 *Moreau envelope.* Let f be a CCP function on $\mathbb{R}^n$. For $\beta > 0$, define the *Moreau envelope* of f of parameter β as

$${}^{\beta}f(x) = \inf_{z\in\mathbb{R}^n}\left\{f(z) + \frac{1}{2\beta}\|z-x\|^2\right\}.$$

Show that
(a) ${}^{\beta}f(x)$ is convex and proper,
(b) $\nabla{}^{\beta}f = \beta^{-1}(\mathbb{I} - \mathrm{Prox}_{\beta f})$,
(c) ${}^{\beta}f(x)$ is closed, and
(d) ${}^{\beta}f$ is $1/\beta$-smooth.
Hint. For (a), establish closedness with ${}^{\beta}f(x) = f(\mathrm{Prox}_{\beta f}(x)) + \frac{1}{2\beta}\|\mathrm{Prox}_{\beta f}(x) - x\|^2$ and the fact that $\mathrm{Prox}_{\beta f}(x)$ is well defined. For (b), note that

$${}^{\beta}f(x) = \frac{1}{2\beta}\|x\|^2 - \frac{1}{\beta}\sup_{z\in\mathbb{R}^n}\left\{\langle x,z\rangle - \beta f(z) - \frac{1}{2}\|z\|^2\right\},$$

and the supremum can be written as a conjugate. Take the gradient of both sides. For (c), use the fact that ${}^{\beta}f$ is differentiable and therefore continuous. For (d), use the Moreau identity to show $\beta\nabla{}^{\beta}f$ is a proximal operator of a convex function.

2.11 *Moreau envelope as a smooth approximation.* Let f be a CCP function on $\mathbb{R}^n$ and $\beta > 0$. Show that $\lim_{\beta\to 0} {}^{\beta}f(x) \to f(x)$ for all $x \in \mathbb{R}^n$.
Hint. First show that $u \in \partial f(x)$ if and only if $f(x) + f^*(u) = \langle u, x\rangle$. Then argue that for any $x \in \mathbb{R}^n$ (possibly $x \notin \operatorname{dom} f$)

$$f(x) = \sup_u \{-f^*(u) + \langle x, u\rangle\} = \sup_{(y,u)\in\partial f} \{f(y) + \langle u, x - y\rangle\}.$$

So there exists a sequence $(y^0, u^0), (y^1, u^1), \ldots$ in ∂f such that

$$f(y^k) + \langle u^k, x - y^k\rangle \to f(x).$$

Remark. This result, along with the smoothness property of Exercise 2.10 allows us to view ${}^{\beta}f$ as a smooth approximation of f. The interpretation of the Moreau envelope as a smooth, regularized function is due to Attouch [Att77, Lemme 1], [Att84, Theorem 2.64]. However, the analogous notion for monotone operators, known as the Moreau–Yosida approximation, was used earlier by Brezis [Bre71, Lemma 3], and the Moreau envelope itself was presented earlier yet by Moreau [Mor65]. The result of this problem was first presented by Friedlander, Goodwin, and Hoheisel [FGH21, Proposition 4].

2.12 *PPM is GD.* Show that $\operatorname{argmin} f = \operatorname{argmin} {}^{\beta}f$ for any $\beta > 0$. Also show that the PPM with f is equivalent to gradient descent with respect to ${}^{\beta}f$ for some $\beta > 0$.
Hint. Use Exercise 2.10.
Remark. This problem illustrates that the Moreau envelope is also useful as a conceptual tool for drawing connections.

2.13 *Projection onto convex sets.* Consider the *convex feasibility problem*

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad x \in C \cap D,$$

where C and D are nonempty closed convex sets. Assume $C \cap D \neq \emptyset$.

(a) The convex feasibility problem is equivalent to the optimization problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & \frac{1}{2}\operatorname{dist}^2(x, D) \\ \text{subject to} & x \in C. \end{array}$$

Show that the proximal gradient method with stepsize 1 applied to this problem is

$$x^{k+1} = \Pi_C \Pi_D x^k,$$

which is called the *alternating projections method*.

(b) The convex feasibility problem is also equivalent to the optimization problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \tfrac{\theta}{2}\operatorname{dist}^2(x, C) + \tfrac{1-\theta}{2}\operatorname{dist}^2(x, D),$$

where $\theta \in (0, 1)$. Show that the gradient method with stepsize 1 applied to this problem is

$$x^{k+1} = \theta\Pi_C x^k + (1 - \theta)\Pi_D x^k,$$

which is called the *parallel projections method*.

(c) Show that $x^k \to x^\star \in C \cap D$ for both methods.

Hint. Note that $\frac{1}{2}\operatorname{dist}^2(x, C)$ is a Moreau envelope of δ_C.
Remark. See [BB96, BBL97, ER11] for an overview on convex feasibility problems.

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

2.14 *Banach fixed-point theorem.* Let $\mathbb{T}\colon \mathbb{R}^n \to \mathbb{R}^n$ be contractive. Show that $\mathbb{T}$ has a unique fixed point, that is, show that a fixed point of $\mathbb{T}$ exists and is unique.
Hint. Consider an FPI and show that $x^0, x^1, \dots$ is a Cauchy sequence.
Remark. This result is called the *Banach fixed-point theorem* [Ban22].

2.15 *Strong monotonicity and unique zero.* Show that if $\mathbb{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal μ-strongly monotone for some $\mu > 0$, then $\mathbb{T}$ has exactly one zero.
Hint. Use the Banach fixed-point theorem.

2.16 *Contraction factor of gradient descent.* Assume f is CCP, μ-strongly convex, L-smooth, and twice continuously differentiable. Show that $I - \alpha\nabla f$ is $\max\{|1-\alpha\mu|, |1-\alpha L|\}$-contractive for $0 < \alpha < 2/L$.
Hint. The fundamental theorem of calculus tells us

$$(I - \alpha\nabla f)(x) - (I - \alpha\nabla f)(y) = \int_0^1 (I - \alpha\nabla^2 f(tx + (1-t)))(x-y)\, dt.$$

Use the instance of Jensen's inequality

$$\left\| \int_0^1 v(t)\, dt \right\| \leq \int_0^1 \|v(t)\|\, dt,$$

where $v(t) \in \mathbb{R}^n$ for $t \in [0,1]$.
Remark. The result still holds when f is not continuously differentiable. See §13.

2.17 *Convergence of dual ascent.* Show that dual ascent converges in the sense of $x^k \to x^\star$ and $u^k \to u^\star$, where $x^\star$ and $u^\star$ are primal and dual solutions, under the stated conditions.
Hint. Use Theorem 1 to establish $u^k \to u^\star$ and write x^k as a continuous function of u^k.
Remark. The stated conditions are f is CCP and μ-strongly convex, total duality holds, and $0 < \alpha < 2\mu/\sigma^2_{\max}(A)$.

2.18 *Method of multipliers primal solution convergence.* Show that the method of multipliers converges in the sense of $x^k \to x^\star$ under the stated conditions and strict convexity. Use the following fact: if h is a CCP function that is differentiable on $D \subseteq \mathbb{R}^n$, then $\nabla h\colon D \to \mathbb{R}^n$ is a continuous function, that is, differentiability and continuous differentiability coincide.
Remark. The stated conditions are f is CCP, $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq \emptyset$, a dual solution exists, $\alpha > 0$, and $\mathbf{L}_\alpha(x,u) = f(x) + \langle u, Ax - b\rangle + \frac{\alpha}{2}\|Ax-b\|^2$.
Hint. Consider the primal problem

$$\begin{array}{ll} \underset{u\in\mathbb{R}^m,\, v\in\mathbb{R}^n}{\text{minimize}} & f^*(v) + b^\intercal u \\ \text{subject to} & -v - A^\intercal u = 0 \end{array}$$

generated by the Lagrangian $\tilde{\mathbf{L}}(v,u,x) = f^*(v) + b^\intercal u - \langle x, v + A^\intercal u\rangle$, and use Slater's constraint qualification to show that $\mathcal{R}(A^\intercal) \cap \operatorname{ri}\operatorname{dom} f^* \neq \emptyset$ implies strong duality and the existence of a primal solution for the primal-dual problem pair generated by $\mathbf{L}$. Use Exercise 2.5 to write $x^k = \sigma(u^k)$, where $\sigma\colon \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.
Remark. The derivation of (2.6) or Exercise 1.5 establishes $\operatorname{argmin}_x \mathbf{L}_\alpha(x, u^k) \neq \emptyset$, that is, $x^{k+1} \in \operatorname{argmin}_x \mathbf{L}_\alpha(x,u^k)$ is well defined for any $u^k \in \mathbb{R}^m$.

2.19 *Contraction factor of dual ascent.* Consider dual ascent. Assume f is μ-strongly convex, L-smooth, CCP, and $0 < \alpha < 2\mu/\sigma^2_{\max}(A)$. Using Exercise 2.16, show that dual ascent converges with contraction factor

$$\max\{|1 - \alpha\sigma^2_{\max}(A)/\mu|, |1 - \alpha\sigma^2_{\min}(A)/L|\}.$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

2.20 *Lyapunov analysis without summability.* Let $\mathbb{T}: \mathbb{R}^n \to \mathbb{R}^n$ be θ-averaged, and consider the fixed-point iteration $x^{k+1} = \mathbb{T}x^k$. Consider the Lyapunov function

$$V^k = k\frac{1-\theta}{\theta}\|x^k - x^{k-1}\|^2 + \|x^k - x^\star\|^2.$$

Show that

$$V^{k+1} \leq V^k$$

for $k = 0, 1, \ldots$. Use this inequality, instead of the summability argument, to prove Theorem 1.

2.21 *When there is no fixed point.* Assume $\mathbb{T}: \mathbb{R}^n \to \mathbb{R}^n$ is averaged and $\operatorname{Fix}\mathbb{T} = \emptyset$. Prove that sequence $x^{k+1} = \mathbb{T}x^k$ satisfies $\|x^k\| \to \infty$.

Hint. Assume for contradiction that $\|x^k\| \not\to \infty$, which implies, by the Bolzano–Weierstrass theorem, that there is a subsequence $k_j \to \infty$ such that $x^{k_j} \to \bar{x}$ for some limit $\bar{x}$. Next, show $\|x^{k+1} - x^k\| \to c$ for some $c \geq 0$. Consider the cases $c = 0$ and $c > 0$ separately. In the $c > 0$ case, show $\mathbb{T}^{k+1}\bar{x} - \mathbb{T}^k\bar{x} = \mathbb{T}^k\bar{x} - \mathbb{T}^{k-1}\bar{x}$ and argue that $\|\mathbb{T}^k\bar{x}\| \to \infty$, where

$$\mathbb{T}^k = \underbrace{\mathbb{T} \circ \cdots \circ \mathbb{T}}_{k \text{ times}}.$$

Remark. Interestingly, this result, first proved by Roehrig and Sine [RS81], does depend on the finite-dimensionality of $\mathbb{R}^n$. If $\mathbb{T}: \mathcal{H} \to \mathcal{H}$ is an averaged operator on an infinite-dimensional Hilbert space $\mathcal{H}$, Browder and Petryshyn showed that $\limsup_{k\to\infty} \|x^k\| = \infty$ [BP66], but Edelstein provided a counterexample for which $\liminf_{k\to\infty} \|x^k\| = 0$ [Ede64, BGMS20].

2.22 *FPI with quasi-nonexpansive operators.* We say $\mathbb{S}$ is *quasi-nonexpansive* if

$$\|\mathbb{S}x - x^\star\|^2 \leq \|x - x^\star\|^2$$

for all $x^\star \in \operatorname{Fix}\mathbb{S}$. We say $\mathbb{T}$ is *θ-quasi-averaged* if $\mathbb{T} = (1-\theta)\mathbb{I} + \theta\mathbb{S}$ for some quasi-nonexpansive operator $\mathbb{S}$. Assume $\mathbb{T}: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and θ-quasi-averaged with $\theta \in (0,1)$. Assume $\operatorname{Fix}\mathbb{T} \neq \emptyset$. Show that $x^{k+1} = \mathbb{T}x^k$ with any starting point $x^0 \in \mathbb{R}^n$ converges to one fixed point, that is,

$$x^k \to x^\star$$

for some $x^\star \in \operatorname{Fix}\mathbb{T}$.

2.23 *Gradient descent with varying stepsize.* Consider the problem of minimizing

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x),$$

where f is an L-smooth CCP function. Then

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k),$$

where $\alpha_0, \alpha_1, \ldots \in \mathbb{R}$, is called *gradient descent with varying stepsize*. Assume $\operatorname{argmin} f \neq \emptyset$ and

$$0 < \inf_{k=0,1,\ldots} \alpha_k \leq \sup_{k=0,1,\ldots} \alpha_k < 2/L.$$

Show

$$x^k \to x^\star \in \operatorname{argmin} f.$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Hint. Adapt the proof of Theorem 1 to fit the current setup.

2.24 Show (2.9) and (2.10).

2.25 *Conic programs with DRS.* Consider the problem of

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & c^\intercal x \\ \text{subject to} & Ax = b \\ & x \in K, \end{array}$$

where $K \subset \mathbb{R}^n$ is a nonempty closed convex set. When K is a nonempty closed convex cone, the problem is said to be a *conic program*. Assume $A \in \mathbb{R}^{m\times n}$, where A has rank m and $b \in \mathbb{R}^m$. Show that the FPI with DRS is

$$\begin{aligned} x^{k+1/2} &= \Pi_K(z^k) \\ x^{k+1} &= D(2x^{k+1/2} - z^k) + v \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2}, \end{aligned}$$

where $D = I - A^\intercal(AA^\intercal)^{-1}A$ and $v = A^\intercal(AA^\intercal)^{-1}b - \alpha Dc$.

2.26 *Convergence of DRS.* Consider the FPI with DRS. Theorem 1 implies $z^k \to z^\star$ for any $\alpha > 0$, provided that a fixed point exists. Show that this implies $x^k \to x^\star$, and $x^{k+1/2} \to x^\star$. Is $\|x^{k+3/2} - x^{k+1/2}\| \to 0$ and $\|x^{k+1} - x^k\| \to 0$ true?

2.27 *When PRS does not converge.* Consider the operators $\mathbb{A} = \mathbb{N}_{\{0\}}$ and $\mathbb{B} = 0$. Show that although a fixed point of PRS does correspond to a solution, the FPI with PRS does not converge. This example also demonstrates that the FPI with the reflected resolvent need not converge.

2.28 *Backward-backward is alternating minimization.* Consider the monotone inclusion problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A} + \mathbb{B})x.$$

The *backward-backward method* is

$$x^{k+1} = \mathbb{J}_{\alpha\mathbb{A}}\mathbb{J}_{\alpha\mathbb{B}}x^k,$$

where $\alpha > 0$. Show that when $\mathbb{A} = \partial f$ and $\mathbb{B} = \partial g$, where f and g are CCP functions, we have

$$\begin{aligned} y^{k+1} &= \underset{y\in\mathbb{R}^n}{\text{argmin}} \left\{ f(x^k) + g(y) + \frac{1}{2\alpha}\|x^k - y\|^2 \right\} \\ x^{k+1} &= \underset{x\in\mathbb{R}^n}{\text{argmin}} \left\{ f(x) + g(y^{k+1}) + \frac{1}{2\alpha}\|x - y^{k+1}\|^2 \right\}. \end{aligned}$$

and that fixed points correspond to minimizers of

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) + g(y) + \tfrac{1}{2\alpha}\|x - y\|^2. \tag{2.20}$$

Finally, show that the backward-backward method converges.

Remark. This result was first published by Bauschke, Combettes, and Reich [BCR05].

2.29 *Consensus + proximable is proximable.* Let r be a CCP function on $\mathbb{R}^n$, C be the consensus set as defined in (2.19), and

$$g(x_1, \ldots, x_m) = \delta_C(x_1, \ldots, x_m) + \sum_{i=1}^m r(x_i).$$

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

Show that we can evaluate $\text{Prox}_{\alpha g}$ with

$$\text{Prox}_{\alpha g}(y_1,\ldots,y_m) = (x,\ldots,x), \quad x = \text{Prox}_{\alpha r}\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right).$$

Also, what is the proximal operator of $h(x_1,\ldots,x_m) = \delta_C(x_1,\ldots,x_m) + r(x_1)$?

2.30 Let $\eta \in (0,1)$ and consider the monotone inclusion problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (2(1-\eta)\mathbb{I} + \mathbb{A} + \mathbb{B})x,$$

where $\mathbb{A}$ and $\mathbb{B}$ are maximal monotone, and assume $\mathbb{A}+\mathbb{B}$ is maximal. Show that the solution can be found through the FPI $z^{k+1} = \mathbb{T}z^k$ with

$$\mathbb{T} = \frac{1}{2}\mathbb{I} + \frac{1}{2}(2\eta\mathbb{J}_{\mathbb{A}} - \mathbb{I})(2\eta\mathbb{J}_{\mathbb{B}} - \mathbb{I}).$$

Hint. Show $\text{Zer}\,(2(1-\eta)\mathbb{I} + \mathbb{A} + \mathbb{B}) = \frac{1}{\eta}\text{Zer}\,(\mathbb{A}^{(\eta)} + \mathbb{B}^{(\eta)})$, where $\mathbb{A}^{(\eta)} = \mathbb{A} \circ \frac{1}{\eta}\mathbb{I} + \frac{1-\eta}{\eta}\mathbb{I}$ and $\mathbb{B}^{(\eta)}$ is defined likewise.

Remark. Since $\text{Zer}\,(2(1-\eta)\mathbb{I} + \mathbb{A} + \mathbb{B}) = \mathbb{J}_{\frac{1}{2(1-\eta)}(\mathbb{A}+\mathbb{B})}(0)$, a unique solution exists. This method is called the averaged alternating modified reflections (AAMR) [AAC18, AAC19].

2.31 *Further properties of the proximal operator.* Let f be a CCP function on $\mathbb{R}^n$. Show:

(a) $f(\text{Prox}_{\alpha f}(x))$ is a nonincreasing function of $\alpha \in (0,\infty)$ (for a fixed $x \in \mathbb{R}^n$).

(b) $\lim_{\alpha\to\infty} f(\text{Prox}_{\alpha f}(x)) = \inf_x f(x)$ (including the case $\inf_x f(x) = -\infty$).

(c) $f(\text{Prox}_{\alpha f}(x)) \le f(x)$ for any $\alpha > 0$.

(d) $\lim_{\alpha\to 0^+} f(\text{Prox}_{\alpha f}(x)) = f(x)$ for all $x \in \text{dom}\, f$.

Hint. For (a), argue that

$$\alpha f\left(\text{Prox}_{\alpha f}(x)\right) + \frac{1}{2}\left\|\text{Prox}_{\alpha f}(x) - x\right\|^2 \le \alpha f\left(\text{Prox}_{\beta f}(x)\right) + \frac{1}{2}\left\|\text{Prox}_{\beta f}(x) - x\right\|^2$$
$$\beta f\left(\text{Prox}_{\beta f}(x)\right) + \frac{1}{2}\left\|\text{Prox}_{\beta f}(x) - x\right\|^2 \le \beta f\left(\text{Prox}_{\alpha f}(x)\right) + \frac{1}{2}\left\|\text{Prox}_{\alpha f}(x) - x\right\|^2$$

for $\alpha,\beta \in \mathbb{R}$. For (b), let $\varepsilon > 0$ and $M > \inf_x f(x)$. Let $x_{M,\varepsilon}$ be a point such that $f(x_{M,\varepsilon}) < M + \varepsilon/2$. Then

$$f(x_{M,\varepsilon}) + \frac{1}{2\alpha}\|x_{M,\varepsilon} - x\|^2 < M + \varepsilon$$

for large enough α. For (d), show

$$\alpha f(x) \ge \alpha f(\text{Prox}_{\alpha f}(x)) + \frac{1}{2}\|\text{Prox}_{\alpha f}(x) - x\|^2$$

and let $\alpha \to 0$.

Remark. The result of (d) is not necessarily true when $x \notin \text{dom}\, f$. For example, consider $f = \delta_{\{0\}}$ and $x = 1$.

Remark. In general, one can show $\lim_{\alpha\to 0^+}\text{Prox}_{\alpha f}(x) = \Pi_{\overline{\text{dom}\, f}}(x)$ [FGH21, Proposition 5].

2.32 *Proximable inequality constraints.* Let f be a CCP function on $\mathbb{R}^n$ and informally assume f is proximable. Through the following steps, show that $\delta_{\{x\in\mathbb{R}^n \,|\, f(x)\le 0\}}$ is proximable. Show:

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

(a) For maximal monotone $\mathbb{A}$, $\alpha, \beta \in (0, \infty)$,

$$\mathbb{J}_{\alpha\mathbb{A}} x = \mathbb{J}_{\beta\mathbb{A}} \left(\frac{\beta}{\alpha} x + \left(1 - \frac{\beta}{\alpha}\right) \mathbb{J}_{\alpha\mathbb{A}} x \right) \qquad \forall x \in \mathbb{R}^n,$$

and

$$\|\mathbb{J}_{\alpha\mathbb{A}} x - \mathbb{J}_{\beta\mathbb{A}} x\| \le \left|1 - \frac{\beta}{\alpha}\right| \|\mathbb{J}_{\alpha\mathbb{A}} x - x\| \qquad \forall x \in \mathbb{R}^n.$$

(b) For a fixed $x \in \mathbb{R}^n$, $f(\mathrm{Prox}_{\alpha f}(x))$ is a nonincreasing continuous function of $\alpha \in (0, \infty)$.

(c) Assume that $\mathrm{dom} f = \mathbb{R}^n$ and that there exists an $x \in \mathbb{R}^n$ such that $f(x) < 0$. Let $\alpha^\star = \inf\{\alpha > 0 \,|\, f(\mathrm{Prox}_{\alpha f}(x)) \le 0\}$. Then

$$\Pi_{\{x \in \mathbb{R}^n \,|\, f(x) \le 0\}}(x) = \begin{cases} x & \text{if } f(x) \le 0 \\ \mathrm{Prox}_{\alpha^\star f}(x) & \text{otherwise.} \end{cases}$$

(d) Assume $\mathrm{dom} f = \mathbb{R}^n$ and $f(x) > 0$. Also assume $l^0, u^0 \in \mathbb{R}$ satisfy $f(\mathrm{Prox}_{l^0 f}(x)) > 0 \ge f(\mathrm{Prox}_{u^0 f}(x))$. The iteration

$$(l^{k+1}, u^{k+1}) = \begin{cases} (l^k, \frac{l^k + u^k}{2}) & \text{if } f\left(\mathrm{Prox}_{\frac{l^k+u^k}{2} f}(x)\right) \le 0 \\ (\frac{l^k + u^k}{2}, u^k) & \text{otherwise} \end{cases}$$

converges in the sense that $l^k \to \alpha^\star$ and $u^k \to \alpha^\star$.

Hint. Show that $(\mathrm{Prox}_{\alpha^\star f}(x), \alpha^\star)$ is a saddle point of

$$\mathbf{L}(z, \lambda) = \frac{1}{2}\|z - x\|^2 + \lambda f(z) - \delta_{\mathbb{R}_+}(\lambda),$$

which implies that $\mathrm{Prox}_{\alpha^\star f}(x)$ is a solution to the primal problem generated by $\mathbf{L}$.

Hint. Use Exercise 2.31.

Remark. The result of this problem was first presented by Friedlander, Goodwin, and Hoheisel [FGH21, Corollary 13].

2.33 Consider the problem

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m, \end{array}$$

where $f_0, \dots, f_m$ are CCP. Assume all forms of total duality. Show that

$$\begin{aligned} x^{k+1/2} &= \mathrm{Prox}_{\alpha f_0}\left(\frac{1}{m} \sum_{i=1}^m z_i^k \right) \\ x_i^{k+1} &= \Pi_{\{x \in \mathbb{R}^n \,|\, f_i(x) \le 0\}}(2x^{k+1/2} - z_i^k) \\ z_i^{k+1} &= z_i^k + x_i^{k+1} - x^{k+1/2} \qquad \text{for } i = 1, \dots, m \end{aligned}$$

converges in the sense that $x^{k+1/2} \to x^\star$ and $x_i^{k+1} \to x^\star$ for $i = 1, \dots, m$.

Hint. Use Exercise 2.29.

2.34 *Indicator function of a subspace.* Let $V \subseteq \mathbb{R}^n$ be a subspace and

$$V^\perp = \{u \in \mathbb{R}^n \,|\, \langle u, v \rangle = 0 \;\forall\, v \in V\}$$

be its orthogonal complement. Show:

(a) $(\delta_V)^* = \delta_{V^\perp}$,

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press

(b) $\mathbb{N}_V(v) = V^\perp$ for all $v \in V$, and
(c) $\Pi_V + \Pi_{V^\perp} = \mathbb{I}$.

2.35 *Indicator function of a convex cone.* Let $K \subseteq \mathbb{R}^n$ be a nonempty closed *convex cone*, that is, K is a nonempty closed set satisfying

$$x_1, x_2 \in K \quad \Rightarrow \quad \theta_1 x_1 + \theta_2 x_2 \in K$$

for all $\theta_1, \theta_2 \geq 0$. Let

$$K^* = \{u \in \mathbb{R}^n \mid \langle u, x\rangle \geq 0 \; \forall\, x \in K\}$$

be the *dual cone* of K. Show:
(a) $(\delta_K)^* = \delta_{-K^*}$,
(b) $\mathbb{N}_K(x) = \{u \in -K^* \mid \langle u, x\rangle = 0\}$ for all $x \in K$, and
(c) $\Pi_K + \Pi_{-K^*} = \mathbb{I}$.
Remark. This problem subsumes Exercise 2.34.

2.36 *Consensus technique for operators.* Show that the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in \sum_{i=1}^m \mathbb{A}_i x,$$

where $\mathbb{A}_1, \ldots, \mathbb{A}_m$ are (multi-valued) operators, is equivalent to

$$\underset{x_1,\ldots,x_m\in\mathbb{R}^n}{\text{find}} \quad 0 \in \begin{bmatrix} \mathbb{A}_1(x_1) \\ \vdots \\ \mathbb{A}_m(x_m) \end{bmatrix} + \mathbb{N}_C(x_1, \ldots, x_m),$$

where $C = \{(x_1, \ldots, x_m) \in \mathbb{R}^{nm} \mid x_1 = \cdots = x_m\}$ is the consensus set.
Hint. Show $C^\perp = \{(u_1, \ldots, u_m) \in \mathbb{R}^{nm} \mid u_1 + \cdots + u_m = 0\}$ and use Exercise 2.34.

2.37 *Variable metric DRS.* Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{find}} \quad 0 \in (\mathbb{A} + \mathbb{B})x,$$

where $\mathbb{A}$ and $\mathbb{B}$ are maximal monotone. Assume $\operatorname{Zer}(\mathbb{A} + \mathbb{B}) \neq \emptyset$. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Show that the FPI with *variable metric DRS*

$$\begin{aligned} x^{k+1/2} &= \mathbb{J}_{M^{-1}\mathbb{B}}(z^k) \\ x^{k+1} &= \mathbb{J}_{M^{-1}\mathbb{A}}(2x^{k+1/2} - z^k) \\ z^{k+1} &= z^k + x^{k+1} - x^{k+1/2} \end{aligned}$$

converges.

2.38 *PPXA.* Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^m g_i(x),$$

where $g_1, \ldots, g_m$ are CCP functions on $\mathbb{R}^n$. Let $\theta_1, \ldots, \theta_m \in \mathbb{R}$ be such that $\theta_i > 0$ for $i = 1, \ldots, m$ and $\theta_1 + \cdots + \theta_m = 1$. Define the weighted average

$$\bar{z}_\theta^k = \theta_1 z_1^k + \cdots + \theta_m z_m^k$$

and denote

$$\bar{\mathbf{z}}_\theta^k = (\bar{z}_\theta^k, \ldots, \bar{z}_\theta^k) \in \mathbb{R}^{mn}.$$

The algorithm *parallel proximal algorithm* (PPXA) is

$$\begin{aligned} x_i^{k+1} &= \mathrm{Prox}_{(1/\theta_i)g_i}(2\bar{z}_\theta^k - z_i^k) \qquad \text{for } i = 1,\dots,m,\\ \mathbf{z}^{k+1} &= \mathbf{z}^k + \mathbf{x}^{k+1} - \bar{\mathbf{z}}_\theta^k. \end{aligned}$$

Assume $\bigcap_{i=1}^m \operatorname{int}\operatorname{dom} g_i \neq \emptyset$ and that a solution exists. Show that PPXA converges in the sense that there exists a solution $x^\star$ such that

$$(x_1^k,\dots,x_m^k) \to (x^\star,\dots,x^\star).$$

Hint. Consider the variable metric DRS with

$$M = \begin{bmatrix} \theta_1 I & & \\ & \ddots & \\ & & \theta_m I \end{bmatrix} \in \mathbb{R}^{mn\times mn},$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, and use

$$\mathbb{J}_{M^{-1}\partial f}(x) = \operatorname*{argmin}_{z\in\mathbb{R}^d} \left\{ f(z) + \frac{1}{2}\|z - x\|_M^2 \right\}.$$

Remark. PPXA was presented by Combettes and Pesquet [CP08, CP11b].

https://doi.org/10.1017/9781009160865.003 Published online by Cambridge University Press