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RESEARCH ARTICLE

A Proof of the Extended Delta Conjecture

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Abstract
We prove the extended delta conjecture of Haglund, Remmel and Wilson, a combinatorial formula for $\Delta_{h_l}\Delta'_{e_k} e_n$, where Δ'_{e_k} and Δ_{h_l} are Macdonald eigenoperators and e_n is an elementary symmetric function. We actually prove a stronger identity of infinite series of GL_m characters expressed in terms of LLT series. This is achieved through new results in the theory of the Schiffmann algebra and its action on the algebra of symmetric functions.

Contents

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1. Introduction

We prove the *extended delta conjecture* of Haglund, Remmel and Wilson [14] by adapting methods from our work in [2] on a generalized shuffle theorem and proving new results about the action of the elliptic Hall algebra on symmetric functions. As in [2], we reformulate the conjecture as the polynomial truncation of an identity of infinite series of GL_m characters, expressed in terms of LLT series. We then prove the stronger infinite series identity using a Cauchy identity for nonsymmetric Hall–Littlewood polynomials.

The conjecture stemmed from studies of the diagonal coinvariant algebra DR_n in two sets of n variables, whose character as a doubly graded S_n module has remarkable links with both classical combinatorial enumeration and the theory of Macdonald polynomials. It was shown in [17] that this character is neatly given by the formula $\Delta'_{e_{n-1}} e_n$, where e_n is the n-th elementary symmetric function, and for any symmetric function f, Δ'_f is a certain eigenoperator on Macdonald polynomials whose eigenvalues depend on f.

The *shuffle theorem*, conjectured in [13] and proven by Carlsson and Mellit in [4], gives a combinatorial expression for $\Delta'_{e_{n-1}} e_n$ in terms of Dyck paths—that is, lattice paths from $(0, n)$ to $(n, 0)$ that lie weakly below the line segment connecting these two points.

An expanded investigation led Haglund, Remmel and Wilson [14] to the *delta conjecture*, a combinatorial prediction for $\Delta'_{e_k} e_n$, for all $0 \leq k < n$. This led to a flurry of activity (e.g., [6, 11, 14, 15, 16, 21, 22, 23, 26, 28]), including a conjecture by Zabrocki [27] that $\Delta'_{e_k} e_n$ captures the character of the superdiagonal coinvariant ring SDR_n, a deformation of DR_n involving the addition of a set of anticommuting variables.

The delta conjecture has been extended in two directions. One gives a *compositional* generalization, recently proved by D'Adderio and Mellit [7]. The other involves a second eigenoperator Δ_{h_l}, where h_l is the l-th homogeneous symmetric function. The *extended delta conjecture* [14, Conjecture 7.4] is, for $l \geq 0$ and $1 \leq k \leq n$,

$$\Delta_{h_l} \Delta'_{e_{k-1}} e_n = \langle z^{n-k} \rangle \sum_{\substack{\lambda \in \mathbf{D}_{n+l} \\ P \in \mathbf{L}_{n+l,l}(\lambda)}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^{\mathrm{wt}_+(P)} \prod_{i \colon r_i(\lambda) = r_{i-1}(\lambda)+1} \left(1 + z\, t^{-r_i(\lambda)}\right), \tag{1}$$

in which λ is a Dyck path and P is a certain type of labelling of λ (see §2 for full definitions). D'Adderio, Iraci and Wyngaerd proved the Schröder case and the $t = 0$ specialization of the conjecture [5, 6]; Qiu and Wilson [21] reformulated the conjecture and established the $q = 0$ specialization as well.

Let us briefly outline the steps by which we prove equation (1).

Feigin–Tsymbaliuk [8] and Schiffmann–Vasserot [25] constructed an action of the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann [3] on the algebra of symmetric functions. The operators Δ_f and Δ'_f are part of the $\mathcal{E}$ action. In Theorem 4.4.1, we use this to reformulate the left-hand side of equation (1) as the polynomial part of an explicit infinite series of virtual GL_m characters with coefficients in $\mathbb{Q}(q, t)$. The proof of Theorem 4.4.1 relies on a symmetry (Proposition 4.3.3) between distinguished elements of $\mathcal{E}$ introduced by Negut [19] and their transposes.

In Theorem 5.1.1, we also reformulate the right-hand side of equation (1) as the polynomial part of an infinite series, in this case expressed in terms of the LLT series introduced by Grojnowski and Haiman in [12]. This given, we ultimately arrive at Theorem 6.3.6—an identity of infinite series of GL_m characters which implies the extended delta conjecture by taking the polynomial part on each side.

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Although the extended delta conjecture and the compositional delta conjecture both imply the delta conjecture, they generalize it in different directions, and our methods are quite different from those of D'Adderio and Mellit. It would be interesting to know whether a common generalization is possible.

2. The extended delta conjecture

The extended delta conjecture equates a 'symmetric function side,' involving the action of a Macdonald operator on an elementary symmetric function, with a 'combinatorial side.' We begin by recalling the definitions of these two quantities.

2.1. Symmetric function side

Integer partitions are written $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$, sometimes with trailing zeroes allowed. We set $|\lambda| = \lambda_1 + \cdots + \lambda_l$ and let $\ell(\lambda)$ be the number of nonzero parts. We identify a partition λ with its French style Ferrers shape, the set of lattice squares (or *boxes*) with northeast corner in the set

$$\{(i,j) \mid 1 \leq j \leq \ell(\lambda),\ 1 \leq i \leq \lambda_j\}. \tag{2}$$

The *shape generator* of λ is the polynomial

$$B_\lambda(q,t) = \sum_{(i,j)\in\lambda} q^{i-1}\, t^{j-1}. \tag{3}$$

Let $\Lambda = \Lambda_{\mathbf{k}}(X)$ be the algebra of symmetric functions in an infinite alphabet of variables $X = x_1, x_2, \ldots$, with coefficients in the field $\mathbf{k} = \mathbb{Q}(q,t)$. We follow the notation of Macdonald [18] for the graded bases of Λ. Basis elements are indexed by a partition λ and have homogeneous degree $|\lambda|$. Examples include the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$, complete homogeneous symmetric functions $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$, power-sums $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, monomial symmetric functions m_λ and Schur functions s_λ.

As is conventional, $\omega \colon \Lambda \to \Lambda$ denotes the $\mathbf{k}$-algebra involution defined by $\omega s_\lambda = s_{\lambda^*}$, where λ^* denotes the transpose of λ, and $\langle -, - \rangle$ denotes the symmetric bilinear inner product such that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

The basis of modified Macdonald polynomials, $\tilde{H}_\mu(X;q,t)$, is defined [9] from the integral form Macdonald polynomials $J_\mu(X;q,t)$ of [18] using the device of *plethystic evaluation*. For an expression A in terms of indeterminates, such as a polynomial, rational function or formal series, $p_k[A]$ is defined to be the result of substituting a^k for every indeterminate a occurring in A. We define $f[A]$ for any $f \in \Lambda$ by substituting $p_k[A]$ for p_k in the expression for f as a polynomial in the power-sums p_k so that $f \mapsto f[A]$ is a homomorphism. The variables q, t from our ground field $\mathbf{k}$ count as indeterminates. The modified Macdonald polynomials are defined by

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right], \tag{4}$$

where

$$n(\mu) = \sum_i (i-1)\mu_i\,. \tag{5}$$

For any symmetric function $f \in \Lambda$, let $f[B]$ denote the eigenoperator on the basis $\{\tilde{H}_\mu\}$ of Λ such that

$$f[B]\, \tilde{H}_\mu = f[B_\mu(q,t)]\, \tilde{H}_\mu\,. \tag{6}$$

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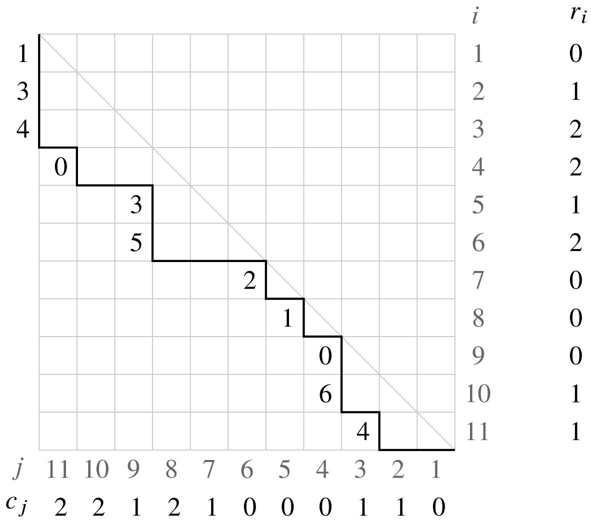

Figure 1. *A path λ and partial labelling $P \in \mathbf{L}_{11,2}(\lambda)$, with* $\operatorname{area}(\lambda) = 10$, $\operatorname{dinv}(P) = 15$, $x^{\mathrm{wt}_+(P)} = x_1^2 x_2 x_3^2 x_4^2 x_5 x_6$ *and* $x^{\mathrm{wt}(P)} = x_0^2 x_1^2 x_2 x_3^2 x_4^2 x_5 x_6$.

The left-hand side of equation (1) is expressed in the notation of [14], where $\Delta_f = f[B]$ and $\Delta'_f = f[B-1]$. Hence, the **symmetric function side of the extended delta conjecture** is

$$h_l[B]e_{k-1}[B-1]e_n\,. \tag{7}$$

2.2. *The combinatorial side*

The right-hand side of the extended delta conjecture (1) is a combinatorial generating function that counts labelled lattice paths.

Definition 2.2.1. A *Dyck path* is a south-east lattice path lying weakly below the line segment connecting the points $(0, N)$ and $(N, 0)$. The set of such paths is denoted $\mathbf{D}_N$. The *staircase path* δ is the Dyck path alternating between south and east steps.

Each $\lambda \in \mathbf{D}_N$ has $\operatorname{area}(\lambda) = |\delta/\lambda|$ defined to be the number of lattice squares lying above λ and below δ. Let $r_i(\lambda)$ be the area contribution from squares in the i-th row, numbered from north to south; in other words, r_i is the distance from the i-th south step of λ to the i-th south step of δ. Note that

$$r_1(\lambda) = 0, \qquad r_i(\lambda) \le r_{i-1}(\lambda) + 1 \quad \text{for } i > 1, \quad \text{and} \quad \sum_{i=1}^{N} r_i(\lambda) = |\delta/\lambda|\,. \tag{8}$$

Definition 2.2.2. A *labelling* $P = (P_1, \dots, P_N) \in \mathbb{N}^N$ attaches a label in $\mathbb{N} = \{0, 1, \dots\}$ to each south step of $\lambda \in \mathbf{D}_N$ so that the labels increase from north to south along vertical runs of south steps, as shown in Figure 1. The set of labellings is denoted by $\mathbf{L}_N(\lambda)$, or simply $\mathbf{L}(\lambda)$. Given $0 \le l < N$, a *partial labelling* of $\lambda \in \mathbf{D}_N$ is a labelling where 0 occurs exactly l times and never on the run at $x = 0$. We denote the set of these partial labellings by $\mathbf{L}_{N,l}(\lambda)$.

Each labelling $P \in \mathbf{L}(\lambda)$ is assigned a statistic $\operatorname{dinv}(P)$, defined to be the number of pairs $(i < j)$ such that either

$$\begin{cases} r_i(\lambda) = r_j(\lambda) \text{ and } P_i < P_j \quad \text{or} \\ r_i(\lambda) = r_j(\lambda) + 1 \text{ and } P_i > P_j\,. \end{cases} \tag{9}$$

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The *weight* of a labelling P is defined so zero labels do not contribute by

$$x^{\mathrm{wt}_+(P)} = \prod_{i\in[N]:\ P_i\neq 0} x_{P_i}\,. \tag{10}$$

This is equivalent to letting $x_0 = 1$ in $x^{\mathrm{wt}(P)} := \prod_{i\in[N]} x_{P_i}$.

The above defines the right-hand side of (1), with $\langle z^{n-k}\rangle$ denoting the coefficient of z^{n-k}.

Remark 2.2.3. In [14], a Dyck path is a northeast lattice path lying weakly above the line segment connecting $(0,0)$ and (N,N), and labellings increase from south to north along vertical runs. After reflecting the picture about a horizontal line, our conventions on paths, labellings and the definition of $\mathrm{dinv}(P)$ match those in [14]. Separately, [13] uses the same conventions that we do for Dyck paths but defines labellings to increase from south to north and defines $\mathrm{dinv}(P)$ with the inequalities in equation (9) reversed. However, since the sum

$$\sum_{P\in\mathbf{L}(\lambda)} q^{\mathrm{dinv}(P)} x^{\mathrm{wt}(P)} \tag{11}$$

is a symmetric function [13], it is unchanged if we reverse the ordering on labels; after which, the conventions in [13] agree with those used here.

We prefer another slight modification based on the following lemma which was mentioned in [14] without details.

Lemma 2.2.4. *For any Dyck path $\lambda \in \mathbf{D}_N$, we have*

$$\prod_{\substack{1<i\leq N\\ r_i(\lambda)=r_{i-1}(\lambda)+1}} (1+z\,t^{-r_i(\lambda)}) = \prod_{\substack{1<i\leq N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+z\,t^{-c_i(\lambda)})\,, \tag{12}$$

where $c_i(\lambda) = r_i(\lambda^*)$ is the contribution to $|\delta/\lambda|$ from boxes in the i-th column, numbered from right to left.

Proof. The condition $r_i(\lambda) = r_{i-1}(\lambda) + 1$ means that λ has consecutive south steps in rows $i-1$ and i with no intervening east step. Similarly, $c_i(\lambda) = c_{i-1}(\lambda) + 1$ if and only if λ has consecutive east steps in columns $i-1$ and i (numbered right to left). Consider the word formed by listing the steps in λ in the south-east direction from $(0,N)$ to $(N,0)$, as shown here for the example in Figure 1.

S S S E S E E S S E E E S E S E S S E S E E

Treating south and east steps as left and right parentheses, each south step pairs with an east step to its right, and we have $r_i(\lambda) = c_j(\lambda)$ if the i-th south step (numbered left to right) pairs with the j-th east step (numbered right to left). Furthermore, the leftmost member of each double south step pairs with the rightmost member of a double east step, as indicated in the word displayed above.

Since each index $i-1$ such that $r_i(\lambda) = r_{i-1}(\lambda) + 1$ pairs with an index $j-1$ such that $c_j(\lambda) = c_{j-1}(\lambda) + 1$, we have

$$\prod_{\substack{1<i\leq N\\ r_i(\lambda)=r_{i-1}(\lambda)+1}} (1+z\,t^{-r_{i-1}(\lambda)-1}) = \prod_{\substack{1<j\leq N\\ c_j(\lambda)=c_{j-1}(\lambda)+1}} (1+z\,t^{-c_{j-1}(\lambda)-1})\,. \tag{13}$$

Now, equation (12) follows. □

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Setting $N = n + l$ and $m = k + l$, the right-hand side of equation (1), or the **combinatorial side of the extended delta conjecture**, is equal to

$$\langle z^{N-m}\rangle \sum_{\substack{\lambda\in\mathbf{D}_N\\ P\in\mathbf{L}_{N,l}(\lambda)}} t^{|\delta/\lambda|}\, q^{\operatorname{dinv}(P)}\, x^{\operatorname{wt}_+(P)} \prod_{\substack{1<i\leq N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+z\,t^{-c_i(\lambda)})\,. \tag{14}$$

3. Background on the Schiffmann algebra $\mathcal{E}$

From work of Feigin and Tsymbaliuk [8] and Schiffmann and Vasserot [25], we know that the operators $f[B]$ in equation (7) form part of an action of the elliptic Hall algebra $\mathcal{E}$ of Burban and Schiffmann [3, 24], or *Schiffmann algebra* for short, on the algebra of symmetric functions. In [2], we used this action to express the symmetric function side of a generalized shuffle theorem as the polynomial part of an explicit infinite series of GL_l characters. Here, we derive a similar expression (Theorem 4.4.1) for the symmetric function side (7) of the extended delta conjecture.

For this purpose, we need a deeper study of the Schiffmann algebra than we did in [2], where a fragment of the theory was enough. We start with a largely self-contained description of $\mathcal{E}$ and its action on Λ, although we occasionally refer to [2] for the restatements of results from [3, 24, 25] in our notation and for some proofs. A precise translation between our notation and that of [3, 24, 25] can be found in [2, eq. (25)]. In the presentation of $\mathcal{E}$ and its action on Λ, we freely use plethystic substitution, defined in §2.1. Indeed, the ability to do so is a principal reason why we prefer the notation used here to that in the foundational papers on the Schiffmann algebra.

3.1. Description of $\mathcal{E}$

Let $\mathbf{k} = \mathbb{Q}(q,t)$, as in §2. The Schiffmann algebra $\mathcal{E}$ is generated by a central Laurent polynomial subalgebra $F = \mathbf{k}[c_1^{\pm1}, c_2^{\pm1}]$ and a family of subalgebras $\Lambda_F(X^{m,n})$ isomorphic to the algebra of symmetric functions $\Lambda_F(X)$ over F, one for each pair of coprime integers (m,n). These are subject to defining relations spelled out below.

For any algebra A containing a copy of Λ, there is an *adjoint action* of Λ on A arising from the Hopf algebra structure of Λ. Using two formal alphabets X and Y to distinguish between the tensor factors in $\Lambda\otimes\Lambda \cong \Lambda(X)\Lambda(Y)$, the coproduct and antipode for the Hopf algebra structure are given by the plethystic substitutions

$$\Delta f = f[X+Y], \qquad S(f) = f[-X]. \tag{15}$$

The adjoint action of $f\in\Lambda$ on $\zeta\in A$ is then given by

$$(\operatorname{Ad} f)\,\zeta = \sum_i f_i\,\zeta\,g_i, \quad \text{where} \quad f[X-Y] = \sum_i f_i(X)g_i(Y) \tag{16}$$

since the formula on the right is another way to write $(1\otimes S)\Delta f = \sum_i f_i\otimes g_i$. More explicitly, we have

$$(\operatorname{Ad} p_n)\,\zeta = [p_n,\zeta] \quad \text{and} \quad (\operatorname{Ad} h_n)\,\zeta = \sum_{j+k=n} (-1)^k h_j\,\zeta\,e_k. \tag{17}$$

The last formula can be expressed for all n at once as a generating function identity

$$(\operatorname{Ad}\Omega[zX])\,\zeta = \Omega[zX]\,\zeta\,\Omega[-zX], \tag{18}$$

where

$$\Omega(X) = \sum_{n=0}^{\infty} h_n(X). \tag{19}$$

We fix notation for the quantities

$$M = (1-q)(1-t), \qquad \widehat{M} = (1-(qt)^{-1})M, \tag{20}$$

which play a role in the presentation of $\mathcal{E}$ and will be referred to again later.

3.1.1. Basic structure and symmetries

The algebra $\mathcal{E}$ is $\mathbb{Z}^2$ graded with the central subalgebra F in degree $(0,0)$ and $f(X^{m,n})$ in degree (dm, dn) for $f(X)$ of degree d in $\Lambda(X)$.

The universal central extension $\widehat{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\mathbb{Z})$ acts on the set of tuples

$$\{(m,n,\theta) \in (\mathbb{Z}^2 \setminus \mathbf{0}) \times \mathbb{R} \mid \theta \text{ is a value of } \arg(m+in)\}, \tag{21}$$

lifting the $\mathrm{SL}_2(\mathbb{Z})$ action on pairs (m,n), with the central subgroup $\mathbb{Z}$ generated by the 'rotation by 2π' map $(m,n,\theta) \mapsto (m,n,\theta+2\pi)$. The group $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ acts on $\mathcal{E}$ by $\mathbf{k}$-algebra automorphisms, compatibly with the action of $\mathrm{SL}_2(\mathbb{Z})$ on the grading group $\mathbb{Z}^2$. Before giving the defining relations of $\mathcal{E}$, we specify how $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ acts on the generators.

For each pair of coprime integers (m,n), we introduce a family of alphabets $X_\theta^{m,n}$, one for each value θ of $\arg(m+in)$, related by

$$X_{\theta+2\pi}^{m,n} = c_1^m c_2^n X_\theta^{m,n}. \tag{22}$$

We make the convention that $X^{m,n}$ without a subscript means $X_\theta^{m,n}$ with $\theta \in (-\pi,\pi]$. For comparison, the implied convention in [2] is $\theta \in [-\pi,\pi)$. The subalgebra $\Lambda_F(X^{m,n}) = \Lambda_F(X_\theta^{m,n})$ only depends on (m,n) and so does not depend on the choice of branch for the angle θ. When we refer to a subalgebra $\Lambda_{\mathbf{k}}(X^{m,n})$, which does depend on the branch, the convention $\theta \in (-\pi,\pi]$ applies.

The $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ action is now given by $\rho \cdot f(X_\theta^{m,n}) = f(X_{\theta'}^{m',n'})$ for $f(X) \in \Lambda_{\mathbf{k}}(X)$, where $\rho \in \widehat{\mathrm{SL}_2(\mathbb{Z})}$ acts on the indexing data in equation (21) by $\rho \cdot (m,n,\theta) = (m',n',\theta')$. Note that, if m,n are coprime, then so are m',n'. The action on F factors through the action of $\mathrm{SL}_2(\mathbb{Z})$ on the group algebra $\mathbf{k}\cdot\mathbb{Z}^2 \cong F$.

For instance, the 'rotation by 2π' element $\rho \in \widehat{\mathrm{SL}_2(\mathbb{Z})}$ fixes F and has $\rho \cdot f(X_\theta^{m,n}) = f(X_{\theta+2\pi}^{m,n}) = f[c_1^m c_2^n X_\theta^{m,n}]$. Thus, ρ coincides with multiplication by $c_1^r c_2^s$ in degree (r,s) and automatically preserves all relations that respect the $\mathbb{Z}^2$ grading.

We now turn to the defining relations of $\mathcal{E}$. Apart from the relations implicit in $F = \mathbf{k}[c_1^{\pm1}, c_2^{\pm1}]$ being central and each $\Lambda_F(X^{m,n})$ being isomorphic to $\Lambda_F(X)$, these fall into three families: Heisenberg relations, internal action relations and axis-crossing relations.

3.1.2. Heisenberg relations

Each pair of subalgebras $\Lambda_F(X^{m,n})$ and $\Lambda_F(X^{-m,-n})$ in degrees along opposite rays in $\mathbb{Z}^2$ satisfy Heisenberg relations

$$[p_k(X_\theta^{-m,-n}),\ p_l(X_{\theta+\pi}^{m,n})] = \delta_{k,l}\, k\, p_k[(c_1^m c_2^n - 1)/\widehat{M}], \tag{23}$$

where $\widehat{M}$ is given by equation (20). As an exercise, the reader can check, using equation (22), that the relations in equation (23) are consistent with swapping the roles of $\Lambda_F(X^{m,n})$ and $\Lambda_F(X^{-m,-n})$.

3.1.3. Internal action relations

The internal action relations describe the adjoint action of each $\Lambda_F(X^{m,n})$ on $\mathcal{E}$. For simplicity, we write these relations and also the axis-crossing relations below, with $\Lambda_F(X^{1,0})$ distinguished. The full set of relations is understood to be given by closing the stated relations under the $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ action.

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Bearing in mind that $X^{m,n}$ means $X^{m,n}_\theta$ with $\theta \in (-\pi, \pi]$, the relations for the internal action of $\Lambda_F(X^{1,0})$ are:

$$\begin{aligned}(\mathrm{Ad}\, f(X^{1,0}))\, p_1(X^{m,1}) &= (\omega f)[z]\Big|z^k \mapsto p_1(X^{m+k,1}) \\ (\mathrm{Ad}\, f(X^{1,0}))\, p_1(X^{m,-1}) &= (\omega f)[-z]\Big|z^k \mapsto p_1(X^{m+k,-1}).\end{aligned} \tag{24}$$

3.1.4. Axis-crossing relations

Again distinguishing $\Lambda_F(X^{1,0})$ and taking angles on the branch $\theta \in (-\pi, \pi]$, the final set of relations is the closure under the $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ action of

$$[p_1(X^{b,-1}),\, p_1(X^{a,1})] = -\frac{e_{a+b}[-\widehat{M} X^{1,0}]}{\widehat{M}} \quad \text{for } a+b>0. \tag{25}$$

More generally, rotating this relation by π determines $[p_1(X^{b,-1}), p_1(X^{a,1})]$ for $a+b<0$, and the Heisenberg relations determine it when $a+b=0$. Combining these gives

$$[p_1(X^{b,-1}),\, p_1(X^{a,1})] = -\frac{1}{\widehat{M}} \begin{cases} e_{a+b}[-\widehat{M} X^{1,0}] & a+b>0 \\ 1 - c_1^{-b} c_2 & a+b=0 \\ -c_1^{-b} c_2 e_{-(a+b)}[-\widehat{M} X^{-1,0}] & a+b<0\,. \end{cases} \tag{26}$$

3.1.5. Further remarks

Define *upper* and *lower half* subalgebras $\mathcal{E}^{*,>0}, \mathcal{E}^{*,<0} \subseteq \mathcal{E}$ to be generated by the $\Lambda_F(X^{m,n})$ with $n>0$ or $n<0$, respectively. Using the $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ image of the relations in equation (25), one can express any $e_k[-\widehat{M} X^{m,n}]$ for $n>0$ in terms of iterated commutators of the elements $p_1(X^{a,1})$. This shows that $\{p_1(X^{a,1}) \mid a \in \mathbb{Z}\}$ generates $\mathcal{E}^{*,>0}$ as an F-algebra. Similarly, $\{p_1(X^{a,-1}) \mid a \in \mathbb{Z}\}$ generates $\mathcal{E}^{*,<0}$.

The internal action relations give the adjoint action of $\Lambda_F(X^{1,0})$ on the space spanned by $\{p_1(X^{a,\pm 1}) \mid a \in \mathbb{Z}\}$. Using the formula $(\mathrm{Ad}\, f)(\zeta_1\zeta_2) = \sum ((\mathrm{Ad}\, f_{(1)})\zeta_1)((\mathrm{Ad}\, f_{(2)})\zeta_2)$, where $\Delta f = \sum f_{(1)} \otimes f_{(2)}$ in Sweedler notation, this determines the adjoint action of $\Lambda_F(X^{1,0})$ on $\mathcal{E}^{*,>0}$ and $\mathcal{E}^{*,<0}$. The Heisenberg relations give the adjoint action of $\Lambda_F(X^{1,0})$ on $\Lambda_F(X^{-1,0})$, while $\Lambda_F(X^{1,0})$ acts trivially on itself, with $(\mathrm{Ad}\, f)\, g = f[1]\, g$.

Together, these determine the adjoint action of $\Lambda_F(X^{1,0})$ on the whole algebra $\mathcal{E}$. By symmetry, the same holds for the adjoint action of any $\Lambda_F(X^{m,n})$.

3.1.6. Anti-involution

One can check from the defining relations above that $\mathcal{E}$ has a further symmetry given by an involutory antiautomorphism (product reversing automorphism)

$$\begin{gathered}\Phi \colon \mathcal{E} \to \mathcal{E} \\ \Phi(g(c_1, c_2)) = g(c_2^{-1}, c_1^{-1}), \quad \Phi(f(X^{m,n}_\theta)) = f(X^{n,m}_{\pi/2-\theta}).\end{gathered} \tag{27}$$

Note that Φ is compatible with reflecting degrees in $\mathbb{Z}^2$ about the line $x=y$. Together with $\widehat{\mathrm{SL}_2(\mathbb{Z})}$, it generates a $\widehat{\mathrm{GL}_2(\mathbb{Z})}$ action on $\mathcal{E}$ for which $\rho \in \widehat{\mathrm{GL}_2(\mathbb{Z})}$ is an anti-automorphism if $\widehat{\mathrm{GL}_2(\mathbb{Z})} \to \mathrm{GL}_2(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\}$ sends ρ to -1.

3.2. Action of $\mathcal{E}$ on Λ

We write $f^\bullet$ for the operator of multiplication by a function f to better distinguish between operator expressions such as $(\omega f)^\bullet$ and $\omega \cdot f^\bullet$. For f a symmetric function, $f^\perp$ denotes the $\langle -, - \rangle$ adjoint of $f^\bullet$.

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Here and again later on, we use an overbar to indicate inverting the variables in any expression; for example,

$$\overline{M} = (1 - q^{-1})(1 - t^{-1}). \tag{28}$$

We extend the notation in equation (6) accordingly, setting

$$f[\overline{B}]\,\tilde{H}_\mu = f[B_\mu(q^{-1}, t^{-1})]\,\tilde{H}_\mu\,. \tag{29}$$

Proposition 3.2.1 [2, Prop 3.3.1]. *There is an action of $\mathcal{E}$ on Λ characterized uniquely by the following properties.*

(i) *The central parameters c_1, c_2 act as scalars*

$$c_1 \mapsto 1, \quad c_2 \mapsto (q\,t)^{-1}. \tag{30}$$

(ii) *The subalgebras $\Lambda_{\mathbf{k}}(X^{\pm 1,0})$ act as*

$$f(X^{1,0}) \mapsto (\omega f)[B - 1/M], \quad f(X^{-1,0}) \mapsto (\omega f)[\overline{1/M - B}]. \tag{31}$$

(iii) *The subalgebras $\Lambda_{\mathbf{k}}(X^{0,\pm 1})$ act as*

$$f(X^{0,1}) \mapsto f[-X/M]^\bullet, \quad f(X^{0,-1}) \mapsto f(X)^\perp. \tag{32}$$

We will make particular use of operators representing the action on Λ of elements $p_1(X^{a,1})$ and $p_1(X^{1,a})$ in $\mathcal{E}$. For the first, we need the operator ∇, defined in [1] as an eigenoperator on the modified Macdonald basis by

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu, \tag{33}$$

where $n(\mu)$ is given by equation (5) and μ^* denotes the transpose partition.

For the second, we introduce the doubly infinite generating series

$$D(z) = \omega\Omega[z^{-1}X]^\bullet (\omega\Omega[-zMX])^\perp\,, \tag{34}$$

where $\Omega(X)$ is given by equation (19).

Definition 3.2.2. For $a \in \mathbb{Z}$, we define operators on $\Lambda = \Lambda_{\mathbf{k}}(X)$:

$$E_a = \nabla^a e_1(X)^\bullet\, \nabla^{-a}, \tag{35}$$

$$D_a = \langle z^{-a} \rangle D(z). \tag{36}$$

The operators D_a are the same as in [2] and differ by a sign $(-1)^a$ from those in [1, 10].

Proposition 3.2.3. *In the action of $\mathcal{E}$ on Λ given by Proposition 3.2.1:*

(i) *The element $p_1[-MX^{1,a}] = -Mp_1(X^{1,a}) \in \mathcal{E}$ acts as the operator D_a;*
(ii) *The element $p_1[-MX^{a,1}] = -Mp_1(X^{a,1}) \in \mathcal{E}$ acts as the operator E_a.*

Proof. Part (i) is proven in [2, Prop 3.3.4].

By equation (32), $p_1[-MX^{0,1}]$ acts on Λ as multiplication by $p_1[X] = e_1(X)$. It was shown in [2, Lemma 3.4.1] that the action of $\mathcal{E}$ on Λ satisfies the symmetry $\nabla f(X^{m,n})\nabla^{-1} = f(X^{m+n,n})$. More generally, this implies $\nabla^a f(X^{m,n})\nabla^{-a} = f(X^{m+an,n})$ for every integer a. Hence, $p_1[-MX^{a,1}]$ acts as $\nabla^a p_1[-MX^{0,1}]\nabla^{-a} = \nabla^a e_1(X)^\bullet\, \nabla^{-a}$. □

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3.3. $\mathbf{GL}_l$ characters and the shuffle algebra

As usual, the weight lattice of GL_l is $\mathbb{Z}^l$, with Weyl group $W = S_l$ permuting the coordinates. A weight λ is dominant if $\lambda_1 \geq \cdots \geq \lambda_l$. A polynomial weight is a dominant weight λ such that $\lambda_l \geq 0$. In other words, polynomial weights of GL_l are integer partitions of length at most l.

As in [2], §2.3, we identify the algebra of virtual GL_l characters over $\mathbf{k}$ with the algebra of symmetric Laurent polynomials $\mathbf{k}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]^{S_l}$. If λ is a polynomial weight, the irreducible character χ_λ is equal to the Schur function $s_\lambda(x_1, \ldots, x_l)$. Given a virtual GL_l character $f(x) = f(x_1, \ldots, x_l) = \sum_\lambda c_\lambda \chi_\lambda$, the partial sum over polynomial weights λ is a symmetric polynomial in l variables, which we denote by $f(x)_{\mathrm{pol}}$ (this is different from the polynomial terms of $f(x)$ considered as a Laurent polynomial). We use the same notation for infinite formal sums $f(x)$ of irreducible GL_l characters, in which case $f(x)_{\mathrm{pol}}$ is a symmetric formal power series.

The Weyl symmetrization operator for GL_l is

$$\boldsymbol{\sigma}(\phi(x_1, \ldots, x_l)) = \sum_{w \in S_l} w\left(\frac{\phi(x)}{\prod_{i<j}(1 - x_j/x_i)}\right). \tag{37}$$

For dominant weights λ, the Weyl character formula can be written $\chi_\lambda = \boldsymbol{\sigma}(x^\lambda)$. More generally, if $\phi(x) = \phi(x_1, \ldots, x_l)$ is a Laurent polynomial over any field $\mathbf{k}$, then $\boldsymbol{\sigma}(\phi(x))$ is a virtual GL_l character over $\mathbf{k}$.

The Hall–Littlewood symmetrization operator is defined by

$$\mathbf{H}_q^l(\phi(x)) = \boldsymbol{\sigma}\left(\frac{\phi(x)}{\prod_{i<j}(1 - q\, x_i/x_j)}\right). \tag{38}$$

If $\phi(x) = \phi(x_1, \ldots, x_l)$ is a rational function over a field $\mathbf{k}$ containing $\mathbb{Q}(q)$, then $\mathbf{H}_q^l(\phi(x))$ is a symmetric rational function over $\mathbf{k}$. If $\phi(x)$ is a Laurent polynomial, we can also regard $\mathbf{H}_q^l(\phi(x))$ as an infinite formal sum of GL_l characters with coefficients in $\mathbf{k}$, by interpreting the factors $1/(1 - q\, x_i/x_j)$ as geometric series $1 + q\, x_i/x_j + (q\, x_i/x_j)^2 + \cdots$. We always understand $\mathbf{H}_q^l(\phi(x))$ in this sense when taking the polynomial part $\mathbf{H}_q^l(\phi(x))_{\mathrm{pol}}$.

We also use the two-parameter symmetrization operator

$$\mathbf{H}_{q,t}^l(\phi(x)) = \mathbf{H}_q^l\left(\phi(x) \prod_{i<j} \frac{(1 - q\,t\,x_i/x_j)}{(1 - t\,x_i/x_j)}\right) = \boldsymbol{\sigma}\left(\frac{\phi(x)\prod_{i<j}(1 - q\,t\,x_i/x_j)}{\prod_{i<j}\big((1 - q\,x_i/x_j)(1 - t\,x_i/x_j)\big)}\right). \tag{39}$$

Again, if $\phi(x)$ is a rational function over $\mathbf{k} = \mathbb{Q}(q,t)$, then $\mathbf{H}_{q,t}^l(\phi(x))$ is a symmetric rational function over $\mathbf{k}$, while if $\phi(x)$ is a Laurent polynomial, or more generally a Laurent polynomial times a rational function which has a power series expansion in the x_i/x_j for $i < j$, we can also interpret $\mathbf{H}_{q,t}^l(\phi(x))$ as an infinite formal sum of GL_l characters, similarly to equation (38). This series interpretation always applies when taking $\mathbf{H}_{q,t}^l(\phi(x))_{\mathrm{pol}}$.

Fixing $\mathbf{k} = \mathbb{Q}(q,t)$ once again, let $T = T(\mathbf{k}[z^{\pm 1}])$ be the tensor algebra on the Laurent polynomial ring in one variable, that is, the noncommutative polynomial algebra with generators corresponding to the basis elements z^a of $\mathbf{k}[z^{\pm 1}]$ as a vector space. Identifying $T^m = T^m(\mathbf{k}[z^{\pm 1}])$ with $\mathbf{k}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$, the product in T is given by 'concatenation',

$$f \cdot g = f(z_1, \ldots, z_k)\, g(z_{k+1}, \ldots, z_{k+l}), \quad \text{for } f \in T^k, g \in T^l. \tag{40}$$

The Feigin–Tsymbaliuk *shuffle algebra* [8] is the quotient $S = T/I$, where I is the graded two-sided ideal whose degree l component $I^l \subseteq T^l$ is the kernel of the symmetrization operator $\mathbf{H}_{q,t}^l$ in variables $z_1, \ldots, z_l$, as explained further in [2, §3.5].

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Let $\mathcal{E}^+ \subseteq \mathcal{E}$ be the subalgebra generated by the $\Lambda_{\mathbf{k}}(X^{m,n})$ for $m > 0$. We leave out the central subalgebra F since the relations of $\mathcal{E}^+$ (as we will see in a moment) do not depend on the central parameters.

The image of $\mathcal{E}^+$ under the antiautomorphism Φ in §3.1.6 is the subalgebra $\Phi(\mathcal{E}^+)$ generated by the $\Lambda_{\mathbf{k}}(X^{m,n})$ for $n > 0$. Note that our convention $\theta \in (-\pi, \pi]$ when the subscript is omitted yields $\Phi(f(X^{m,n})) = f(X^{n,m})$ for $\Lambda_{\mathbf{k}}(X^{m,n}) \subseteq \mathcal{E}^+$ since the branch cut is in the third quadrant.

Schiffmann and Vasserot [25] proved the following result. See [2, §3.5] for more details on the translation of their theorem into our notation.

Proposition 3.3.1 [25, Theorem 10.1]. *There is an algebra isomorphism $\psi\colon S \to \mathcal{E}^+$ and an anti-isomorphism $\psi^{\mathrm{op}} = \Phi \circ \psi\colon S \to \Phi(\mathcal{E}^+)$, given on the generators by $\psi(z^a) = p_1[-MX^{1,a}]$ and $\psi^{\mathrm{op}}(z^a) = p_1[-MX^{a,1}]$.*

To be clear, on monomials in m variables, representing elements of tensor degree m in S, the maps in Proposition 3.3.1 are given by

$$\psi(z_1^{a_1} \cdots z_m^{a_m}) = p_1[-MX^{1,a_1}] \cdots p_1[-MX^{1,a_m}] \tag{41}$$

$$\psi^{\mathrm{op}}(z_1^{a_1} \cdots z_m^{a_m}) = p_1[-MX^{a_m,1}] \cdots p_1[-MX^{a_1,1}]. \tag{42}$$

Later, we will need the following formula for the action of $\psi(\phi(z))$ on $\Lambda(X)$.

Proposition 3.3.2 [2, Proposition 3.5.2]. *Let $\phi(z) = \phi(z_1, \ldots, z_l)$ be a Laurent polynomial representing an element of tensor degree l in S, and let $\zeta = \psi(\phi(z)) \in \mathcal{E}^+$ be its image under the map in equation (41). With $\mathcal{E}$ acting on Λ as in Proposition 3.2.1, we have*

$$\omega(\zeta \cdot 1)(x_1, \ldots, x_l) = \mathbf{H}^l_{q,t}(\phi(x))_{\mathrm{pol}}. \tag{43}$$

4. Schiffmann algebra reformulation of the symmetric function side

4.1. Distinguished elements $D_{\mathbf{b}}$ and $E_{\mathbf{a}}$

Negut [19] defined a family of distinguished elements $D_{\mathbf{b}} \in \mathcal{E}^+$, indexed by $\mathbf{b} \in \mathbb{Z}^l$, which in the case $l = 1$ reduce to the elements in Proposition 3.2.3(i). Here, a remarkable symmetry between these elements and their images $E_{\mathbf{a}}$ under the anti-involution Φ will play a crucial role. After defining the Negut elements, we derive this symmetry in Proposition 4.3.3 with the help of a commutator formula of Negut [20].

Definition 4.1.1 (see also [2, §3.6]). Given $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$, set

$$\phi(z) = \frac{z_1^{b_1} \cdots z_l^{b_l}}{\prod_{i=1}^{l-1}(1 - q\,t\,z_i/z_{i+1})}, \tag{44}$$

and let $\nu(z) = \nu(z_1, \ldots, z_l)$ be a Laurent polynomial satisfying $\mathbf{H}^l_{q,t}(\nu(z)) = \mathbf{H}^l_{q,t}(\phi(z))$. Such a $\nu(z)$ exists by [19, Proposition 6.1] and represents a well-defined element of the shuffle algebra S. The *Negut element* $D_{\mathbf{b}}$ and the *transposed Negut element* $E_{\mathbf{a}}$, where $\mathbf{a} = (b_l, \ldots, b_1)$ is the reversed sequence of indices, are defined by

$$D_{\mathbf{b}} = D_{b_1,\ldots,b_l} = \psi(\nu(z)) \in \mathcal{E}^+ \tag{45}$$

$$E_{\mathbf{a}} = E_{b_l,\ldots,b_1} = \Phi(D_{\mathbf{b}}) = \psi^{\mathrm{op}}(\nu(z)) \in \Phi(\mathcal{E}^+). \tag{46}$$

We should point out that, strictly speaking, the Negut elements in the case $l = 1$ are defined to be elements $D_a = p_1[-MX^{1,a}]$ and $E_a = p_1[-MX^{a,1}]$ of $\mathcal{E}$, while in Definition 3.2.2, we used the

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notation D_a and E_a for operators on Λ. However, by Proposition 3.2.3, these Negut elements act as the operators with the same name, so no confusion should ensue.

Later, we will use the following product formulas, which are immediate from Definition 4.1.1.

$$D_{b_1,\dots,b_l}\, D_{b_{l+1},\dots,b_n} = D_{b_1,\dots,b_n} - q\,t\, D_{b_1,\dots,b_l+1,b_{l+1}-1,\dots,b_n}\,, \tag{47}$$

$$E_{a_n,\dots,a_{l+1}}\, E_{a_l,\dots,a_1} = E_{a_n,\dots,a_1} - q\,t\, E_{a_n,\dots,a_{l+1}-1,a_l+1,\dots,a_1}\,. \tag{48}$$

As noted in §3.1.5, the internal action relations determine the action of $\Lambda_{\mathbf{k}}(X^{0,1})$ on $\Phi(\mathcal{E}^+)$. Using the anti-isomorphism between $\Phi(\mathcal{E}^+)$ and the shuffle algebra, we can make this more explicit.

Lemma 4.1.2. *Let $\phi(z) = \phi(z_1,\dots,z_n)$ be a Laurent polynomial representing an element of tensor degree n in S. Then*

$$(\operatorname{Ad} f(X^{1,0}))\,\psi^{\mathrm{op}}(\phi(z)) = \psi^{\mathrm{op}}\big((\omega f)(z_1,\dots,z_n)\cdot\phi(z)\big). \tag{49}$$

As a particular consequence, we have

$$(\operatorname{Ad} f(X^{1,0}))E_{a_n,\dots,a_1} = \psi^{\mathrm{op}}\left(\frac{(\omega f)(z_1,\dots,z_n)\cdot z_1^{a_1}\cdots z_n^{a_n}}{\prod_{i=1}^{n-1}(1-q\,t\,z_i/z_{i+1})}\right). \tag{50}$$

Proof. This follows immediately from the rule in §3.1.5 for $\operatorname{Ad} f$ acting on a product. □

4.2. Commutator identity

We use a formula for the commutator of elements D_a and $D_{\mathbf{b}}$ and a similar identity for E_a and $E_{\mathbf{b}}$. This commutation relation was proved geometrically by Negut in [20], but to keep things self-contained, we provide an elementary algebraic proof. It is convenient to express the formula using the notation

$$\sum_{i=a}^{b}{}^{\#} f_i = \begin{cases} \sum_{i=a}^{b} f_i & \text{for } a \le b+1 \\ -\sum_{i=b+1}^{a-1} f_i & \text{for } a \ge b+1. \end{cases} \tag{51}$$

As a mnemonic device, note that both cases can be interpreted as $\sum_{i=a}^{\infty} f_i - \sum_{i=b+1}^{\infty} f_i$.

Proposition 4.2.1 [20, Proposition 4.7]. *For any $a \in \mathbb{Z}$ and $\mathbf{b} = (b_1,\dots,b_l) \in \mathbb{Z}^l$, we have*

$$[D_a, D_{b_1,b_2,\dots,b_l}] = -M\sum_{i=1}^{l}\sum_{k=a+1}^{b_i}{}^{\#} D_{b_1,\dots,b_{i-1},k,a+b_i-k,b_{i+1},\dots,b_l} \tag{52}$$

$$[E_{b_l,\dots,b_2,b_1}, E_a] = -M\sum_{i=1}^{l}\sum_{k=a+1}^{b_i}{}^{\#} E_{b_l,\dots,b_{i+1},a+b_i-k,k,b_{i-1},\dots,b_1}. \tag{53}$$

We will need the following lemma for the proof. The notation $\Omega(X)$ is defined in equation (19). Since plethystic substitution into $\Omega(X)$ is characterized by

$$\Omega[a_1+a_2+\cdots-b_1-b_2-\cdots] = \frac{\prod_i(1-b_i)}{\prod_i(1-a_i)}, \tag{54}$$

we have

$$\Omega[Mz] = \frac{(1-q\,z)(1-t\,z)}{(1-z)(1-q\,t\,z)} \quad \text{and} \quad \Omega[-Mz] = \frac{(1-z)(1-q\,t\,z)}{(1-q\,z)(1-t\,z)}. \tag{55}$$

Lemma 4.2.2. *For any $f(z) = f(z_1, \ldots, z_m)$ antisymmetric in z_i and z_{i+1}, we have*

$$\mathbf{H}^m_{q,t}\big(\Omega[M\, z_i/z_{i+1}]\, f(z)\big) = 0\,. \tag{56}$$

Proof. The definition of $\mathbf{H}^m_{q,t}$ and equation (55) imply that

$$\mathbf{H}^m_{q,t}\big(\Omega[M\, z_i/z_{i+1}]\, f(z)\big) = \sum_{w\in S_m} w\left(f(z) \prod_{j\neq k} \frac{1}{1 - z_j/z_k} \prod_{\substack{j<k\\(j,k)\neq(i,i+1)}} \Omega[-M\, z_j/z_k]\right), \tag{57}$$

which vanishes since $f(z)$ is antisymmetric in z_i and z_{i+1}. $\square$

Proof of Proposition 4.2.1. Identity (53) for $[E_{b_l,\ldots,b_1}, E_a]$ follows from equation (52) by applying the antihomomorphism Φ, so we only prove equation (52), which can be written

$$D_a\, D_{\mathbf{b}} - D_{\mathbf{b}}\, D_a + M \sum_{i=1}^{l} \sum_{k=a+1}^{b_i}{}^{\#} D_{b_1,\ldots,b_{i-1},k,a+b_i-k,b_{i+1},\ldots,b_l} = 0. \tag{58}$$

Using Definition 4.1.1 and the isomorphism $\psi\colon S \to \mathcal{E}^+$, we can prove equation (58) by showing that a rational function representing the left-hand side is in the kernel of the symmetrization operator $\mathbf{H}^{l+1}_{q,t}$. For this, we can work directly with the rational functions $\phi(z)$ in equation (44); there is no need to replace them explicitly with Laurent polynomials having the same symmetrization.

Let $\phi(z)$ be the function in equation (44) for $D_{\mathbf{b}}$, and set

$$\phi(\hat{z}_i) = \phi(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{l+1}) = \frac{z_1^{b_1} \cdots z_{i-1}^{b_{i-1}} z_{i+1}^{b_i} \cdots z_{l+1}^{b_l}}{(1 - q\, t\, z_{i-1}/z_{i+1}) \prod_{\substack{1\leq j\leq l\\ j\neq i,i-1}} (1 - q\, t\, z_j/z_{j+1})}\,. \tag{59}$$

To prove equation (58), we want to show

$$\mathbf{H}^{l+1}_{q,t}\left(z_1^a \phi(\hat{z_1}) - \phi(\hat{z_{l+1}}) z_{l+1}^a + M \frac{\sum_{i=1}^{l} \sum_{k=a+1}^{b_i}{}^{\#} z_1^{b_1} \cdots z_{i-1}^{b_{i-1}} z_i^k z_{i+1}^{a+b_i-k} z_{i+2}^{b_{i+1}} \cdots z_{l+1}^{b_l}}{\prod_{j=1}^{l} (1 - q\, t\, z_j/z_{j+1})}\right) = 0\,. \tag{60}$$

Since $z_i^a \phi(\hat{z}_i) - \phi(\hat{z}_{i+1}) z_{i+1}^a$ is antisymmetric in z_i and z_{i+1}, Lemma 4.2.2 implies

$$\sum_{i=1}^{l} \mathbf{H}^{l+1}_{q,t}\Big(\Omega[M\, z_i/z_{i+1}] (z_i^a \phi(\hat{z}_i) - \phi(\hat{z}_{i+1}) z_{i+1}^a)\Big) = 0. \tag{61}$$

The first formula in equation (55) is algebraically the same as

$$\Omega[M\, z] = 1 - \frac{M}{(1 - z^{-1})(1 - q\, t\, z)}.$$

After substituting this into equation (61), the linearity of $\mathbf{H}^{l+1}_{q,t}$ gives

$$\mathbf{H}^{l+1}_{q,t}\left(\sum_{i=1}^{l} \left(z_i^a \phi(\hat{z}_i) - \phi(\hat{z_{i+1}}) z_{i+1}^a - M \frac{z_i^a \phi(\hat{z}_i) - \phi(\hat{z_{i+1}}) z_{i+1}^a}{(1 - z_{i+1}/z_i)(1 - q\, t\, z_i/z_{i+1})}\right)\right) = 0. \tag{62}$$

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The terms $z_i^a\phi(\hat{z_i}) - \phi(\hat{z_{i+1}})z_{i+1}^a$ telescope, reducing this to

$$\mathbf{H}_{q,t}^{l+1}\Bigg(z_1^a\phi(\hat{z_1}) - \phi(\hat{z_{l+1}})z_{l+1}^a - M\sum_{i=1}^{l}\frac{z_i^a\phi(\hat{z_i}) - \phi(\hat{z_{i+1}})z_{i+1}^a}{(1-z_{i+1}/z_i)(1-q\,t\,z_i/z_{i+1})}\Bigg) = 0. \tag{63}$$

If we use the convention $z_0 = 0$ and $z_{l+2} = \infty$, collecting terms in $z_i^a\phi(\hat{z_i})$ and some further algebra manipulations give

$$\begin{aligned}
\sum_{i=1}^{l}\frac{z_i^a\phi(\hat{z_i}) - \phi(\hat{z_{i+1}})z_{i+1}^a}{(1-\frac{z_{i+1}}{z_i})(1-q\,t\frac{z_i}{z_{i+1}})} &= \sum_{i=1}^{l+1}\left[\frac{1}{(1-\frac{z_{i+1}}{z_i})(1-q\,t\frac{z_i}{z_{i+1}})} - \frac{1}{(1-\frac{z_i}{z_{i-1}})(1-q\,t\frac{z_{i-1}}{z_i})}\right]z_i^a\phi(\hat{z_i})\\
&= \sum_{i=1}^{l+1}\frac{z_i^a\phi(\hat{z_i})(1-q\,t\frac{z_{i-1}}{z_{i+1}})}{(1-q\,t\frac{z_{i-1}}{z_i})(1-q\,t\frac{z_i}{z_{i+1}})}\left(\frac{1}{1-\frac{z_{i+1}}{z_i}} - \frac{1}{1-\frac{z_i}{z_{i-1}}}\right)\\
&= \sum_{i=1}^{l+1}\frac{\dfrac{z_i^a\phi(\hat{z_i})(1-q\,t\frac{z_{i-1}}{z_{i+1}})}{(1-q\,t\frac{z_{i-1}}{z_i})(1-q\,t\frac{z_i}{z_{i+1}})} - \dfrac{z_{i+1}^a\phi(\hat{z_{i+1}})(1-q\,t\frac{z_i}{z_{i+2}})}{(1-q\,t\frac{z_i}{z_{i+1}})(1-q\,t\frac{z_{i+1}}{z_{i+2}})}}{1-\frac{z_{i+1}}{z_i}}.
\end{aligned}$$

Expanding the definition (59) of $\phi(\hat{z_i})$ for each i yields

$$\frac{z_i^a\phi(\hat{z_i})(1-q\,t\,z_{i-1}/z_{i+1})}{(1-q\,t\,z_{i-1}/z_i)(1-q\,t\,z_i/z_{i+1})} = \frac{z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}z_i^a z_{i+1}^{b_i}\cdots z_{l+1}^{b_l}}{\prod_{j=1}^{l}(1-q\,t\,z_j/z_{j+1})}$$

so that

$$\begin{aligned}
\sum_{i=1}^{l}\frac{z_i^a\phi(\hat{z_i}) - \phi(\hat{z_{i+1}})z_{i+1}^a}{(1-z_{i+1}/z_i)(1-q\,t\,z_i/z_{i+1})} &= \frac{\sum_{i=1}^{l} z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}\cdot\dfrac{z_i^a z_{i+1}^{b_i} - z_i^{b_i}z_{i+1}^a}{1-z_{i+1}/z_i}\cdot z_{i+2}^{b_{i+1}}\cdots z_{l+1}^{b_l}}{\prod_{j=1}^{l}(1-q\,t\,z_j/z_{j+1})}\\
&= \frac{-\sum_{i=1}^{l} z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}\cdot\Big(\sum_{k=a+1}^{b_i}{}^{\#}z_i^k z_{i+1}^{a+b_i-k}\Big)\cdot z_{i+2}^{b_{i+1}}\cdots z_{l+1}^{b_l}}{\prod_{j=1}^{l}(1-q\,t\,z_j/z_{j+1})}.
\end{aligned}$$

Identity (60) follows by substituting this back into equation (63). □

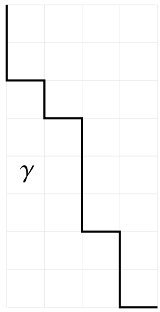
4.3. Symmetry identity for $D_{\mathbf{b}}$ and $E_{\mathbf{a}}$

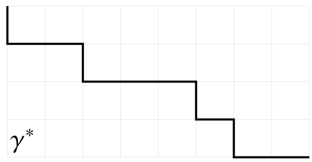
Next, we will prove an identity between certain instances of the Negut elements $D_{\mathbf{b}} \in \mathcal{E}^+$ and transposed Negut elements $E_{\mathbf{a}} \in \Phi(\mathcal{E}^+)$. Before stating the identity, we need to describe how the indices $\mathbf{a}$ and $\mathbf{b}$ will correspond.

Definition 4.3.1. A south-east lattice path γ from $(0,n)$ to $(m,0)$, for positive integers m,n, is *admissible* if it starts with a south step and ends with an east step; that is, γ has a step from $(0,n)$ to $(0,n-1)$ and one from $(m-1,0)$ to $(m,0)$. Define $\mathbf{b}(\gamma) = (b_1,\ldots,b_m)$ by taking $b_i =$ (vertical run of γ at $x = i-1$) for $i = 1,\ldots,m$ and $\mathbf{a}(\gamma) = (a_n,\ldots,a_1)$ with $a_j =$ (horizontal run of γ at $y = j-1$) for $j = 1,\ldots,n$. Set $D_\gamma = D_{\mathbf{b}(\gamma)}$ and $E_\gamma = E_{\mathbf{a}(\gamma)}$.

Note that, if γ^* is the transpose of an admissible path γ with $\mathbf{b}(\gamma) = (b_1,\ldots,b_m)$ and $\mathbf{a}(\gamma) = (a_n,\ldots,a_1)$, as above, then $\mathbf{a}(\gamma^*) = (b_m,\ldots,b_1)$ and $\mathbf{b}(\gamma^*) = (a_1,\ldots,a_n)$, and $E_\gamma = \Phi(D_{\gamma^*})$.

Example 4.3.2. Paths γ and γ^* below are both admissible. Path γ is from $(0,8)$ to $(4,0)$ with $\mathbf{b}(\gamma) = (2,1,3,2)$ and $\mathbf{a}(\gamma) = (0,1,1,0,0,1,0,1)$, whereas γ^* is from $(0,4)$ to $(8,0)$ and has $\mathbf{a}(\gamma^*) = (2,3,1,2)$ and $\mathbf{b}(\gamma^*) = (1,0,1,0,0,1,1,0)$.





Proposition 4.3.3. *For every admissible path γ, we have $D_\gamma = E_\gamma$.*

Proof. Let γ be an admissible path γ from $(0,n)$ to $(m,0)$, where m,n are positive integers.

We first establish the case when $n=1$. In this case, $E_\gamma = E_m = p_1[-MX^{m,1}]$ and $D_\gamma = D_{10^{m-1}}$. If $m=1$, these are $E_1 = D_1 = p_1[-MX^{1,1}]$. In general, equation (24) implies $E_m = p_1[-MX^{m,1}] = (\operatorname{Ad} p_1(X^{1,0}))^{m-1} p_1[-MX^{1,1}] = (\operatorname{Ad} p_1(X^{1,0}))^{m-1} D_1$, while equation (17) and the commutator identity (52) imply $(\operatorname{Ad} p_1(X^{1,0}))D_{10^k} = [p_1(X^{1,0}),\, D_{10^k}] = -(1/M)[D_0, D_{10^k}] = D_{10^{k+1}}$, and therefore $(\operatorname{Ad} p_1(X^{1,0}))^{m-1} D_1 = D_{10^{m-1}}$.

Using the involution Φ, we can deduce the $m=1$ case from the $n=1$ case:

$$D_\gamma = D_n = \Phi(E_n) = \Phi(D_{1,0^{n-1}}) = E_{0^{n-1},1} = E_\gamma. \tag{64}$$

For $m,n>1$, we proceed by induction, assuming that the result holds for paths from $(0,n')$ to $(m',0)$ when $m' \le m$ and $n' \le n$ and $(m',n') \ne (m,n)$.

For a given m,n, there are finitely many admissible paths γ, and thus a finite-dimensional space V of linear combinations $\sum_\gamma c_\gamma D_\gamma$ involving these paths. Let $V' \subseteq V$ denote the subspace consisting of linear combinations which form the left-hand side of a valid instance of the identity

$$\sum_\gamma c_\gamma D_\gamma = \sum_\gamma c_\gamma E_\gamma. \tag{65}$$

Note that $V' = V$ if and only if $D_\gamma = E_\gamma$ for all the paths γ in question.

We will use the induction hypothesis to construct enough instances of equation (65) to reduce each D_γ modulo V' to a scalar multiple of D_{γ_0}, where γ_0 is the path with a south run from $(0,n)$ to $(0,0)$ followed by an east run to $(m,0)$. We will then prove one more instance of equation (65) for which the left-hand side reduces to a nonzero scalar multiple of D_{γ_0}, showing that $V' = V$.

Suppose now that $\gamma \ne \gamma_0$. Then γ contains an east step from (m_1-1, n_2) to (m_1,n_2) and a south step from (m_1,n_2) to (m_1, n_2-1) for some $m_1+m_2 = m$ and $n_1+n_2 = n$. In particular, $\gamma = \nu\cdot\eta$ for shorter admissible paths ν and η, where $\nu\cdot\eta$ is defined to be the lattice path obtained by placing ν and η end to end; thus, $\nu\cdot\eta$ traces a copy of ν from $(0, n_1+n_2)$ to (m_1,n_2) and then traces a copy of η from (m_1,n_2) to $(m_1+m_2, 0)$.

Define $\gamma' = \nu\cdot'\eta$ to be the admissible path obtained from $\nu\cdot\eta$ by replacing the east-south corner at (m_1,n_2) with a south-east corner at (m_1-1, n_2-1); γ' contains a south step from (m_1-1,n_2) to (m_1-1,n_2-1) and an east step from (m_1-1,n_2-1) to (m_1,n_2-1).

The product formulas (47) and (48) imply that the elements corresponding to the paths constructed in this way satisfy

$$D_\nu D_\eta = D_{\nu\cdot\eta} - q\,t\,D_{\nu\cdot'\eta} \qquad\text{and}\qquad E_\nu E_\eta = E_{\nu\cdot\eta} - q\,t\,E_{\nu\cdot'\eta}\,. \tag{66}$$

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By induction, $D_\nu = E_\nu$ and $D_\eta = E_\eta$, so equation (66) implies $D_\gamma - q\,t\,D_{\gamma'} = E_\gamma - q\,t\,E_{\gamma'}$. In other words, in terms of the space V' defined above, we have $D_\gamma \equiv q\,t\,D_{\gamma'} (\mathrm{mod}\ V')$. Using this repeatedly, we obtain $D_\gamma \equiv (q\,t)^{h(\gamma)} D_{\gamma_0} (\mathrm{mod}\ V')$ for every path γ, where $h(\gamma)$ is the area enclosed by the path γ and the x and y axes.

To complete the proof it suffices to establish one more identity of the form (65), for which the congruences $D_\gamma \equiv (q\,t)^{h(\gamma)} D_{\gamma_0} (\mathrm{mod}\ V')$ reduce the left-hand side to a nonzero scalar multiple of D_{γ_0}.

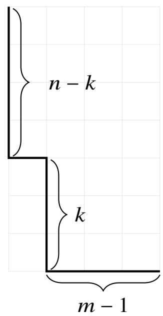
We can assume by induction that $D_{n,0^{m-2}} = E_{0^{n-1},m-1}$ since this case has the same n and a smaller m. Taking the commutator with $p_1(X^{1,0})$ on both sides gives

$$-\frac{1}{M}[D_0, D_{n,0^{m-2}}] = [p_1(X^{1,0}), D_{n,0^{m-2}}] = (\mathrm{Ad}\, p_1(X^{1,0}))E_{0^{n-1},m-1}. \tag{67}$$

Using equation (52) on the left-hand side and equation (50) on the right-hand side gives

$$\sum_{k=0}^{n-1} D_{(n-k,k,0^{m-2})} = \sum_{k=0}^{n-1} E_{(0^{n-1},m-1)+\varepsilon_{n-k}}. \tag{68}$$

Now, for $1 \le k \le n-1$, we have $D_{(n-k,k,0^{m-2})} = D_\gamma$ and $E_{(0^{n-1},m-1)+\varepsilon_{n-k}} = E_\gamma$ for an admissible path with $h(\gamma) = k$, as displayed below.



This shows that equation (68) is an instance of equation (65). The previous congruences reduce the left-hand side of (68) to $(1 + q\,t + \cdots + (q\,t)^{n-1})D_{\gamma_0}$. Since the coefficient is nonzero, we have now established a set of instances of equation (65) whose left-hand sides span V. □

Corollary 4.3.4. *For any indices $a_1, \ldots, a_l$, we have*

$$E_{a_l,\ldots,a_2,a_1} \cdot 1 = E_{a_l,\ldots,a_2,0} \cdot 1. \tag{69}$$

Proof. To rephrase, we are to show that $E_{a_l,\ldots,a_2,a_1} \cdot 1$ does not depend on a_1. The symmetry $f(X^{m,n}) \mapsto f(X^{m+rn,n})$ of $\Phi(\mathcal{E}^+)$ sends $E_{a_l,\ldots,a_1}$ to $E_{a_l+r,\ldots,a_1+r}$. By [2, Lemma 3.4.1], the action of $\mathcal{E}$ on Λ satisfies $\nabla^r f(X^{m,n})\nabla^{-r} = f(X^{m+rn,n})$, and since $\nabla(1) = 1$, this gives $\nabla^r E_{a_l,\ldots,a_2,a_1} \cdot 1 = E_{a_l+r,\ldots,a_2+r,a_1+r} \cdot 1$. Hence, we can reduce to the case that $a_i > 0$ for all i.

By [2, Lemma 3.6.2], we have that $D_{b_1,\ldots,b_n,0,\ldots,0} \cdot 1$ is independent of the number of trailing zeroes. In the case that $b_i \ge 0$ for all i and $b_1 > 0$, this and Proposition 4.3.3 imply that $E_{a_l,\ldots,a_1} \cdot 1$ is independent of a_1, provided that $a_i \ge 0$ for all i and $a_1 > 0$. However, we already saw that this suffices. □

4.4. Shuffling the symmetric function side of the extended delta conjecture

We can now give the promised reformulation of (7).

Theorem 4.4.1. *For $0 \le l < m \le N$, we have*

$$\big(\omega(h_l[B]e_{m-l-1}[B-1]e_{N-l})\big)(x_1, \ldots, x_m) = \mathbf{H}^m_{q,t}\,(\phi(x))_{\mathrm{pol}}\,, \tag{70}$$

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where

$$\phi(x) = \frac{x_1 \cdots x_m}{\prod_i (1 - q\, t\, x_i/x_{i+1})} h_{N-m}(x_1, \ldots, x_m) \overline{e_l(x_2, \ldots, x_m)}, \tag{71}$$

and $\overline{e_l(x_2, \ldots, x_m)} = e_l(x_2^{-1}, \ldots, x_m^{-1})$ *by our convention on the use of the overbar.*

Proof. For any symmetric function f set $g(X) = (\omega f)[X + 1/M]$; then equation (31) gives an identity in Λ for every $\zeta \in \mathcal{E}$

$$f[B]\, \zeta \cdot 1 = g(X^{1,0})\, \zeta \cdot 1 = \sum ((\mathrm{Ad}\, g_{(1)}(X^{1,0}))\, \zeta)\, g_{(2)}(X^{1,0}) \cdot 1, \tag{72}$$

where $g[X + Y] = \sum g_{(1)}(X) g_{(2)}(Y)$ in Sweedler notation and we used the general formula $g\,\zeta = \sum ((\mathrm{Ad}\, g_{(1)})\zeta) g_{(2)}$. Since $g[X + Y] = (\omega f)[X + Y + 1/M]$, and $h[B] \cdot 1 = h[0] \cdot 1$ for any $h(X)$, the right-hand side of equation (72) is equal to

$$\begin{aligned} &\sum ((\mathrm{Ad}\, (\omega f)_{(1)}(X^{1,0}))\, \zeta)\, (\omega f)_{(2)}[X^{1,0} + 1/M] \cdot 1 \\ &\qquad = \sum ((\mathrm{Ad}\, (\omega f)_{(1)}(X^{1,0}))\, \zeta)\, (\omega f)_{(2)}[0] \cdot 1 = ((\mathrm{Ad}\, (\omega f)(X^{1,0}))\, \zeta) \cdot 1. \end{aligned} \tag{73}$$

Let $n = N - l$. Taking $\zeta = E_{a_n, \ldots, a_1}$ and using equation (50), this gives

$$f[B] E_{a_n, \ldots, a_1} \cdot 1 = f(z_n, \ldots, z_1) \Big|\, z_n^{r_n} \cdots z_1^{r_1} \mapsto E_{a_n + r_n, \ldots, a_2 + r_2, a_1 + r_1} \cdot 1. \tag{74}$$

By Corollary 4.3.4, the right-hand side is a function of $f(z_n, \ldots, z_2, 1)$ since the substitution for the monomial $z^{\mathbf{r}}$ does not depend on the exponent r_1. Expressing $f(z_n, \ldots, z_2, 1)$ as $f[z_n + \cdots + z_2 + 1]$ and then substituting $f[X - 1]$ for $f(X)$ yields

$$f[B - 1] E_{a_n, \ldots, a_1} \cdot 1 = f[z_n + \cdots + z_2] \Big|\, z_n^{r_n} \cdots z_2^{r_2} \mapsto E_{a_n + r_n, \ldots, a_2 + r_2, a_1} \cdot 1. \tag{75}$$

By [19, Proposition 6.7], $E_{0^n} = \Phi(D_{0^n}) = \Phi(e_n[-MX^{1,0}]) = e_n[-MX^{0,1}]$ (see also [2, Proposition 3.6.1]).

Using equation (75), we therefore obtain

$$\begin{aligned} e_{k-1}[B - 1] e_n = e_{k-1}[z_n + \cdots + z_2] \Big|\, z_n^{r_n} \cdots z_2^{r_2} \mapsto E_{r_n, \ldots, r_2, 0} \cdot 1 \\ = \sum_{|I| = k-1} E_{\varepsilon_I, 0} \cdot 1 = \sum_{|I| = k-1} E_{\varepsilon_I, 1} \cdot 1\,, \end{aligned} \tag{76}$$

where the sum is over subsets $I \subseteq [n - 1]$ and $\varepsilon_I = \sum_{i \in I} \varepsilon_i$. The terms in the last sum are just $E_{\mathbf{a}(\nu)} \cdot 1$ for paths ν from $(0, n)$ to $(k, 0)$ with single east steps on any $k - 1$ chosen lines $y = j$ for $j \in [n - 1]$, and a final east step at $y = 0$. Denote the set of these admissible paths by $\mathcal{P}_{k,n}$. For instance, with $n = 8$ and $k = 4$, the path γ in Example 4.3.2 corresponds to $E_\gamma = E_{0,1,1,0,0,1,0,1}$.

By equation (74), applying $h_l[B]$ to equation (76) gives

$$h_l[B] e_{k-1}[B - 1] e_n = \sum_{\nu \in \mathcal{P}_{k,n}} \sum_{\substack{\mathbf{r} \in \mathbb{N}^n \\ |\mathbf{r}| = l}} E_{\mathbf{r} + \mathbf{a}(\nu)} \cdot 1\,. \tag{77}$$

This last expression is the sum of $E_\gamma \cdot 1$ over admissible paths γ from $(0, n)$ to $(k + l, 0)$, together with a choice of $k - 1$ indices $j \in [n - 1]$ for which γ has at least one east step on the line $y = j$. We can consider these indices as distinguishing $k - 1$ east-south corners in γ. However, we can also distinguish these corners by their x coordinates, that is, by a set of $k - 1$ indices $i \in [k + l - 1]$ for which γ has at least

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one south step on the line $x = i$. Setting $m = k + l$ and using Proposition 4.3.3, this yields the identity

$$h_l[B]e_{m-l-1}[B-1]e_n = \sum_{\substack{\mathbf{s}\in\mathbb{N}^m:|\mathbf{s}|=n-k \\ I\subseteq[2,m],|I|=l}} D_{\mathbf{s}+(1^m)-\varepsilon_I} \cdot 1\,. \tag{78}$$

Now, since

$$\sum_{\substack{\mathbf{s}\in\mathbb{N}^m:|\mathbf{s}|=n-k \\ I\subseteq[2,m],|I|=l}} x^{\mathbf{s}+(1^m)-\varepsilon_I} = x_1 x_2 \cdots x_m h_{n-k}(x_1,\dots,x_m)\overline{e_l(x_2,\dots,x_m)}\,, \tag{79}$$

the definition of $D_{\mathbf{b}}$ and Proposition 3.3.2 imply that

$$\omega\Bigg(\sum_{\substack{\mathbf{s}\in\mathbb{N}^m:|\mathbf{s}|=n-k \\ I\subseteq[2,m],|I|=l}} D_{\mathbf{s}+(1^m)-\varepsilon_I}\cdot 1\Bigg)(x_1,\dots,x_m) = \mathbf{H}^m_{q,t}(\phi(x))_{\mathrm{pol}} \tag{80}$$

with $\phi(x)$ given by equation (71). □

Remark 4.4.2. For any $\mathbf{b} \in \mathbb{Z}^m$, [2, Corollary 3.7.2] gives that the Schur expansion of $\omega(D_{\mathbf{b}}\cdot 1)$ involves only $s_\lambda(X)$ with $\ell(\lambda) \le m$. Hence, although Theorem 4.4.1 is a statement in m variables, it determines $\omega(h_l[B]e_{m-l-1}[B-1]e_{N-l})$ by equation (78).

5. Reformulation of the combinatorial side

5.1. Statement of the reformulation

We reformulate (14) by explicitly extracting the coefficient of z^{N-m}. The most natural form of the resulting expression involves a generating function $N_{\beta/\alpha}$ for q-weighted tableaux rather than partially labelled paths. For now, we work only with the tableau description of $N_{\beta/\alpha}$, but in §6.2, we will see that $N_{\beta/\alpha}$ is a truncation of an LLT series introduced by Grojnowski and Haiman in [12].

The q-weight in our reformulation involves two auxiliary statistics: for $\eta, \tau \in \mathbb{N}^m$, define

$$d(\eta,\tau) = \sum_{1\le j<r\le m} \big|[\eta_j, \eta_j+\tau_j] \cap [\eta_r, \eta_r+\tau_r-1]\big|, \tag{81}$$

with $[a,b] = \{a,\dots,b\}$ and $[b] = [1,b]$, and for a vector η of length n and $I \subseteq [n]$, define

$$h_I(\eta) = |\{(r<s) : r\in I, s\notin I, \eta_s = \eta_r + 1\}|\,, \tag{82}$$

where $(r<s)$ denotes a pair of positions (r,s) in η with $1 \le r < s \le n$.

Our reformulation of (14) is stated in the following theorem, proven at the end of this section.

Theorem 5.1.1. *For $0 \le l < m \le N$, we have*

$$\langle z^{N-m}\rangle \sum_{\substack{\lambda\in\mathbf{D}_N \\ P\in\mathbf{L}_{N,l}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N \\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+z\,t^{-c_i(\lambda)})\, q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)}$$
$$= \sum_{\substack{J\subseteq[m-1] \\ |J|=l}} \sum_{\substack{\tau,(0,\mathbf{a})\in\mathbb{N}^m \\ |\tau|=N-m}} t^{|\mathbf{a}|} q^{d((0,\mathbf{a}),\tau)+h_J(\mathbf{a})} N_{((0,\mathbf{a})+(1^m)+\tau)/((\mathbf{a},0)+\varepsilon_J)}(X;q)\,, \tag{83}$$

where $N_{\beta/\alpha}$ is given by Definition 5.2.1 below.

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5.2. *Definition of $N_{\beta/\alpha}$*

For $\alpha, \beta \in \mathbb{Z}^l$ such that $\alpha_j \le \beta_j$ for all j, define β/α to be the tuple of single row skew shapes $(\beta_j)/(\alpha_j)$ such that the x coordinates of the right edges of boxes a in the j-th row are the integers $\alpha_j + 1, \dots, \beta_j$. The boxes just outside the j-th row, adjacent to the left and right ends of the row, then have x coordinates α_j and $\beta_j + 1$. We consider these two boxes to be adjacent to the ends of an empty row, with $\alpha_j = \beta_j$, as well.

Given a tuple of skew row shapes β/α, three boxes (u, v, w) form a w_0-*triple* when box v is in row r of β/α, boxes u and w are in or adjacent to a row j with $j > r$ and the x-coordinates i_u, i_v, i_w of these boxes satisfy $i_u = i_v$ and $i_w = i_v + 1$. These triples are a special case of σ-triples defined for any $\sigma \in S_l$ in [2]. We denote the number of w_0-triples in β/α by $h_{w_0}(\beta/\alpha)$. The reader can verify that

$$h_{w_0}(\beta/\alpha) = \sum_{r<j} \big|[\alpha_r + 1, \beta_r] \cap [\alpha_j, \beta_j]\big| . \tag{84}$$

For a totally ordered alphabet $\mathcal{A}$, a *row strict tableau* of shape β/α is a map $S\colon \beta/\alpha \to \mathcal{A}$ that is strictly increasing on each row. The set of these maps is denoted by $\mathrm{RST}(\beta/\alpha, \mathcal{A})$. For convenience, given $\alpha, \beta \in \mathbb{Z}^l$ with some $\alpha_j > \beta_j$, we set $\mathrm{RST}(\beta/\alpha, \mathcal{A}) = \varnothing$.

A w_0-triple (u, v, w) is an *increasing* w_0-triple in S if $S(u) < S(v) < S(w)$, with the convention that $S(u) = -\infty$ if u is adjacent to the left end of a row of β/α and $S(w) = \infty$ if w is adjacent to the right end of a row. Let $h_{w_0}(S)$ be the number of increasing w_0-triples in S.

For $S \in \mathrm{RST}(\beta/\alpha, \mathbb{N})$, define

$$x^{\mathrm{wt}_+(S)} = \prod_{u \in \beta/\alpha,\, S(u) \neq 0} x_{S(u)} \qquad \text{and} \qquad x^{\mathrm{wt}(S)} = \prod_{u \in \beta/\alpha} x_{S(u)} . \tag{85}$$

Definition 5.2.1. For $\alpha, \beta \in \mathbb{N}^m$, define

$$N_{\beta/\alpha} = N_{\beta/\alpha}(X; q) = \sum_{S \in \mathrm{RST}(\beta/\alpha, \mathbb{Z}_{>0})} q^{h_{w_0}(S)} x^{\mathrm{wt}(S)} . \tag{86}$$

Note that, if $\alpha_j > \beta_j$ for any j, then $N_{\beta/\alpha} = 0$ by our convention that $\mathrm{RST}(\beta/\alpha, \mathcal{A}) = \varnothing$.

Remark 5.2.2. It is shown in [2, Proposition 4.5.2] and its proof that, for $\alpha, \beta \in \mathbb{N}^m$, $\omega N_{\beta/\alpha}$ is a symmetric function whose Schur expansion involves only s_λ where $\ell(\lambda) \le m$.

5.3. *Transforming the combinatorial side*

To prove equation (83), we first associate each Dyck path with a tuple of row shapes recording vertical runs.

Definition 5.3.1. The *LLT data* associated to a path $\lambda \in \mathbf{D}_N$ are

$$\beta = (1, c_2(\lambda) + 1, \dots, c_N(\lambda) + 1) \;\text{ and }\; \alpha = (c_2(\lambda), \dots, c_N(\lambda), 0) ,$$

where $c_i(\lambda)$ counts lattice squares between λ and the line segment connecting $(0, N)$ to $(N, 0)$ in column i, numbered from right to left, as in Lemma 2.2.4.

Figure 2 shows the LLT data β, α associated to the path λ in Figure 1. Note that β_i (resp. α_i) is the furthest (resp. closest) distance from the diagonal to the path λ on the line $x = N - i$ so that $\beta_i - \alpha_i$ is the number of south steps of λ on that line.

This association allows us to relate q-weighted sums over partial labellings to the $N_{\beta/\alpha}$.

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$$S = \begin{array}{ccccc}
-\infty & \boxed{1} & \boxed{3} & \boxed{4} & \infty \\
 & & -\infty & \boxed{0} & \infty \\
 & & -\infty & \infty & \\
 & -\infty & \boxed{3} & \boxed{5} & \infty \\
 & & -\infty & \infty & \\
 & -\infty & \infty & & \\
-\infty & \boxed{2} & \infty & & \\
-\infty & \boxed{1} & \infty & & \\
-\infty & \boxed{0} & \boxed{6} & \infty & \\
 & -\infty & \boxed{4} & \infty & \\
 & -\infty & \infty & &
\end{array}$$

Figure 2. *For $\beta = (12211123233)$, $\alpha = (11000121220)$, there are $h_{w_0}(\beta/\alpha) = 29$ w_0-triples in β/α. The row strict tableau S of shape β/α has $h_{w_0}(S) = 15$ increasing w_0-triples, $x^{\mathrm{wt}_+(S)} = x_1^2 x_2 x_3^2 x_4^2 x_5 x_6$, and $x^{\mathrm{wt}(S)} = x_0^2 x_1^2 x_2 x_3^2 x_4^2 x_5 x_6$.*

Lemma 5.3.2. *For $\lambda \in \mathbf{D}_N$ and its associated LLT data α, β, we have*

$$\sum_{P \in \mathbf{L}_{N,l}(\lambda)} q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} = \sum_{\substack{I \subseteq [N-1] \\ |I| = l}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)\,. \tag{87}$$

Proof. There is a natural weight-preserving bijection mapping $P \in \mathbf{L}_N(\lambda)$ to $S \in \mathrm{RST}(\beta/\alpha, \mathbb{N})$, where the labels of column $x = i$ of P, read north to south, are placed into row $N - i$ of β/α, west to east. See Figures 1 and 2. Moreover, $\mathrm{dinv}(P) = h_{w_0}(S)$. To see this, let $\hat{P}$ be the same labelling as P but with the ordering on letters taken to be $0 > 1 > 2 \cdots$. It is proven in [2, Proposition 6.1.1] that $\mathrm{dinv}_1(\hat{P}) = h_{w_0}(S)$, where $\mathrm{dinv}_1(\hat{P})$ was introduced in [13] and matches $\mathrm{dinv}(P)$ as discussed in Remark 2.2.3. The bijection restricts to a bijection from $\mathbf{L}_{N,l}(\lambda)$ to the subset of tableaux $S \in \mathrm{RST}(\beta/\alpha, \mathbb{N})$ with exactly l 0's, none in row N. This gives

$$\sum_{P \in \mathbf{L}_{N,l}(\lambda)} q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} = \sum_{\substack{I \subseteq [N-1] \\ |I| = l}} \sum_{\substack{S \in \mathrm{RST}(\beta/\alpha, \mathbb{N}) \\ 0 \text{ in rows } i \in I}} q^{h_{w_0}(S)} x^{\mathrm{wt}_+(S)}\,. \tag{88}$$

The claim then follows from Definition 5.2.1 and the following Lemma. □

Lemma 5.3.3. *For $\alpha, \beta \in \mathbb{N}^N$ and $S \in \mathrm{RST}(\beta/\alpha, \mathbb{N})$, let $I \subseteq [N]$ be the rows of S containing a zero and let T be the tableau in $\mathrm{RST}(\beta/(\alpha + \varepsilon_I), \mathbb{Z}_{>0})$ obtained by deleting all zeros from S. Then*

$$h_{w_0}(T) = h_{w_0}(S) - h_I(\alpha)\,, \tag{89}$$

where $h_I(\alpha)$ is defined in equation (82).

Proof. Consider an increasing w_0-triple (u, v, w) of S; the entries satisfy $S(u) < S(v) < S(w)$, v lies in some row r and both u and w lie in a row $j > r$. When $r \notin I$, either $j \notin I$ so that (u, v, w) is an increasing w_0-triple of T with the same entries as S, or $j \in I$ and $S(u) = 0$ changes to $T(u) = -\infty$ where still (u, v, w) is an increasing w_0-triple of T. However, if $r \in I$, $S(v) = 0$ changes to $T(v) = -\infty$ and

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thus (u, v, w) is not an increasing w_0-triple of T. Note the increasing condition implies that this happens only when $j \notin I$ and $\alpha_r = \alpha_j - 1$ since $S(u) < 0 < S(w)$. Thus (89) follows. □

Definition 5.3.4. Given $\mathbf{a} = (a_1, \dots, a_{m-1}) \in \mathbb{N}^{m-1}$ and $\tau = (\tau_1, \dots, \tau_m) \in \mathbb{N}^m$, we define two sequences $\beta_{\mathbf{a}\tau}$ and $\alpha_{\mathbf{a}\tau}$ of length $|\tau| + m$ as follows.

The sequence $\beta_{\mathbf{a}\tau}$ is the concatenation of sequences $(1, 2, \dots, \tau_1+1)$ and $(a_{i-1}+1, a_{i-1}+2, \dots, a_{i-1}+\tau_i+1)$ for $i = 2, \dots, m$. The sequence $\alpha_{\mathbf{a}\tau}$ is the same as $\beta_{\mathbf{a}\tau}$ except in the positions corresponding to the ends of the concatenated subsequences. In these positions, we change the entries $\tau_1 + 1, a_1 + \tau_2 + 1, \dots, a_{m-1}+\tau_m+1$ in $\beta_{\mathbf{a}\tau}$ to $a_1, a_2, \dots, a_{m-1}, 0$. Equivalently, $\alpha_{\mathbf{a}\tau}$ is the same as the sequence obtained by subtracting 1 from all entries of $\beta_{\mathbf{a}\tau}$ and shifting one place to the left, deleting the first entry and adding a zero at the end.

Example 5.3.5. For $\mathbf{a} = (130012)$ and $\tau = (2311022)$,

$$\begin{aligned} (0,\mathbf{a}) + (1^m) + \tau &= (\quad\;\; 3 \qquad\quad\;\; 5 \qquad 5 \qquad 2 \quad 1 \qquad\;\; 4 \qquad\;\; 5) \\ \beta_{\mathbf{a}\tau} &= (1\ 2\ 3 \quad 2\ 3\ 4\ 5 \quad 4\ 5 \quad 1\ 2 \quad 1 \quad 2\ 3\ 4 \quad 3\ 4\ 5) \\ \alpha_{\mathbf{a}\tau} &= (1\ 2\ 1 \quad 2\ 3\ 4\ 3 \quad 4\ 0 \quad 1\ 0 \quad 1 \quad 2\ 3\ 2 \quad 3\ 4\ 0) \\ (\mathbf{a}, 0) &= (\quad\;\; 1 \qquad\quad\;\; 3 \qquad 0 \qquad 0 \quad 1 \qquad\;\; 2 \qquad\;\; 0) \end{aligned} . \tag{90}$$

The wider spaces show the division into blocks of size $\tau_i + 1$. The last entry of $\alpha_{\mathbf{a}\tau}$ in each block is a_i, and the next block in $\alpha_{\mathbf{a}\tau}$ and $\beta_{\mathbf{a}\tau}$ starts with $a_i + 1$.

Lemma 5.3.6. *For $0 \le l < m \le N$,*

$$\begin{aligned} \langle z^{N-m} \rangle \sum_{\substack{\lambda \in \mathbf{D}_N \\ P \in \mathbf{L}_{N,l}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1 < i \le N \\ c_i(\lambda) = c_{i-1}(\lambda)+1}} (1 + z\, t^{-c_i(\lambda)})\, q^{\operatorname{dinv}(P)} x^{\operatorname{wt}_+(P)} \\ = \sum_{\substack{I \subseteq [N-1] \\ |I| = l}} \sum_{\substack{\tau,\, (0,\mathbf{a}) \in \mathbb{N}^m \\ |\tau| = N-m}} t^{|\mathbf{a}|} q^{h_I(\alpha_{\mathbf{a}\tau})} \mathcal{N}_{\beta_{\mathbf{a}\tau}/(\alpha_{\mathbf{a}\tau}+\varepsilon_I)}(X; q)\, . \end{aligned} \tag{91}$$

Proof. Use Lemma 5.3.2 to rewrite the left-hand side of equation (91) as

$$\langle z^{N-m} \rangle \sum_{\lambda \in \mathbf{D}_N} t^{|\delta/\lambda|} \prod_{\substack{1 < i \le N \\ c_i(\lambda) = c_{i-1}(\lambda)+1}} (1 + z\, t^{-c_i(\lambda)}) \sum_{\substack{I \subseteq [N-1] \\ |I| = l}} q^{h_I(\alpha)} \mathcal{N}_{\beta/(\alpha+\varepsilon_I)}, \tag{92}$$

where $\beta = (1^N) + (0, c_2(\lambda), \dots, c_N(\lambda))$, $\alpha = (c_2(\lambda), \dots, c_N(\lambda), 0)$ are the LLT data for λ. Note that a tuple $\mathbf{c} = (c_1, c_2, \dots, c_N) \in \mathbb{N}^N$ is the sequence of column heights $c_i(\lambda)$ of a path $\lambda \in \mathbf{D}_N$ if and only if $c_s \le c_{s-1} + 1$ for all $s > 1$ and $c_1 = 0$; in this case, $|\delta/\lambda| = |\mathbf{c}|$. Replace $\mathbf{D}_N$ in equation (92) by these tuples, and expand the product to obtain

$$\begin{aligned} \langle z^{N-m} \rangle \sum_{A \subseteq [N] \setminus \{1\}} \sum_{\substack{c_i \le c_{i-1}+1\ \forall i \\ c_i = c_{i-1}+1\ \forall i \in A}} t^{|\mathbf{c}| - \sum_{i \in A} c_i} z^{|A|} \sum_{\substack{I \subseteq [N-1] \\ |I| = l}} q^{h_I(\alpha)} \mathcal{N}_{\beta/(\alpha+\varepsilon_I)} \\ = \sum_{\substack{\{1\} \subseteq J \subseteq [N] \\ |J| = m}} \sum_{c_j = c_{j-1}+1\ \forall j \notin J} t^{\sum_{j \in J} c_j} \sum_{\substack{I \subseteq [N-1] \\ |I| = l}} q^{h_I(\alpha)} \mathcal{N}_{\beta/(\alpha+\varepsilon_I)}\, , \end{aligned} \tag{93}$$

where the equality comes from reindexing with $J = [N] \setminus A$ and noting that we can drop the condition $c_j \le c_{j-1} + 1\ \forall j \in J$ because $\mathcal{N}_{\beta/(\alpha+\varepsilon_I)} = 0$ if any $(\alpha + \varepsilon_I)_j \ge \alpha_j > \beta_j$.

If we replace the sum over J by a sum over $\{\tau \in \mathbb{N}^m : |\tau| = N - m\}$ using $J = \{1, \tau_1 + 2, \tau_1 + \tau_2 + 3, \dots, \tau_1 + \cdots + \tau_{m-1} + m\}$, then, for fixed J (or fixed τ), the sum over $\mathbf{c}$ can be replaced by a sum over

$$\mathbf{c} = (0, 1, 2, \dots, \tau_1, a_1, a_1 + 1, \dots, a_1 + \tau_2, a_2, \dots, a_{m-1} + \tau_m) \tag{94}$$

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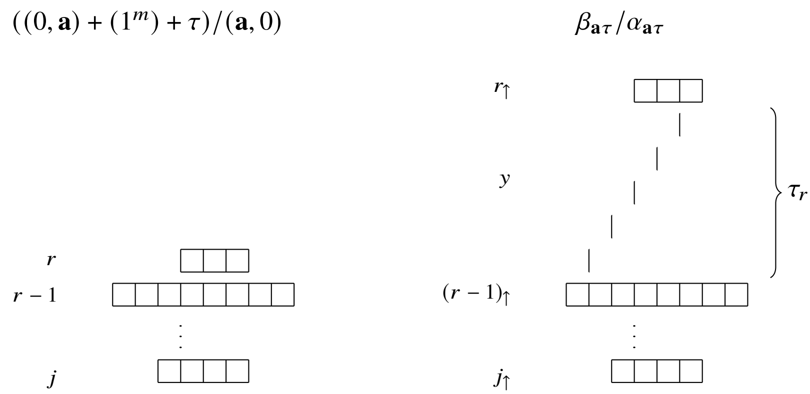

Figure 3. *Comparing the tuples of rows* $\beta_{\mathbf{a}\tau}/\alpha_{\mathbf{a}\tau}$ *and* $((0,\mathbf{a}) + (1^m) + \tau)/(\mathbf{a}, 0)$ *for* $\mathbf{a} \in \mathbb{N}^{m-1}$ *and* $\tau \in \mathbb{N}^m$*. Here,* $a_j = 2$, $a_{r-1} = 0$, $a_r = 3$ *and* $\tau_r = 5$*.*

for $\mathbf{a}$ ranging over $\mathbb{N}^{m-1}$. Note that $\sum_{j\in J} c_j = |\mathbf{a}|$. With this encoding of $\mathbf{c}$, we have $\beta/\alpha = \beta_{\mathbf{a}\tau}/\alpha_{\mathbf{a}\tau}$ in the notation of Definition 5.3.4, and the quantity in equation (93) becomes the right-hand side of equation (91). □

We make a final adjustment to the right-hand side of equation (91). This sum runs over tuples $\beta_{\mathbf{a}\tau}/(\alpha_{\mathbf{a}\tau} + \varepsilon_I)$ with $|\tau|$ necessarily empty rows which can be removed at the cost of a q factor. We introduce some notation depending on a given $\mathbf{a} \in \mathbb{N}^{m-1}$, $\tau = (\tau_1, \dots, \tau_m) \in \mathbb{N}^m$, and the associated $\beta_{\mathbf{a}\tau}/\alpha_{\mathbf{a}\tau}$ from Definition 5.3.4. First, we set $j_\uparrow = j + \sum_{x\le j} \tau_x$ for $j \in [m]$, so the entry of $\beta_{\mathbf{a}\tau}$ in position $j_\uparrow$ is $a_{j-1} + \tau_j + 1$, or $\tau_1 + 1$ if $j = 1$, and the entry of $\alpha_{\mathbf{a}\tau}$ in the same position is a_j or 0 if $j = m$. For a subset $J \subseteq [m]$, we set $J_\uparrow = \{j_\uparrow : j \in J\}$. In positions $i \notin [m]_\uparrow$, the sequences $\beta_{\mathbf{a}\tau}$ and $\alpha_{\mathbf{a}\tau}$ agree, so row i is empty in $\beta_{\mathbf{a}\tau}/\alpha_{\mathbf{a}\tau}$. The tuple of row shapes obtained by deleting these empty rows from $\beta_{\mathbf{a}\tau}/\alpha_{\mathbf{a}\tau}$ is $((0,\mathbf{a}) + (1^m) + \tau)/(\mathbf{a}, 0)$, where row $j \in [m]$ corresponds to row $j_\uparrow$ of $\beta_{\mathbf{a}\tau}/\alpha_{\mathbf{a}\tau}$; note that rows $(j-1)_\uparrow$ and $j_\uparrow$ are separated by τ_j empty rows. See Figure 3.

Lemma 5.3.7. *For* $J \subseteq [m]$, $\mathbf{a} \in \mathbb{N}^{m-1}$ *and* $\tau \in \mathbb{N}^m$*, let* $I = J_\uparrow$*. Then*

$$N_{\beta_{\mathbf{a}\tau}/(\alpha_{\mathbf{a}\tau}+\varepsilon_I)} = q^{d((0,\mathbf{a}),\tau) - h'_J(\mathbf{a},\tau)} N_{((0,\mathbf{a})+(1^m)+\tau)/((\mathbf{a},0)+\varepsilon_J)}\,, \tag{95}$$

where $h'_J(\mathbf{a},\tau) = \big|\{(j<r) : j \in J, r \in [m], a_j \in [a_{r-1}, a_{r-1} + \tau_r - 1]\}\big|$ *with* $a_0 = 0$*, and* $d((0,\mathbf{a}),\tau)$ *is defined by equation (81).*

Proof. Set $a_0 = 0$. We can assume $a_j + (\varepsilon_J)_j \le a_{j-1} + \tau_j + 1$ for all $j \in [m]$ since otherwise both sides of equation (95) vanish by Definition 5.2.1. Hence, each side is a q-generating function for row strict tableaux on tuples of single row skew shapes; rows of $\beta_{\mathbf{a}\tau}/(\alpha_{\mathbf{a}\tau} + \varepsilon_I)$ on the left-hand side differ from the right-hand side only by the removal of empty rows $r \notin [m]_\uparrow$. Thus, the two sides agree up to a factor q^d, where d counts w_0-triples of $\beta_{\mathbf{a}\tau}/(\alpha_{\mathbf{a}\tau} + \varepsilon_I)$ involving one of these empty rows.

To evaluate d, consider such an empty row $(b)/(b)$, coming from $b \in \{a_{r-1}+1, \dots, a_{r-1}+\tau_r\}$ for some $r \in [m]$. The adjacent boxes on the left and right of this empty row form a w_0-triple, increasing in every tableau, with one box in each nonempty lower row $j_\uparrow$, of the form $(a_{j-1} + \tau_j + 1)/(a_j + (\varepsilon_J)_j)$, such that $b \in [a_j + (\varepsilon_J)_j + 1, a_{j-1} + \tau_j + 1]$. Hence,

$$\begin{aligned} d &= \sum_{1\le j<r\le m} \big|[a_j + (\varepsilon_J)_j, a_{j-1} + \tau_j] \cap [a_{r-1}, a_{r-1} + \tau_r - 1]\big| \\ &= \sum_{1\le j<r\le m} \big|[a_j, a_{j-1} + \tau_j] \cap [a_{r-1}, a_{r-1} + \tau_r - 1]\big| - \sum_{\substack{1\le j<r\le m\\ j\in J}} \big|\{a_j\} \cap [a_{r-1}, a_{r-1} + \tau_r - 1]\big|. \end{aligned}$$

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The sum after the minus sign is $h'_J(\mathbf{a},\tau)$. To prove that the remaining sum is $d((0,\mathbf{a}),\tau)$, first rewrite it as

$$\sum_{1\le j<r\le m}\left(\big|[a_j,\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|-\big|[a_{j-1}+\tau_j+1,\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|\right), \tag{96}$$

using the fact that $a_j\le a_{j-1}+\tau_j+1$ by assumption. Next, observe that since $a_0=0\le a_{r-1}$,

$$\big|[a_{r-1},\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|=\big|[a_0,\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|.$$

Adding $\sum_{1<j<r}\big|[a_{j-1},\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|$ to both sides, it follows that

$$\sum_{1\le j<r}\big|[a_j,\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|=\sum_{1\le j<r}\big|[a_{j-1},\infty)\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|.$$

Hence, formula (96) is unchanged upon replacing $[a_j,\infty)$ with $[a_{j-1},\infty)$ and is thus equal to

$$\sum_{1\le j<r\le m}\big|[a_{j-1},a_{j-1}+\tau_j]\cap[a_{r-1},a_{r-1}+\tau_r-1]\big|=d((0,\mathbf{a}),\tau).$$ □

Proof of Theorem 5.1.1. Consider a summand $t^{|\mathbf{a}|}q^{h_I(\alpha_{\mathbf{a}\tau})}N_{\beta_{\mathbf{a}\tau}/(\alpha_{\mathbf{a}\tau}+\varepsilon_I)}$ on the right-hand side of identity (91) for $I\subseteq[N-1]$, $\mathbf{a}\in\mathbb{N}^{m-1}$, $\tau\in\mathbb{N}^m$. It vanishes unless $I=J_\uparrow$ for some $J\subseteq[m-1]$ since $N_{\beta/(\alpha+\varepsilon_I)}=0$ when $(\alpha+\varepsilon_I)_i>\beta_i$ for some index i. For $I=J_\uparrow$, we can use Lemma 5.3.7 to replace this summand with $t^{|\mathbf{a}|}q^{d((0,\mathbf{a}),\tau)+h_I(\alpha_{\mathbf{a}\tau})-h'_J(\mathbf{a},\tau)}N_{((0,\mathbf{a})+(1^m)+\tau)/((\mathbf{a},0)+\varepsilon_J)}$.

It now suffices to prove that, for $\alpha=\alpha_{\mathbf{a}\tau}$,

$$h_I(\alpha)=h'_J(\mathbf{a},\tau)+h_J(\mathbf{a}). \tag{97}$$

We recall that $N=m_\uparrow$ and note that $[N]\setminus I=([N]\setminus[m]_\uparrow)\sqcup([m]_\uparrow\setminus I)=([N]\setminus[m]_\uparrow)\sqcup([m]\setminus J)_\uparrow$. Hence, $h_I(\alpha)=\big|\{(x<y):x\in I,y\in[N]\setminus I,\alpha_y=\alpha_x+1\}\big|=|S_1|+|S_2|$ for

$$\begin{aligned}S_1&=\{(x<y):x\in J_\uparrow,\ y\in[N]\setminus[m]_\uparrow,\ \alpha_y=\alpha_x+1\},\\ S_2&=\{(x<y):x\in J_\uparrow,\ y\in([m]\setminus J)_\uparrow,\ \alpha_y=\alpha_x+1\}.\end{aligned}$$

Since $\alpha_{m_\uparrow}=0$ implies $(x<m_\uparrow)\notin S_2$ for all $x<m_\uparrow$, we use that $a_u=\alpha_{u_\uparrow}$ for every $u\in[m-1]$ to see that

$$h_J(\mathbf{a})=\big|S_2\big|=\big|\{(j<r):j\in J,\ r\in[m-1]\setminus J,\ a_r=a_j+1\}\big|. \tag{98}$$

Furthermore, $\{(j<r):j\in J,r\in[m],a_{r-1}+1\le a_j+1\le a_{r-1}+\tau_r\}$ and S_1 are equinumerous, as we can see by letting a pair $(j<r)$ in the first set correspond to the pair $(j_\uparrow<y)$ in S_1, where y is the unique row index in the range $(r-1)_\uparrow<y<r_\uparrow$ such that $\alpha_y=\alpha_{j_\uparrow}+1=a_j+1$, as illustrated in Figure 3. □

6. Stable unstraightened extended delta theorem

6.1. Overview

By Theorems 4.4.1 and 5.1.1, the **extended delta conjecture is equivalent to**

$$\begin{aligned}&\mathbf{H}_q^m\left(\frac{\prod_{i+1<j\le m}(1-q\,t\,x_i/x_j)}{\prod_{i<j\le m}(1-t\,x_i/x_j)}x_1\cdots x_m h_{N-m}(x_1,\dots,x_m)\overline{e_l(x_2,\dots,x_m)}\right)_{\mathrm{pol}}\\&\qquad=\sum_{\substack{J\subseteq[m-1]\\|J|=l}}\ \sum_{\substack{(0,\mathbf{a}),\tau\in\mathbb{N}^m\\|\tau|=N-m}}t^{|\mathbf{a}|}q^{d((0,\mathbf{a}),\tau)+h_J(\mathbf{a})}\left(\omega N_{\beta/\alpha}\right)(x_1,\dots,x_m;q),\end{aligned} \tag{99}$$

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where $\beta = (0, \mathbf{a}) + (1^m) + \tau$, $\alpha = (\mathbf{a}, 0) + \varepsilon_J$ and $\left(\omega N_{\beta/\alpha}\right)(x_1, \dots, x_m; q)$ is $\omega N_{\beta/\alpha}(X;q)$ evaluated in m variables.

Although this is an identity in only m variables, it does amount to the extended delta conjecture by Remarks 4.4.2 and 5.2.2: Both $\omega(h_l[B]e_{m-l-1}[B-1]e_{N-l})$ and $\omega N_{\beta/\alpha}(X;q)$ for the α, β arising in equation (99) are linear combinations of Schur functions s_λ with $\ell(\lambda) \leq m$.

By Proposition 6.2.2 (below, proven in [2]), the functions $\omega N_{\beta/\alpha}$ on the right-hand side of equation (99) are the polynomial parts of the 'LLT series' introduced in [12], making each side of equation (99) the polynomial part of an infinite series of GL_m characters. We then prove equation (99) as a consequence of a stronger identity between these infinite series.

Hereafter, we use x to abbreviate the alphabet $x_1, \dots, x_m$.

6.2. LLT series

We will work with the (twisted) nonsymmetric Hall–Littlewood polynomials as in [2]. For a GL_m weight $\lambda \in \mathbb{Z}^m$ and $\sigma \in S_m$, the twisted nonsymmetric Hall–Littlewood polynomial $E^\sigma_\lambda(x;q)$ is an element of $\mathbb{Z}[q^{\pm 1}][x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ defined using an action of the Hecke algebra on this ring; we refer the reader to [2, §4.3] for the precise definition, citing properties as needed. We also have their variants

$$F^\sigma_\lambda(x;q) = \overline{E^{\sigma w_0}_{-\lambda}(x;q)}\,, \tag{100}$$

recalling that $\overline{f(x_1, \dots, x_m; q)} = f(x_1^{-1}, \dots, x_m^{-1}; q^{-1})$.

For any weights $\alpha, \beta \in \mathbb{Z}^m$ and a permutation $\sigma \in S_m$, the *LLT series* $\mathcal{L}^\sigma_{\beta/\alpha}(x;q) = \mathcal{L}^\sigma_{\beta/\alpha}(x_1, \dots, x_m; q)$ is defined in [2, §4.4] by

$$\langle \chi_\lambda \rangle \mathcal{L}^{\sigma^{-1}}_{\beta/\alpha}(x;q^{-1}) = \langle E^\sigma_\beta \rangle\, \chi_\lambda \cdot E^\sigma_\alpha\,. \tag{101}$$

Alternatively, [2, Proposition 4.4.2] gives the following expression in terms of the Hall–Littlewood symmetrization operator in equation (38):

$$\mathcal{L}^\sigma_{\beta/\alpha}(x;q) = \mathbf{H}^m_q\big(w_0(F^{\sigma^{-1}}_\beta(x;q)\overline{E^{\sigma^{-1}}_\alpha(x;q)})\big)\,, \tag{102}$$

where w_0 denotes the permutation of maximum length here and after. We will only need the LLT series for $\sigma = w_0$ and $\sigma = id$, although most of what follows can be generalized to any σ.

In addition to the above formulas, we have the following combinatorial expressions for the polynomial truncations of LLT series as tableau generating functions with q weights that count triples. As usual, a *semistandard tableau* on a tuple of skew row shapes $\nu = \beta/\alpha$ is a map $T\colon \nu \to [m]$ which is weakly increasing on rows. Let $\mathrm{SSYT}(\nu)$ denote the set of these, and define $x^{\mathrm{wt}(T)} = \prod_{b \in \nu} x_{T(b)}$.

Proposition 6.2.1 [2, Remark 4.5.5 and Corollary 4.5.7]. *If $\alpha_i \leq \beta_i$ for all i, then*

$$\mathcal{L}^{w_0}_{\beta/\alpha}(x;q)_{\mathrm{pol}} = \sum_{T \in \mathrm{SSYT}(\beta/\alpha)} q^{h'_{w_0}(T)} x^{\mathrm{wt}(T)}\,, \tag{103}$$

where $h'_{w_0}(T)$ is the number of w_0-triples (u, v, w) of β/α such that $T(u) \leq T(v) \leq T(w)$.

Proposition 6.2.2 [2, Proposition 4.5.2]. *For any $\alpha, \beta \in \mathbb{Z}^m$,*

$$\mathcal{L}^{w_0}_{\beta/\alpha}(x;q)_{\mathrm{pol}} = \left(\omega N_{\beta/\alpha}\right)(x;q)\,. \tag{104}$$

6.3. Extended delta theorem

We now give several lemmas on nonsymmetric Hall–Littlewood polynomials, then conclude by using the Cauchy formula for these polynomials to prove Theorem 6.3.6, below, yielding the stronger series identity that implies equation (99).

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Lemma 6.3.1. *For* $\mathbf{a} \in \mathbb{N}^{m-1}$ *and* $w_0 \in S_m$ *and* $\tilde{w}_0 \in S_{m-1}$ *the permutations of maximum length, we have*

$$E^{w_0}_{(\mathbf{a},0)}(x_1, \dots, x_m; q) = E^{\tilde{w}_0}_{\mathbf{a}}(x_1, \dots, x_{m-1}; q) \tag{105}$$

$$F^{w_0}_{(0,\mathbf{a})}(x_1, \dots, x_m; q) = F^{\tilde{w}_0}_{\mathbf{a}}(x_2, \dots, x_m; q)\,. \tag{106}$$

Proof. By [2, Lemma 4.3.4], we have $E^{w_0}_{(\mathbf{a},0)}(x_1, \dots, x_m; q) = E^{\tilde{w}_0}_{\mathbf{a}}(x_1, \dots, x_{m-1}; q)E^{id}_{(0)}(x_m; q)$ and $E^{id}_{(0,-\mathbf{a})}(x_1, \dots, x_m; q) = E^{id}_{(0)}(x_1; q)E^{id}_{-\mathbf{a}}(x_2, \dots, x_m; q)$. The claim then follows from the definition $F^{\sigma}_{\mathbf{a}} = \overline{E^{w_0\sigma}_{-\mathbf{a}}}$ and noting that $E^{id}_{(0)}(x_m; q) = 1 = F^{id}_{(0)}(x_1; q)$. □

Inverting all variables and specializing $\sigma = w_0$ in [2, Lemma 4.5.1] yields the following lemma.

Lemma 6.3.2. *For* $l \le m$, $\mathbf{a} \in \mathbb{Z}^m$, *we have*

$$\overline{e_l(x)}\;\overline{E^{w_0}_{\mathbf{a}}(x; q)} = \sum_{I \subseteq [m] : |I| = l} q^{h_I(\mathbf{a})}\,\overline{E^{w_0}_{\mathbf{a}+\varepsilon_I}(x; q)}\,, \tag{107}$$

where $h_I(\mathbf{a}) = \left|\{(i < j) \mid a_j = a_i + 1, i \in I, j \notin I\}\right|$, *as defined in equation (82).*

Lemma 6.3.3. *For every* $\lambda \in \mathbb{Z}^m$ *and* $\sigma \in S_m$, *we have*

$$F^{\sigma}_{\lambda}(x; q) = w_0 E^{w_0\sigma}_{w_0\lambda}(x; q^{-1}). \tag{108}$$

Proof. The desired identity follows from

$$w_0 E^{\sigma}_{\lambda}(x_1^{-1}, \dots, x_m^{-1}; q) = E^{w_0\sigma w_0}_{-w_0\lambda}(x; q) \tag{109}$$

by applying w_0 to both sides; substituting $\sigma \mapsto \sigma w_0$, $\lambda \mapsto -\lambda$ and $q \mapsto q^{-1}$ and using the definition of F^{σ}_{λ}.

To prove equation (109), we use the characterization of $E^{\sigma}_{\lambda}(x; q)$ by the recurrence [2, (77)] and initial condition $E^{\sigma}_{\lambda} = x^{\lambda}$ for λ dominant. The change of variables $x^{\mu} \mapsto x^{-w_0(\mu)}$ replaces the Hecke algebra operator $T_i = T_{s_i}$ in the recurrence with $T_{w_0 s_i w_0}$, giving a modified recurrence satisfied by the left-hand side of (109). It is straightforward to verify that the right-hand side of equation (109) satisfies the same modified recurrence. Since both sides reduce to $x^{-w_0(\lambda)}$ for λ dominant, equation (109) holds. □

Lemma 6.3.4. *Given* $\alpha, \beta \in \mathbb{Z}^m$ *and a symmetric Laurent polynomial* $f(x_1, \dots, x_m)$, *we have, for any* $\sigma \in S_m$,

$$\langle E^{w_0\sigma w_0}_{w_0\beta}(x; q^{-1})\rangle\, f(x) \cdot E^{w_0\sigma w_0}_{w_0\alpha}(x; q^{-1}) = \langle F^{\sigma}_{-\alpha}(x; q)\rangle\, f(x) \cdot F^{\sigma}_{-\beta}(x; q). \tag{110}$$

Proof. In fact, we will show that

$$\langle E^{w_0\sigma w_0}_{w_0\beta}(x; q^{-1})\rangle\, f(x) \cdot E^{w_0\sigma w_0}_{w_0\alpha}(x; q^{-1}) = \langle F^{\sigma}_{-\alpha}(x; q)\rangle\, w_0(f(x)) \cdot F^{\sigma}_{-\beta}(x; q), \tag{111}$$

even if we do not assume that $f(x)$ is symmetric. By Lemma 6.3.3, the right-hand side of equation (111) is equal to

$$\langle E^{w_0\sigma}_{-w_0\alpha}(x; q^{-1})\rangle\, f(x) \cdot E^{w_0\sigma}_{-w_0\beta}(x; q^{-1}). \tag{112}$$

By [2, Proposition 4.3.2], the functions $E^{\sigma}_{\lambda}(x; q)$ and $E^{\sigma w_0}_{-\lambda}(x; q)$ are dual bases with respect to to the inner product $\langle -, - \rangle_q$ defined there. Moreover, it is immediate from the construction of the inner product that multiplication by any $f(x)$ is self-adjoint. This gives

$$\langle f(x)E^{w_0\sigma w_0}_{w_0\alpha}(x;q^{-1}),\ E^{w_0\sigma}_{-w_0\beta}(x;q^{-1})\rangle_{q^{-1}} = \langle E^{w_0\sigma w_0}_{w_0\alpha}(x;q^{-1}),\ f(x)E^{w_0\sigma}_{-w_0\beta}(x;q^{-1})\rangle_{q^{-1}}, \tag{113}$$

in which the left-hand side is equal to the left-hand side of equation (111), and the right-hand side is equal to equation (112). □

Lemma 6.3.5. *For w_0 the maximum length permutation in S_m and $\eta \in \mathbb{N}^m$, we have*

$$h_l(x)F^{w_0}_{\eta}(x;q) = \sum_{\substack{\tau\in\mathbb{N}^m\\ |\tau|=l}} q^{d(\eta,\tau)} F^{w_0}_{\eta+\tau}(x;q)\,, \tag{114}$$

recalling from equation (81) that $d(\eta,\tau) = \sum_{j<r} \left|[\eta_j,\eta_j+\tau_j] \cap [\eta_r,\eta_r+\tau_r-1]\right|$.

Proof. Set $\alpha = -\eta-\tau$ and $\beta = -\eta$. By equation (101) and Lemma 6.3.4 (with $\sigma = w_0$), we have

$$\langle h_l(x)\rangle\, \mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(x;q)_{\mathrm{pol}} = \langle E^{w_0}_{w_0\beta}(x;q^{-1})\rangle h_l(x)E^{w_0}_{w_0\alpha}(x;q^{-1}) = \langle F^{w_0}_{-\alpha}(x;q)\rangle h_l(x)F^{w_0}_{-\beta}(x;q). \tag{115}$$

By specializing all but one variable in equation (103) to zero, Proposition 6.2.1 implies that the coefficient of h_l in $\mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(x;q)_{\mathrm{pol}}$ is $q^{h'_{w_0}(T)}$ for T the semistandard tableau of shape $w_0(\beta/\alpha)$ filled with a single letter, where $h'_{w_0}(T)$ is the number of w_0-triples of $w_0(\beta/\alpha) = w_0(-\eta/(-\eta-\tau))$. By equation (84), this number is $d(\eta,\tau)$. □

Theorem 6.3.6. *For $0 \le l < m \le N$ and $w_0 \in S_m$ the maximum length permutation, we have*

$$\begin{aligned}
&\frac{\prod_{i+1<j\le m}(1-qtx_i/x_j)}{\prod_{i<j\le m}(1-tx_i/x_j)} x_1\cdots x_m h_{N-m}(x_1,\dots,x_m)\overline{e_l(x_2,\dots,x_m)}\\
&= \sum_{\substack{(0,\mathbf{a}),\tau\in\mathbb{N}^m\\ I\subseteq[m-1]\\ |\tau|=N-m,|I|=l}} t^{|\mathbf{a}|}q^{d((0,\mathbf{a}),\tau)+h_I(\mathbf{a})}\ w_0\big(F^{w_0}_{(0,\mathbf{a})+\tau+(1^m)}(x_1,\dots,x_m;q)\overline{E^{w_0}_{(\mathbf{a},0)+\varepsilon_I}(x_1,\dots,x_m;q)}\big).
\end{aligned}$$

Proof. Our starting point is the Cauchy formula [2, Theorem 5.1.1] for the twisted nonsymmetric Hall–Littlewood polynomials associated to any $\tilde\sigma \in S_{m-1}$:

$$\frac{\prod_{i<j<m}(1-q\,t\,x_i\,y_j)}{\prod_{i\le j<m}(1-t\,x_i\,y_j)} = \sum_{\mathbf{a}\in\mathbb{N}^{m-1}} t^{|\mathbf{a}|}\, E^{\tilde\sigma}_{\mathbf{a}}(x_1,\dots,x_{m-1};q^{-1})\, F^{\tilde\sigma}_{\mathbf{a}}(y_1,\dots,y_{m-1};q)\,. \tag{116}$$

Take $\tilde\sigma = \tilde w_0$ the maximum length permutation in S_{m-1}, replace x_i by x_i^{-1} and then let $y_j = x_{j+1}$ to get

$$\frac{\prod_{i+1<j\le m}(1-q\,t\,x_j/x_i)}{\prod_{i<j\le m}(1-t\,x_j/x_i)} = \sum_{\mathbf{a}\in\mathbb{N}^{m-1}} t^{|\mathbf{a}|}F^{\tilde w_0}_{\mathbf{a}}(x_2,\dots,x_m;q)\overline{E^{\tilde w_0}_{\mathbf{a}}(x_1,\dots,x_{m-1};q)}\,. \tag{117}$$

By equation (106) and the definition of F^{σ},

$$(x_1\cdots x_m)F^{\tilde w_0}_{\mathbf{a}}(x_2,\dots,x_m;q) = (x_1\cdots x_m)F^{w_0}_{(0,\mathbf{a})}(x_1,\dots,x_m;q) = F^{w_0}_{(0,\mathbf{a})+(1^m)}(x_1,\dots,x_m;q)$$

for $w_0 \in S_m$. Hence,

$$\begin{aligned}
&\frac{\prod_{i+1<j\le m}(1-q\,t\,x_j/x_i)}{\prod_{i<j\le m}(1-t\,x_j/x_i)}(x_1\cdots x_m)\\
&\qquad= \sum_{\mathbf{a}\in\mathbb{N}^{m-1}} t^{|\mathbf{a}|}F^{w_0}_{(0,\mathbf{a})+(1^m)}(x_1,\dots,x_m;q)\overline{E^{\tilde w_0}_{\mathbf{a}}(x_1,\dots,x_{m-1};q)}\,.
\end{aligned}$$

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Multiplying by $h_{N-m}(x_1,\dots,x_m)$ with the help of Lemma 6.3.5 yields

$$\frac{\prod_{i+1<j\le m}(1-q\,t\,x_j/x_i)}{\prod_{i<j\le m}(1-t\,x_j/x_i)}(x_1\cdots x_m)h_{N-m}(x_1,\dots,x_m)$$
$$=\sum_{\substack{(0,\mathbf{a}),\tau\in\mathbb{N}^m\\|\tau|=N-m}} t^{|\mathbf{a}|}q^{d((0,\mathbf{a}),\tau)}F^{w_0}_{\eta+\tau}(x_1,\dots,x_m;q)\overline{E^{\tilde{w}_0}_{\mathbf{a}}(x_1,\dots,x_{m-1};q)},$$

where $\eta=(1^m)+(0,\mathbf{a})$ and we have used that $d(\eta,\tau)=d((0,\mathbf{a}),\tau)$ by equation (81). Now, multiply by $\overline{e_l(x_1,\dots,x_{m-1})}$ and apply equation (107) to get

$$\frac{\prod_{i+1<j\le m}(1-q\,t\,x_j/x_i)}{\prod_{i<j\le m}(1-t\,x_j/x_i)}(x_1\cdots x_m)\overline{e_l(x_1,\dots,x_{m-1})}h_{N-m}(x_1,\dots,x_m)$$
$$=\sum_{\substack{(0,\mathbf{a}),\tau\in\mathbb{N}^m\\|\tau|=N-m}}\sum_{|I|=l} t^{|\mathbf{a}|}q^{d((0,\mathbf{a}),\tau)+h_I(\mathbf{a})}F^{w_0}_{\eta+\tau}(x_1,\dots,x_m;q)\overline{E^{\tilde{w}_0}_{\mathbf{a}+\varepsilon_I}(x_1,\dots,x_{m-1};q)},\qquad(118)$$

where $I\subseteq[m-1]$. The result then follows by using equation (105) on the right-hand side and applying $w_0\in S_m$ to both sides, noting that $w_0(\overline{e_l(x_2,\dots,x_m)})=\overline{e_l(x_1,\dots,x_{m-1})}$. □

Proof of the extended delta conjecture. It suffices to prove the reformulation in equation (99); this follows by applying $\mathbf{H}^m_q$ and equation (102) to the identity of Theorem 6.3.6, taking the polynomial part and using Proposition 6.2.2. □

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Conflict of Interest. The authors have no conflict of interest to declare.

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