SOME RELATIONS BETWEEN VARIOUS TYPES OF NORMALITY OF NUMBERS

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1. Introduction. In this paper certain relations will be proved between ϵ-normality of integers, (k, ϵ)-normality of integers, and normality of real numbers. Also a new type of normality of numbers will be introduced, namely, quasi-normality, as defined below.

DEFINITION 1.1. A simply normal number *is a real number, expressed in some scale B, in which each digit of the scale B occurs with the asymptotic frequency* $1/B$.

DEFINITION 1.2. A normal number *is a real number, expressed in some scale B, in which every sequence of k digits of the scale B,* $k = 1, 2, 3, \ldots$, *occurs with the asymptotic frequency* $1/B^k$.

DEFINITION 1.3. *An integer*

$$m = a_{\mu-1}a_{\mu-2}\ldots a_1a_0 \qquad (a_{\mu-1} \neq 0),$$

where the a_i *are digits of some scale B, is* (k, ϵ)-normal *in the scale B for a given k and a given* $\epsilon > 0$, *if for every k-digit sequence* $b_1b_2\ldots b_k$,

$$\left|\frac{N(m, b_1b_2\ldots b_k)}{\mu - k + 1} - \frac{1}{B^k}\right| < \epsilon,$$

where $N(m, b_1b_2\ldots b_k)$ *is the number of occurrences of* $b_1b_2\ldots b_k$ *in m. For* $k = 1$ *we shall say simply that m is* ϵ-normal *in the scale B.*

DEFINITION 1.4. *A real number y, expressed in some scale B, is* quasi-normal *in the scale B if every number derived from y by selecting those digits whose positions in y form an arithmetic progression is a simply normal number.*

Definition 1.2 was originally given by Borel **(2)** as the characteristic property of normal numbers. Borel actually defined a number x to be normal in the scale B if $x, Bx, B^2x, \ldots,$ are all simply normal in all of the scales B, B^2, $B^3, \ldots$. That Borel's definition is equivalent to Definition 1.2 was first proved by Niven and Zuckerman **(7)**, and later a very simple proof was given by Cassels **(4)**. Definition 1.3 is essentially that of Besicovitch **(1)**, differing only in trivial details which do not affect the validity of Besicovitch's results. Definition 1.4 is that of the writer.

In §2 we shall show, first, how the problem of the (k, ϵ)-normality of almost all of an increasing sequence of integers can be reduced to the case $k = 1$ (Theorem 2.1). Also the following problem is treated: Consider an increasing

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sequence $\{a_n\}$ of positive integers expressed in some fixed scale B, such that, for any given k and any given $\epsilon > 0$, almost all of the integers a_n are (k, ϵ)-normal in the scale B. Under what sufficient condition can we conclude that the number $.a_1a_2a_3\ldots$, formed by writing the integers a_n in order and in juxtaposition after the decimal point, is a normal number? (See Theorem 2.2.) Finally, in §3, we shall show how to construct quasi-normal numbers out of normal numbers (Theorem 3.1). For these quasi-normal numbers the asymptotic frequency of any k-digit sequence is actually determined (Theorem 3.2); and we obtain the result that quasi-normality does not imply normality (Theorem 3.3).

2. Some relations between ϵ-normality, (k, ϵ)-normality, and normality. Besicovitch **(1)** proved that, for any given integer k and any given $\epsilon > 0$, almost all integers are (k, ϵ)-normal, and almost all squares of integers are ϵ-normal. That the squares of almost all integers are also (k, ϵ)-normal is a particular case of a theorem proved by Davenport and Erdös **(5)**. We shall prove that in a quite general way the problem of (k, ϵ)-normality reduces to one of ϵ-normality.

LEMMA 2.1. *Given $\epsilon > 0$ and an integer $k \geqslant 2$. A sufficient condition that an integer*

$$m = a_{\mu-1}a_{\mu-2}\ldots a_1a_0 \qquad (a_{\mu-1} \neq 0), \tag{1}$$

be (k, ϵ)-normal in the scale B is that m be ϵ'-normal in the scale B^r, where $k/r < \epsilon/3$, $\epsilon' B^r < \epsilon/3$, and that m be sufficiently large that $r/\mu < \epsilon/3B^k$.

Proof. Let m be an integer which is e'-normal in the scale B^r, where r, ϵ', and μ satisfy the hypothesis.

The digits of the scale B^r, when represented in the scale B, constitute the complete set of r-digit sequences of the scale B (if we write, where necessary, initial zeros). Let $b_1b_2\ldots b_k$ be any given k-digit sequence of the scale B. Then $b_1b_2\ldots b_k$ occurs in the complete set of r-digit sequences of the scale B exactly $(r - k + 1)B^{r-k}$ times.

Let us represent m in the scale B^r,

$$m = A_{\nu-1}A_{\nu-2}\ldots A_1A_0 \qquad (A_{\nu-1} \neq 0), \tag{2}$$

where $\mu/r \leqslant \nu < \mu/r + 1$. Let us then replace each A_i by the corresponding sequence of r digits of the scale B. We obtain, thus, a representation of m in the form

$$m = 00\ldots 0a_{\mu-1}a_{\mu-2}\ldots a_1a_0, \tag{3}$$

where the number of initial zeros is less than r.

Since m is ϵ'-normal in the scale B^r, every digit of the scale B^r occurs in (2) more than $(B^{-r} - \epsilon')$ times. Hence $b_1b_2\ldots b_k$ occurs in (3) more than

$$(B^{-r} - \epsilon')\frac{\mu}{r}(r - k + 1)\,B^{r-k}$$

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times. In this estimate we disregard possible occurrences of $b_1b_2 \ldots b_k$ beginning in some A_i and ending in A_{i-1}. If we also take account of the fact that, in going from (3) to (1), less than r sequences of k digits can be lost, we see that

$$N(m, b_1b_2 \ldots b_k) > \left(\frac{1}{B^r} - \epsilon'\right)\frac{\mu(r-k+1)}{r}B^{r-k} - r,$$
$$> \left(\frac{1}{B^k} - \frac{\epsilon' B^r}{B^k} - \frac{k}{rB^k} - \frac{r}{\mu}\right)\mu$$
$$> \left(\frac{1}{B^k} - \frac{\epsilon}{B^k}\right)(\mu - k + 1).$$

This is true for every k-digit sequence. It follows that, for every k-digit sequence, $b_1b_2 \ldots b_k$,

$$N(m, b_1b_2 \ldots b_k) < (B^{-k} + \epsilon)(\mu - k + 1),$$

and hence m is (k, ϵ)-normal in the scale B.

THEOREM 2.1. *Let $\{a_n\}$ be an increasing sequence of positive integers having the property that, for any given $\epsilon > 0$ and any given scale B, almost all a_n are ϵ-normal. Then the sequence $\{a_n\}$ has the property that, for any given $\epsilon > 0$, any given $k \geqslant 2$, and any given scale B, almost all a_n are (k, ϵ)-normal.*

Proof. For a given $\epsilon > 0$, a given k, and a given scale B, choose an r and an $\epsilon' > 0$ which satisfy the hypothesis of Lemma 2.1. By the hypothesis of the theorem, almost all a_n are ϵ'-normal in the scale B^r. Choose also a μ which satisfies the hypothesis of the lemma. Almost all a_n have at least μ digits. It follows that almost all a_n satisfy the entire hypothesis of the lemma, and hence are (k, ϵ)-normal in the scale B.

THEOREM 2.2. *Let $\{a_n\}$ be an increasing sequence of positive integers having the property that, for any given k and any given $\epsilon > 0$, almost all a_n are (k, ϵ)-normal in the scale B. Let ν_i denote the number of digits in a_i $(i = 1, 2, 3, \ldots)$, and let*

$$S_n = \sum_{i=1}^{n} \nu_1.$$

Then a sufficient condition that the number $x = .a_1a_2a_3 \ldots$ be normal in the scale B is that

$$n\nu_n = O(S_n). \tag{4}$$

Proof. Let $b_1b_2 \ldots b_3$ be any given sequence of k digits of the scale B. Let m be an integer and let n be such that $S_n \leqslant m < S_{n+1}$. Let $N_m(x, b_1b_2 \ldots b_k)$ denote the number of occurrences of $b_1b_2 \ldots b_k$ in the first m digits of x. Then, for a given $\epsilon > 0$,

$$N_m(x, b_1b_2 \ldots b_k) \geqslant (B^{-k} - \epsilon)\sum_{\lambda}{}'(\nu_\lambda - k + 1),$$

where $\sum'$ is taken over the values of $\lambda \leqslant n$ for which a_λ is (k, ϵ)-normal.

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Let the number of integers among $a_1, a_2, \ldots, a_n$ which are not (k, ϵ)-normal be denoted by ω_n. By hypothesis $\omega_n = o(n)$ as $n \to \infty$. Also $\nu_\lambda \leqslant \nu_n$ for every $\lambda \leqslant n$. Hence

$$N_m(x, b_1b_2 \ldots b_k) \geqslant (B^{-k} - \epsilon)\{\sum_\lambda{}' \nu_\lambda - (n - \omega_n)(k-1)\}$$
$$> (B^{-k} - \epsilon)(S_n - \omega_n\nu_n - nk),$$

and

$$\frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} > (B^{-k} - \epsilon)\frac{S_n - \omega_n\nu_n - nk}{S_{n+1}}$$
$$= (B^{-k} - \epsilon)\left\{1 - \frac{1}{n+1}\cdot\frac{(n+1)\nu_{n+1}}{S_{n+1}}\right\}\left(1 - \frac{\omega_n}{n}\cdot\frac{n\nu_n}{S_n} - \frac{k}{\nu_n}\cdot\frac{n\nu_n}{S_n}\right),$$

which, by (4), approaches $B^{-k} - \epsilon$ as $n \to \infty$. Hence

$$\liminf_{m\to\infty} \frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} \geqslant B^{-k} - \epsilon,$$

and, since ϵ is arbitrary,

$$\liminf_{m\to\infty} \frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} \geqslant B^{-k}.$$

Since this is true for every k-digit sequence, we have

$$\lim_{m\to\infty} \frac{N_m(x, b_1b_2 \ldots b_k)}{m - k + 1} = B^{-k},$$

and x is normal in the scale B.

Remark 1. It is easily seen that a sufficient condition for (4) is that $\nu_n = O(\log n)$. For, noting that

$$\nu_n = 1 + [\log_B a_n] > \frac{\log n}{\log B},$$

we see that

$$\frac{n\nu_n}{S_n} < \frac{(C \log B)\, n \log n}{\log 2 + \log 3 + \ldots + \log n} = \frac{(C \log B)\, n \log n}{n \log n + o(n \log n)},$$

which approaches $C \log B$ as $n \to \infty$.

The condition $\nu_n = O(\log n)$ is satisfied, for example, if $a_n = [f(n)]$, where $f(x)$ is a polynomial with real coefficients, $f(n) > 0$ for positive values of n.

It can easily be shown, also, that (4) holds if $\mu_1 n^\alpha < \nu_n < \mu_2 n^\alpha$ $(n = 1, 2, 3, \ldots)$, where μ_1, μ_2, and α are positive constants.

Remark 2. If, however, the a_n increase too rapidly the conclusion of Theorem 2.2 does not hold. To show this we shall employ a sequence of integers, $\{I_m\}$, expressed in some scale B, which, for each m are constructed as follows: Write first m consecutive equal digits $B - 1$, and follow these at each successive position by the smallest digit of the scale B which does not cause the repetition of a previously occurring sequence of m digits, continuing thus until no longer possible. It is not difficult to see that the integer thus constructed contains every m-digit sequence of the scale B exactly once and hence consists of exactly $B^m + m - 1$ digits. (For the case $B = 2$, see Lessard, Problem 4385, American Mathematical Monthly, 58 (1951), 573-574.) It can also easily be ascertained that, for any given k and any given $\epsilon > 0$, the integers I_m for almost all m, in fact, for all except finitely many m, are (k, ϵ)-normal in the scale B. (Sequences of digits which contain every m-digit sequence of the scale of representation exactly once have been investigated by Goode **(6)** and Rees **(8)**, who give methods of construction different from that above, and by de Bruijn **(3)**, who proves that, for the scale 2, the number of such sequences is $2^{f(m)}$, $f(m) = 2^{m-1}$, if cyclic permutations are accounted distinct.)

Let $J_m = I_m$ if m is not a perfect square, and let J_m be $B^m + m - 1$ consecutive equal digits $B - 1$ if m is a perfect square. Then the sequence $\{J_m\}$ has the property that, for any $k \geqslant 1$ and any $\epsilon > 0$, almost all J_m are (k, ϵ)-normal. But the number $.J_1J_2J_3\ldots$ is not even simply normal, for a quite simple estimate will show that, for the particular digit $(B - 1)$,

$$\limsup_{n\to\infty} \frac{N_n(x, B-1)}{n} \geqslant \frac{B-1}{B},$$

which is greater than $1/B$ if $B > 2$. For $B = 2$, it can be shown, by a closer estimate, that

$$\limsup_{n\to\infty} \frac{N_n(x, 1)}{n} \geqslant \frac{3}{4}.$$

3. Quasi-Normal Numbers. We shall show first that every number which is normal in the scale B is also quasi-normal in the scale B (see Definition 1.4). This follows from the following lemma.

LEMMA 3.1. *If x is normal in the scale B, and k, j, and i are any positive integers, and $b_1b_2\ldots b_k$ is any sequence of k digits of the scale B; then $b_1b_2\ldots b_k$ occurs in x in a position*[1] *congruent to i modulo j with the asymptotic frequency $1/jB^k$.*

Proof. Let r be the smallest integer for which $rj \geqslant k$. Then, by the well known property of normal numbers, every sequence of rj digits occurs in x in a position congruent to 1 modulo rj with the asymptotic frequency $1/rjB^{rj}$. Among the sequences of rj digits each, B^{rj-k} begin with $b_1b_2\ldots b_k$. Hence the asymptotic frequency of $b_1b_2\ldots b_k$ in x in a position congruent to 1 modulo rj is $1/rjB^k$. Applying the same principle to any of the numbers $B^{\rho-1}x$ ($\rho = 1, 2, 3, \ldots, rj$), we find that, for a fixed ρ $(1 \leqslant \rho \leqslant rj)$, the asymptotic frequency of $b_1b_2\ldots b_k$ in x in a position congruent to ρ modulo rj is also $1/rjB^k$. Since

[1]We shall say that a k-digit sequence occurs in a position congruent to i modulo j if the index of the first digit of the sequence is congruent to i modulo j.

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there are r values of ρ that are congruent to i modulo j, it follows that the asymptotic frequency of $b_1b_2 \ldots b_k$ in x in a position congruent to i modulo j is $1/jB^k$.

The statement that every normal number is also quasi-normal is merely the particular case of Lemma 3.1 for $k = 1$.

In the proof of the next theorem we shall make use of two results obtained by Wall **(9)**; first, the equivalence of the normality of a number x in the scale B and the uniform distribution modulo 1 of the sequence $\{B^n x\}$; and, second, the fact that if x is normal in the scale B, then x/s is normal in the scale B for every positive integer s.

It will be convenient to introduce the following notation: If X is any real number and q is a positive integer, then we mean by res X(mod q) the number σ, where $0 \leqslant \sigma < q$ and $(X - \sigma)/q$ is an integer.

THEOREM 3.1. *Let x be a number which is normal in the scale B. Let s be any integer greater than* 1. *Let* $r_j =$ res $[B^j x]$ (mod s). *Let n_j denote the number of digits preceding the jth occurrence of any given k-digit sequence $b_1b_2 \ldots b_k$ in x. Then*

(i) *the number $y = .r_1r_2r_3 \ldots$ is quasi-normal in the scale s;*

(ii) *the number $.r_{n_1}r_{n_2}r_{n_3} \ldots$ is simply normal in the scale s.*

Proof. By Wall's result **(9)**, x/s is normal in the scale B, and, consequently, $\{B^n x/s\}$ is uniformly distributed modulo 1.

The number res $B^n x/s$ (mod 1), which is the fractional part of $B^n x/s$, for each value of n falls into one of the intervals $(0, 1/s)$, $(1/s, 2/s)$, ..., $(1\text{-}1/s, 1)$, namely, into the interval $(\sigma/s, (\sigma + 1)/s)$, where $\sigma =$ res $[B^n x]$ (mod s). Since $\{B^n x/s\}$ is uniformly distributed modulo 1, the number of numbers, res $B^n x/s$ (mod 1), in each of the above intervals is asymptotically equal to n/s, and, consequently, the integers, res $[B^n x]$ (mod s) are asymptotically equally distributed among the integers $0, 1, 2, \ldots, s - 1$. Hence $y = .r_1r_2r_3 \ldots$ is simply normal in the scale s.

Further, $B^{u-t}x$ is normal in the scale B^t, where u and t are any positive integers. Let us take $u \leqslant t$. Then if $x = .a_1a_2a_3 \ldots$ in the scale B, $B^{u-t}x = .00 \ldots 0a_1a_2a_3 \ldots$ in the scale B, where the number of initial zeros is $t - u$. In the scale B^t, $B^{u-t}x = .A_1A_2A_3 \ldots$, where A_1, represented in the scale B is $00 \ldots 0a_1a_2 \ldots a_u$, and A_j, $j > 1$, is $a_{u+(j-2)t+1} \ldots a_{u+(j-1)t}$.

If we write $R_j =$ res $[B^{jt}(B^{u-t}x)]$ (mod s), then, by the preceding argument, $.R_1R_2R_3 \ldots$ is simply normal in the scale s. But $R_j = r_{u+(j-1)t}$. Hence the number $.r_u r_{u+t} r_{u+2t} \ldots$ is simply normal in the scale s and y is quasi-normal in the scale s.

For the proof of (ii), consider those values of n for which the numbers res $B^n x$ (mod 1) lie in the interval of length $1/B^k$ whose left endpoint is $.b_1b_2 \ldots b_k$. Note that these are precisely the values $n_1, n_2 \ldots$, defined in the statement of the theorem. For each of these values of n, the number res $B^n x/s$

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(mod 1) lies in one of s non-overlapping intervals of length $1/sB^k$. Of these intervals, one lies in each of the intervals $(\sigma/s, (\sigma+1)/s)$; and, for each n, the value of σ is determined by

$$\sigma = \text{res}[B^n x](\text{mod } s).$$

Since there are asymptotically n/sB^k numbers res $B^n x/s$ (mod 1) in each of these intervals of length $1/sB^k$, it follows that the values of res $[B^n x]$ (mod s) for the values $n_1, n_2, \ldots$, are asymptotically equally distributed among the integers $0, 1, 2, \ldots, s-1$, and, therefore, the number $.r_{n_1}r_{n_2}r_{n_3}\ldots$ is simply normal in the scale s.

Remark. Note that it follows from Theorem 3.1 that the asymptotic frequency of a given digit of the scale s in y in the positions n_j is $1/sB^k$.

The numbers y, as constructed in Theorem 3.1, not only are quasi-normal, but they possess the additional property (ii), and this additional property enables us to calculate, for this class of quasi-normal numbers, the asymptotic frequency of any k-digit sequence.

THEOREM 3.2. *Let x be normal in the scale B; let s be any integer greater than* 1; *and let $\gamma_1\gamma_2\ldots\gamma_k$ be any given k-digit sequence in the scale s. Then the asymptotic frequency of $\gamma_1\gamma_2\ldots\gamma_k$ in the number y, as defined in Theorem* 3.1, *is equal to*

$$\frac{1}{sB^{k-1}}\prod_{i=2}^{k}\left(\left[\frac{B}{s}\right]+\delta_i\right).$$

Here $\delta_i = 0$ or 1, according as $\mu_i \geqslant m$ or $\mu_i < m$, where $m =$ res B (mod s) *and $\mu_i =$* res $(\gamma_i - \gamma_{i-1}B)$(mod s).

Proof. Let $c_1c_2\ldots c_k$ be any k-digit sequence of the scale B, and let r denote the integer res $[B^n x]$ (mod s), where n is the number of digits in x preceding an occurrence of $c_1c_2\ldots c_k$.

We inquire, how many combinations consisting of a value of r and a sequence $c_1c_2\ldots c_k$ in x will result in the occurrence of the given sequence $\gamma_1\gamma_2\ldots\gamma_k$ in the corresponding position in y?

For each of the digits of the sequence $c_1c_2\ldots c_k$ and each of the values of r, the following relations must hold:

(6) $$rB + c_1 \equiv \gamma_1(\text{mod } s),$$

(7) $$\begin{cases} \gamma_1 B + c_2 \equiv \gamma_2(\text{mod } s), \\ \gamma_2 B + c_3 \equiv \gamma_3(\text{mod } s), \\ \quad - \quad - \quad - \\ \gamma_{k-1}B + c_k \equiv \gamma_k(\text{mod } s), \end{cases}$$

where $0 \leqslant r < s$, $0 \leqslant c_i < B$.

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The left member of (6) takes on the sB values $0, 1, 2, \ldots, sB - 1$, as r and c_1 range independently over the integers from 0 to $s - 1$ and from 0 to $B - 1$, respectively, of which values exactly B are congruent to γ_1 modulo s.

The relations (7) are independent of each other and of (6). For each of the relations (7), it is easily seen that there are either $[B/s] + 1$ or $[B/s]$ values of c_i which satisfy the relation, according as $m > \mu_i$ or $m \leqslant \mu_i$, where m and μ_i are defined as in the statement of the theorem. Hence the number of combinations of r and sequences $c_1 c_2 \ldots c_k$ which result in the occurrence of a given sequence $\gamma_1 \gamma_2 \ldots \gamma_k$ in y is

$$B \prod_{i=2}^{k} \left(\left[\frac{B}{s} \right] + \delta_i \right),$$

where δ_i is defined as in the statement of the theorem.

From the remark following Theorem 3.1, if follows, then, that the asymptotic frequency of $\gamma_1 \gamma_2 \ldots \gamma_k$ in y is equal to

$$\frac{1}{sB^{k-1}} \prod_{i=2}^{k} \left(\left[\frac{B}{s} \right] + \delta_i \right).$$

THEOREM 3.3. *The number y, defined as in Theorem 3.1, is normal in the scale s if and only if s divides B.*

Proof. If s divides B, then $m = 0$, and each factor of the product in Theorem 3.2 is B/s. Thus the asymptotic frequency of any k-digit sequence in y is $1/s^k$, and y is normal in the scale s.

If s does not divide B, then in order that the asymptotic frequency of a given k-digit sequence in y be $1/s^k$, we must have

$$\prod_{i=2}^{k} \left(\left[\frac{B}{s} \right] + \delta_i \right) = \left(\frac{B}{s} \right)^{k-1}.$$

But the left member of this equation is an integer, while the right member is not unless $k = 1$.

Thus we have answered in the negative the question whether a quasi-normal number is necessarily normal. Indeed, with regard to the class of quasi-normal numbers y derived in the manner of Theorem 3.1 from a normal number x, we can say that if s does not divide B, then for no $k > 1$ does any k-digit sequence of y have the proper asymptotic frequency. We note, too, that if $s > B$, then by (7) there are some sequences of digits of the scale s which do not occur in y at all, in particular, any sequence in which a zero is followed by a digit equal to or greater than B.

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