BULL. AUSTRAL. MATH. SOC. 47H20, 47H09
VOL. 72 (2005) [371–379]

A STRONG CONVERGENCE THEOREM FOR CONTRACTION SEMIGROUPS IN BANACH SPACES

HONG-KUN XU

We establish a Banach space version of a theorem of Suzuki [8]. More precisely we prove that if X is a uniformly convex Banach space with a weakly continuous duality map (for example, l^p for $1 < p < \infty$), if C is a closed convex subset of X, and if $\mathcal{F} = \{T(t) : t \geqslant 0\}$ is a contraction semigroup on C such that $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$, then under certain appropriate assumptions made on the sequences $\{\alpha_n\}$ and $\{t_n\}$ of the parameters, we show that the sequence $\{x_n\}$ implicitly defined by

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n$$

for all $n \geqslant 1$ converges strongly to a member of $\mathrm{Fix}(\mathcal{F})$.

1. INTRODUCTION

Let X be a Banach space and let C be a nonempty closed convex subset of X. A (one-parameter) contraction semigroup is a family

$$\mathcal{F} = \{T(t) : t \geqslant 0\}$$

of self-mappings of C such that

(i) $T(0)x = x$ for $x \in C$;
(ii) $T(t+s)x = T(t)T(s)x$ for $t, s \geqslant 0$ and $x \in C$;
(iii) $\lim_{t \to 0^+} T(t)x = x$ for $x \in C$;
(iv) for each $t > 0$, $T(t)$ is nonexpansive; that is,

$$\|T(t)x - T(t)y\| \leqslant \|x - y\|, \quad x, y \in C.$$

We shall denote by F the common fixed point set of $\mathcal{F}$; that is,

$$F := \mathrm{Fix}(\mathcal{F}) = \{x \in C : T(t)x = x, \ t > 0\} = \bigcap_{t>0} \mathrm{Fix}(T(t)).$$

Received 19th May, 2005
Supported in part by the National Research Foundation of South Africa.

(Here $\operatorname{Fix}(T) = \{x \in C : Tx = x\}$ is the set of fixed points of a mapping T.)

Let $T : C \to C$ be a nonexpansive mapping (that is, $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$). Assume that the fixed point set $\operatorname{Fix}(T)$ of T is nonempty. One classical method to study nonexpansive mappings is to use strict contractions to approximate nonexpansive mappings. More precisely, for a fixed point u in C, define for each $0 < t < 1$, a strict contraction T_t by

$$T_t x = tu + (1-t)Tx, \quad x \in C.$$

Let x_t be the fixed point of T_t; thus,

$$x_t = tu + (1-t)Tx_t. \tag{1.1}$$

Browder [1] (Reich [6], respectively) proves that as $t \to 0$, x_t converges strongly to a fixed point of T in a Hilbert space (uniformly smooth Banach space, respectively).

It is an interesting problem to extend Browder's and Reich's results to the contraction semigroup case. However, only partial answers have been obtained.

In [7], Shioji–Takahashi introduce in a Hilbert space the implicit iteration

$$x_n = \alpha_n u + (1-\alpha_n)\sigma_{t_n}(x_n), \quad n \geqslant 1, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ a sequence of positive real numbers divergent to ∞, and for each $t > 0$ and $x \in C$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\,ds.$$

Under certain restrictions to the sequence $\{\alpha_n\}$, Shioji–Takahashi [7] prove strong convergence of $\{x_n\}$ to a member of F. (See also [9].) Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval $(0,t)$. Suzuki [8] is the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geqslant 1, \tag{1.3}$$

for the contraction semigroup case.

Note that in the iteration process (1.3) the iterate x_n at step n is constructed directly from the semigroup (more precisely, from $T(t_n)$). So we can view Suzuki's iteration process (1.3) as an extension of the implicit iteration process (1.1) to contraction semigroups. Suzuki [8] proves strong convergence of his process (1.3) in a Hilbert space with appropriate assumptions imposed upon the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$. It is the purpose of this paper to extend Suzuki's result to the framework of Banach spaces. More precisely, we show that Suzuki's result holds in a uniformly convex Banach space with a weakly continuous duality map (for example, l^p for $1 < p < \infty$). We do not know however if the same result holds in a uniformly convex and uniformly smooth Banach space (for example, L^p for $1 < p < \infty$).

2. Preliminaries

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \Big\{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi\big(\|x\|\big),\ \|x^*\| = \varphi\big(\|x\|\big)\Big\}, \quad x \in X.$$

Following Browder [**2**], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (that is, if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak*ly to $J_\varphi(x)$). It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad t \geqslant 0.$$

Then

$$J_\varphi(x) = \partial\Phi\big(\|x\|\big), \quad x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis.

The first part of the following lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [**5**]; see also [**10**].

Lemma 2.1. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*

(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi\big(\|x + y\|\big) \leqslant \Phi\big(\|x\|\big) + \big\langle y, J_\varphi(x + y)\big\rangle.$$

(ii) *Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty} \Phi\big(\|x_n - y\|\big) = \limsup_{n\to\infty} \Phi\big(\|x_n - x\|\big) + \Phi\big(\|y - x\|\big)$$

for all $x, y \in X$. In particular, X satisfies Opial's property; that is, if $\{x_n\}$ is a sequence weakly convergent to x, then there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad y \in X,\ y \neq x.$$

We also need the demiclosedness principle for nonexpansive mappings in a uniformly convex Banach space (see [**4**]).

Lemma 2.2. *Let X be a uniformly convex Banach space, C a closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with a fixed point. Assume that a sequence $\{x_n\}$ in C is such that $x_n \to x$ weakly and $x_n - Tx_n \to y$ strongly. Then $x - Tx = y$.*

Notation: In the rest of this paper, we use '$\rightharpoonup$' to stand for weak convergence and '$\to$' for strong convergence.

3. Main Result

We begin with restating the main result of Suzuki [8].

Theorem 3.1. [8] *Let C be a closed convex subset of a Hilbert space H. Let*

$$\mathcal{F} = \{T(t) : t \geqslant 0\}$$

be a contraction semigroup on C such that

$$F \equiv \mathrm{Fix}(\mathcal{F}) \neq \emptyset.$$

Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $\alpha_n \in (0,1)$ and $t_n > 0$ for all n, and $\lim_n t_n = \lim_n(\alpha_n/t_n) = 0$. Fix a $u \in C$ and define a sequence $\{u_n\}$ by the implicit iteration process

$$u_n = \alpha_n u + (1-\alpha_n)T(t_n)u_n, \quad n \geqslant 1. \tag{3.1}$$

Then $\{u_n\}$ converges strongly to the element of F which is nearest to u in F.

We now analyse the possibility of establishing a Banach space version of Theorem 3.1. First we observe that the limit of $\{x_n\}$ is pre-identified as the nearest point projection of u onto F. This means that the nearest point projection P_F from H onto the fixed point set F of the semigroup plays crucial rule in Suzuki's proof which does not work for uniformly convex Banach spaces. The crux lies in the fact that nearest point projections in a Hilbert space are nonexpansive. While we can define nearest point projections in a uniformly convex Banach space, they are in general not nonexpansive anymore. To overcome this crux, we need to use sunny nonexpansive retractions in the framework of Banach spaces in place of nearest point projections in the framework of Hilbert spaces. Then we encounter another crux which is the question: in what Banach spaces X, there does exist a sunny nonexpansive retraction from C onto F, where C is a closed convex subset of X and F is the fixed point set $\mathrm{Fix}(\mathcal{F})$ of a contraction semigroup $\mathcal{F} = \{T(t) : t \geqslant 0\}$ defined on C. The next proposition ensures an affirmative answer to this question in a Banach space which is uniformly convex and has a weakly continuous duality map.

But before stating the proposition, let us recall that a map $Q : C \to F$ is said to be a sunny nonexpansive retraction from C onto F if Q is a retraction (that is, $Qx = x$ for $x \in F$), if Q is sunny (that is, $Q\big(x + t(x - Qx)\big) = Qx$ for all $x \in C$ and $t \geqslant 0$ whenever $x + t(x - Qx) \in C$), and if Q is nonexpansive. It is known that if X is a smooth Banach space, then a retraction $Q : C \to F$ is sunny nonexpansive if and only if there holds the inequality (see [4]):

$$\big\langle x - Qx, J(y - Qx)\big\rangle \leqslant 0 \quad \text{for all } x \in C \text{ and } y \in F, \tag{3.2}$$

where J is the (normalised) duality map associated with the gauge $\varphi(t) = t$. Note that the inequality (3.2) is equivalent to the inequality

$$\big\langle x - Qx, J_\varphi(y - Qx)\big\rangle \leqslant 0 \quad \text{for all } x \in C \text{ and } y \in F, \tag{3.3}$$

where φ is an arbitrary gauge. This is because there holds the relation

$$J_\varphi(x) = \Big(\varphi(\|x\|)/\|x\|\Big)J(x)$$

for all $x \neq 0$.

PROPOSITION 3.2. *Let X be a uniformly convex Banach space having a weakly continuous duality map J_φ with gauge φ and let*

$$\mathcal{F} = \{T(t) : t \geqslant 0\}$$

be a contraction semigroup on C such that $F \equiv \mathrm{Fix}(\mathcal{F}) \neq \emptyset$. Then there exists a (unique) sunny nonexpansive retraction Q from C onto F and Q is constructed as follows.

Denote by x_t the unique fixed point of the equation

$$x_t = \frac{1}{t}u + \Big(1 - \frac{1}{t}\Big)\sigma_t(x_t), \quad t > 1. \tag{3.4}$$

Then $Q(u) := s - \lim_{t\to\infty} x_t$ exists and defines a sunny nonexpansive retraction from C onto F.

PROOF: By a result of Bruck [**3**], the uniform convexity of X implies that, for each fixed $s > 0$,

$$\lim_{t\to\infty}\big\|\sigma_t(x) - T(s)\sigma_t(x)\big\| = 0 \tag{3.5}$$

and the limit is attained uniformly for x in any bounded subset of C. It then follows from (3.4) that

$$\big\|x_t - \sigma_t(x_t)\big\| = \frac{1}{t-1}\|u - x_t\| \to 0 \quad \text{as } t \to \infty.$$

This and the fact that the limit in (3.5) is attained uniformly for bounded x in C imply that for all $s > 0$,

$$\begin{aligned}\big\|x_t - T(s)x_t\big\| &\leqslant \big\|x_t - \sigma_t(x_t)\big\| + \big\|\sigma_t(x_t) - T(s)\sigma_t(x_t)\big\| \\ &\qquad + \big\|T(s)\sigma_t(x_t) - T(s)x_t\big\| \\ &\leqslant 2\big\|x_t - \sigma_t(x_t)\big\| + \big\|\sigma_t(x_t) - T(s)\sigma_t(x_t)\big\| \\ &\to 0 \quad \text{as } t \to \infty.\end{aligned} \tag{3.6}$$

Since X is uniformly convex, we have by Lemma 2.2 that $\omega_w(x_t) \subset F$, where $\omega_w(x_t)$ is the weak ω-limit set of $\{x_t\}$ as $t \to \infty$; namely, the set of all points x for which $x_{t_n} \rightharpoonup x$ for some subsequence $\{x_{t_n}\}$ of $\{x_t\}$, where $t_n \to \infty$. We next show that

$$\big\langle u - x_t, J_\varphi(z - x_t)\big\rangle \leqslant 0, \quad z \in F. \tag{3.7}$$

Indeed, observing that $u - x_t = (t-1)\big(x_t - \sigma_t(x_t)\big)$, we have for $z \in F$,

$$\begin{aligned}\big\langle u - x_t, J_\varphi(z - x_t)\big\rangle &= (t-1)\big\langle (I - \sigma_t)x_t, J_\varphi(z - x_t)\big\rangle \\ &= -(t-1)\big\langle (I-\sigma_t)z - (I-\sigma_t)x_t, J_\varphi(z - x_t)\big\rangle \\ &\leqslant 0\end{aligned}$$

since it is easy to see that for each $t > 0$, $I - \sigma_t$ is monotone.

Applying Lemma 2.1(i), we get for $z \in F$,

$$\begin{aligned}\Phi\big(\|x_t - z\|\big) &= \Phi\left(\left\|\Big(1 - \frac{1}{t}\Big)(\sigma_t(x_t) - z) + \frac{1}{t}(u - z)\right\|\right) \\ &\leqslant \Phi\left(\Big(1-\frac{1}{t}\Big)\|\sigma_t(x_t) - z\|\right) + \frac{1}{t}\langle u - z, J_\varphi(x_t - z)\rangle \\ &\leqslant \Big(1 - \frac{1}{t}\Big)\Phi\big(\|x_t - z\|\big) + \frac{1}{t}\big\langle u - z, J_\varphi(x_t - z)\big\rangle.\end{aligned}$$

Hence,

$$\Phi\big(\|x_t - z\|\big) \leqslant \big\langle u - z, J_\varphi(x_t - z)\big\rangle. \tag{3.8}$$

Now take $p \in \omega_w(x_t)$ and assume $x_{t_n} \rightharpoonup p$. Then $p \in F$ by (3.6) and Lemma 2.2. We claim that $x_{t_n} \to p$. Indeed, we have by (3.8),

$$\Phi\big(\|x_{t_n} - p\|\big) \leqslant \big\langle u - p, J_\varphi(x_{t_n} - p)\big\rangle \to 0 \tag{3.9}$$

since J_φ is weakly continuous and $x_{t_n} \rightharpoonup p$. By (3.9), $x_{t_n} \to p$.

To prove that $\{x_t\}$ is strongly convergent, we assume that $x_{t_n} \to p$ and $x_{s_n} \to q$ and have to prove that $p = q$. To see this, we have first of all that $p, q \in F$. Next we have by (3.7)

$$\big\langle u - x_{t_n}, J_\varphi(q - x_{t_n})\big\rangle \leqslant 0$$

and

$$\big\langle u - x_{s_n}, J_\varphi(p - x_{s_n})\big\rangle \leqslant 0.$$

Letting $n \to \infty$ yields

$$\big\langle u - p, J_\varphi(q - p)\big\rangle \leqslant 0 \tag{3.10}$$

and

$$\big\langle u - q, J_\varphi(p - q)\big\rangle \leqslant 0. \tag{3.11}$$

Adding (3.10) and (3.11) gives $\|p - q\|\varphi\big(\|p - q\|\big) \leqslant 0$ which implies that $p = q$.

So we can define $Q : C \to F$ by

$$Q(u) = s - \lim_{t \to \infty} x_t.$$

That $Qu = u$ for $u \in F$ is obvious and hence Q is a retraction from C onto F. To see that Q is also sunny nonexpansive, we take the limit as $t \to \infty$ in (3.7) to get that

$$\big\langle u - Qu, J_\varphi(z - Qu)\big\rangle \leqslant 0, \quad z \in F.$$

By the characteristic inequality (3.3) we see that Q is sunny nonexpansive. ∎

Below is the main result of this paper.

THEOREM 3.3. *Let X be a uniformly convex Banach space having a weakly continuous duality map J_φ with gauge φ, C a closed convex subset of X and*

$$\mathcal{F} = \{T(t) : t \geqslant 0\}$$

a contraction semigroup on C such that $F = \mathrm{Fix}(\mathcal{F}) \neq \emptyset$. Define a sequence $\{x_n\}$ in C implicitly by the fixed point iteration process

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n) x_n, \quad n \geqslant 1, \tag{3.12}$$

where $u \in C$ is an arbitrarily fixed element in C and $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers such that $\alpha_n \in (0,1)$ and $t_n > 0$ for all n, and $\lim_n t_n = \lim_n(\alpha_n/t_n) = 0$. Then $\{x_n\}$ strongly converges to a member of F.

PROOF:

1. $\{x_n\}$ is bounded. Indeed, for $z \in F$,

$$\begin{aligned} \|x_n - z\| &\leqslant \alpha_n\|u - z\| + (1 - \alpha_n)\big\|T(t_n)x_n - z\big\| \\ &\leqslant \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned}$$

This implies that $\|x_n - z\| \leqslant \|u - z\|$ for all n; thus $\{x_n\}$ is bounded.

2. $\omega_w(x_n) \subset F$. (Here $\omega_w(x_n) := \{z : \exists x_{n_j} \rightharpoonup z\}$ is the weak ω-limit set of the sequence $\{x_n\}$.) Indeed, take $z \in \omega_w(x_n)$ and let $x_{n_j} \rightharpoonup z$. Fix a $t > 0$; we may assume $t > t_n$ since $t_n \to 0$. Putting

$$\widetilde{t}_j = t_{n_j},\ m = [t/\widetilde{t}_j],\ \widetilde{x}_j = x_{n_j},\ \widetilde{\alpha}_j = \alpha_{n_j}$$

and noting that

$$T(\widetilde{t}_j)\widetilde{x}_j - \widetilde{x}_j = \widetilde{\alpha}_j(T(\widetilde{t}_j) - u),$$

we derive that

$$\begin{aligned}\|x_{n_j} - T(t)z\| &\leqslant \sum_{k=0}^{m-1}\left\|T\big((k+1)\widetilde{t}_j\big)\widetilde{x}_j - T(k\widetilde{t}_j)\widetilde{x}_j\right\| \\ &\qquad + \|T(m\widetilde{t}_j)\widetilde{x}_j - T(m\widetilde{t}_j)z\| + \|T(m\widetilde{t}_j)z - T(t)z\| \\ &\leqslant m\|T(\widetilde{t}_j)\widetilde{x}_j - \widetilde{x}_j\| + \|\widetilde{x}_j - z\| + \|T(t - m\widetilde{t}_j)z - z\| \\ &\leqslant t(\widetilde{\alpha}_j/\widetilde{t}_j)\|T(\widetilde{t}_j)\widetilde{x}_j - u\| + \|\widetilde{x}_j - z\| \\ &\qquad + \max\left\{\|T(s)z - z\| : 0 \leqslant s \leqslant \widetilde{t}_j\right\}.\end{aligned}$$

By assumption, we find that the first and last terms in the last inequality tend to zero when $j \to \infty$. Hence we deduce that

$$\limsup_{j\to\infty}\|x_{n_j} - T(t)z\| \leqslant \limsup_{j\to\infty}\|x_{n_j} - z\|.$$

We therefore conclude that $T(t)z = z$ by Opial's property (Lemma 2.1(ii)).

3. $\limsup_{n\to\infty}\langle u-q, J_\varphi(x_n - q)\rangle \leqslant 0$ with $q = Q(u)$. (Here Q is the sunny nonexpansive retraction from C onto F as obtained in Proposition 3.2.) As a matter of fact, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ so that

$$\limsup_{n\to\infty}\langle u - q, J_\varphi(x_n - q)\rangle = \lim_{j\to\infty}\langle u - q, J_\varphi(x_{n_j} - q)\rangle$$

and

$$x_{n_j} \rightharpoonup \widetilde{x}.$$

By step 2 above, we have $\widetilde{x} \in F$. From the weak continuity of the duality map J_φ, it follows from (3.3) that

$$\limsup_{n\to\infty}\langle u - q, J_\varphi(x_n - q)\rangle = \langle u - q, J_\varphi(\widetilde{x} - q)\rangle \leqslant 0.$$

4. $x_n \to q$. Indeed, apply Lemma 2.1(i) to get

$$\begin{aligned}\Phi(\|x_n - q\|) &= \Phi\Big(\big\|(1-\alpha_n)(T(t_n)x_n - q) + \alpha_n(u - q)\big\|\Big) \\ &\leqslant \Phi\Big((1-\alpha_n)\|T(t_n)x_n - q\|\Big) + \alpha_n\langle u - q, J_\varphi(x_n - q)\rangle \\ &\leqslant (1-\alpha_n)\Phi(\|x_n - q\|) + \alpha_n\langle u - q, J_\varphi(x_n - q)\rangle.\end{aligned}$$

Thus,

$$\Phi(\|x_n - q\|) \leqslant \langle u - q, J_\varphi(x_n - q)\rangle.$$

This implies by step 3 that

$$\limsup_{n\to\infty}\Phi(\|x_n - q\|) \leqslant \limsup_{n\to\infty}\langle u - q, J_\varphi(x_n - q)\rangle \leqslant 0.$$

Hence $x_n \to q$. ∎

References

[1] F.E. Browder, *Fixed point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1272-1276.

[2] F.E. Browder, 'Convergence theorems for sequences of nonlinear operators in Banach spaces', *Math. Z.* **100** (1967), 201-225.

[3] R.E. Bruck, 'A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces', *Israel J. Math.* **32** (1979), 107–116.

[4] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings* (Marcel Dekker, New York, 1984).

[5] T.C. Lim and H.K. Xu, 'Fixed point theorems for asymptotically nonexpansive mappings', *Nonlinear Anal.* **22** (1994), 1345–1355.

[6] S. Reich, 'Strong convergence theorems for resolvents of accretive operators in Banach spaces', *J. Math. Anal. Appl.* **75** (1980), 287–292.

[7] N. Shioji and W. Takahashi, 'Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces', *Nonlinear Anal.* **34** (1998), 87–99.

[8] T. Suzuki, 'On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces', *Proc. Amer. Math. Soc.* **131** (2002), 2133–2136.

[9] H.K. Xu, 'Approximations to fixed points of contraction semigroups in Hilbert spaces', *Numer. Funct. Anal. Optim.* **19** (1998), 157–163.

[10] H.K. Xu, 'Iterative algorithms for nonlinear operators', *J. London Math. Soc.* **66** (2002), 240–256.

School of Mathematical Sciences
University of KwaZulu-Natal
Westville Campus
Private Bag X54001
Durban 4000
South Africa
e-mail: xuhk@ukzn.ac.za