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MEASURES OF NONCOMPACTNESS IN ULTRAPRODUCTS

WIESŁAWA KACZOR✉, ADAM STACHURA, JUSTYNA WALCZUK and MAGDALENA ZOŁA

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Abstract

We investigate the connection between measures of noncompactness of a bounded subset of a given Banach space and the corresponding measures of noncompactness of an ultrapower of this subset. The Kuratowski, Hausdorff and separation measures of noncompactness are considered. We prove that in the first two cases the measures of a subset are equal to the respective measures of ultrapowers of this subset. In the case of separation measure of noncompactness, the equality is not necessarily fulfilled.

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1. Introduction

Our aim in this paper is to investigate the connection between measures of noncompactness of a bounded subset of a Banach space and the corresponding measures of noncompactness of an ultrapower of this subset. Ideas of ultrapowers and measures of noncompactness have appeared recently in a significant number of results in the fixed point theory of nonexpansive mappings (see for instance [3, 5]). We shall begin by recalling these two concepts.

Let $(X, \|\cdot\|)$ be a Banach space and A be a bounded subset of X. Recall that the Kuratowski measure of noncompactness of a set $A \subset X$ is given by formula

$$\alpha(A) = \inf\{\varepsilon > 0\colon \text{there exists a finite cover of } A \text{ by sets with diameter not greater than } \varepsilon\}.$$

The Hausdorff measure of noncompactness of a set $A \subset X$ is defined as follows:

$$\chi(A) = \inf\{\varepsilon > 0\colon \text{there exists a finite cover of } A \text{ by closed balls with radii not greater than } \varepsilon\}.$$

We also consider the separation measure of noncompactness of a set $A \subset X$ given by

$$\beta(A) = \sup\{r > 0\colon \text{there exists a sequence } (x_n) \text{ of points of } A \text{ such that } \|x_k - x_m\| \geq r \text{ for all } k \neq m\}.$$

(See [2] for details.)

Recall now the construction of an ultrapower of the space X. We use the same notations and terminology as used in [1].

Let I be an infinite set and $\mathcal{U} \subset 2^I$ an ultrafilter containing no finite sets. Let

$$l_\infty(X) = \Big\{(x_i) \in X^I\colon \sup_{i \in I} \|x_i\| < \infty\Big\},$$
$$\mathcal{N}((x_i)) = \lim_{\mathcal{U}} \|x_i\|, \quad (x_i) \in l_\infty(X),$$
$$\ker \mathcal{N} = \{(x_i) \in l_\infty(X)\colon \mathcal{N}((x_i)) = 0\},$$

where $\lim_{\mathcal{U}}$ denotes the limit over the ultrafilter $\mathcal{U}$ (for details see [1]).

The ultrapower of the Banach space X with respect to the ultrafilter $\mathcal{U}$ is defined as the quotient space

$$(X)_{\mathcal{U}} = l_\infty(X)/\ker \mathcal{N}.$$

We shall adopt the notation

$$\widetilde{(x_i)} = (x_i) + \ker \mathcal{N}$$

for any $(x_i) \in l_\infty(X)$.

If $A \subset X$, then the ultrapower of the set A with respect to the ultrafilter $\mathcal{U}$ is defined by

$$(A)_{\mathcal{U}} = \{\widetilde{(x_i)} \in (X)_{\mathcal{U}}\colon x_i \in A \text{ for all } i \in I\}$$

or, equivalently,

$$(A)_{\mathcal{U}} = \{\widetilde{(x_i)} \in (X)_{\mathcal{U}}\colon \{i \in I\colon x_i \in A\} \in \mathcal{U}\}.$$

If $x_i = x$ for all $i \in I$, we shall indicate this by writing $\widetilde{(x)}$ in place of $\widetilde{(x_i)}$.

The following formula is valid for the norm in an ultraproduct $(X)_{\mathcal{U}}$

$$\|\widetilde{(x_i)}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\|.$$

It is well known that $((X)_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}})$ is a Banach space and that the function $F\colon X \to (X)_{\mathcal{U}}$ defined by

$$F(x) = \widetilde{(x)}$$

is an isometric embedding of the space X into $(X)_{\mathcal{U}}$ (see [1]). An excellent exposition of ultraproducts which deals with fixed point theory of nonexpansive mappings can be found in [1].

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2. Main results

Let $(X)_{\mathcal{U}}$ be an ultrapower of a Banach space $(X, \|\cdot\|)$ over a fixed ultrafilter $\mathcal{U} \subset 2^I$. The Kuratowski measure of noncompactness in the spaces $(X, \|\cdot\|)$ and $((X)_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}})$ will be denoted by α and $\alpha_{\mathcal{U}}$, respectively. Analogously, let χ, $\chi_{\mathcal{U}}$, β and $\beta_{\mathcal{U}}$ denote the corresponding Hausdorff and separation measures of noncompactness, respectively. We denote the closed ball of radius $r > 0$ centered at a point x by $B[x, r]$.

LEMMA 2.1. *For all bounded sets $A \subset X$ the following inequality is satisfied:*

$$\alpha(A) \leq \alpha_{\mathcal{U}}((A)_{\mathcal{U}}).$$

PROOF. Let $F(x) = \widetilde{(x)}$ for any $x \in X$. It is easy to show that

$$F(A) \subset (A)_{\mathcal{U}}$$

and consequently

$$\alpha_{\mathcal{U}}(F(A)) \leq \alpha_{\mathcal{U}}((A)_{\mathcal{U}}).$$

To complete the proof it is enough to notice that

$$\alpha(A) = \alpha_{\mathcal{U}}(F(A)),$$

which results from the fact that F is an isometric embedding of the space X into $(X)_{\mathcal{U}}$. □

LEMMA 2.2. *For all bounded sets $A \subset X$ the following inequality is satisfied:*

$$\alpha(A) \geq \alpha_{\mathcal{U}}((A)_{\mathcal{U}}).$$

PROOF. Let us fix $\varepsilon > \alpha(A)$. Thus, there exists a finite family of sets $\{K_1, K_2, \ldots, K_n\}$ such that

$$A \subset \bigcup_{j=1}^{n} K_j \quad \text{and} \quad \operatorname{diam} K_j \leq \varepsilon \quad \text{for any } j = 1, 2, \ldots, n.$$

Consider the family of sets $\{(K_1)_{\mathcal{U}}, (K_2)_{\mathcal{U}}, \ldots, (K_n)_{\mathcal{U}}\}$. We show that

$$(A)_{\mathcal{U}} \subset \bigcup_{j=1}^{n} (K_j)_{\mathcal{U}}.$$

Let $\widetilde{(x_i)}$ be an arbitrary element of $(A)_{\mathcal{U}}$ and let $I_0 = \{i : x_i \in A\}$. Then, $I_0 \in \mathcal{U}$ and for each $j = 1, 2, \ldots, n$, define

$$I_j = \{i \in I : x_i \in K_j\}.$$

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Since

$$A \subset \bigcup_{j=1}^{n} K_j,$$

it is easy to show that

$$\bigcup_{j=1}^{n} I_j \supseteq I_0.$$

Thus, there exists some $j_0 \in \{1, 2, \ldots, n\}$ such that $I_{j_0} \in \mathcal{U}$.

Consider the sequence (x_i'), where

$$x_i' = \begin{cases} x_i & \text{if } i \in I_{j_0}, \\ a & \text{if } i \notin I_{j_0}, \end{cases}$$

and $a \in K_{j_0}$ is a vector arbitrarily chosen from the set K_{j_0}. If $i \in I_{j_0}$ then $x_i' \in K_{j_0}$, and if $i \notin I_{j_0}$ then $x_i' = a \in K_{j_0}$. Hence, $x_i' \in K_{j_0}$ for any $i \in I$. As a consequence, we can see that $\widetilde{(x_i')} \in (K_{j_0})_{\mathcal{U}}$.

For any $r > 0$ we write

$$J_r = \{i \in I : \|x_i - x_i'\| < r\}.$$

Notice that $J_r \supset I_{j_0} \in \mathcal{U}$. Thus, $J_r \in \mathcal{U}$ as a superset of a set belonging to $\mathcal{U}$. It then follows that

$$\lim_{\mathcal{U}} \|x_i - x_i'\| = 0.$$

Therefore, $(x_i - x_i') \in \ker \mathcal{N}$, so $(x_i) - (x_i') \in \ker \mathcal{N}$. Eventually, we have $\widetilde{(x_i')} = \widetilde{(x_i)}$. Since $\widetilde{(x_i')} \in (K_{j_0})_{\mathcal{U}}$, then $\widetilde{(x_i)} \in (K_{j_0})_{\mathcal{U}}$. We have shown that

$$(A)_{\mathcal{U}} \subset \bigcup_{j=1}^{n} (K_j)_{\mathcal{U}}.$$

Knowing that

$$\operatorname{diam}(K_j)_{\mathcal{U}} = \operatorname{diam} K_j \leq \varepsilon, \qquad j = 1, 2, \ldots, n,$$

we conclude that the family $\{(K_1)_{\mathcal{U}}, (K_2)_{\mathcal{U}}, \ldots, (K_n)_{\mathcal{U}}\}$ is a finite cover of the set $(A)_{\mathcal{U}}$ by sets with diameter not greater then ε. Therefore

$$\alpha_{\mathcal{U}}((A)_{\mathcal{U}}) \leq \varepsilon.$$

As $\varepsilon > \alpha(A)$ was chosen arbitrarily, we obtain the desired inequality

$$\alpha(A) \geq \alpha_{\mathcal{U}}((A)_{\mathcal{U}}). \qquad \square$$

COROLLARY 2.3. $\alpha(A) = \alpha_{\mathcal{U}}((A)_{\mathcal{U}})$ *for all bounded sets* $A \subset X$.

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Before we consider the Hausdorff measure of noncompactness, we recall some known facts concerning measures of noncompactness. If ϕ is any measure of noncompactness, $A \subset B \subset X$ and X is a subspace of Y, then we always have $\phi_X(A) \leq \phi_X(B)$. Furthermore, the equality $\phi_Y(B) = \phi_X(B)$ holds when ϕ is either the Kuratowski measure α or the separation measure β. But in the case of the Hausdorff measure of noncompactness we only have $\chi_Y(B) \leq \chi_X(B)$. Thus we cannot use an embedding argument similar to that for Lemma 2.1 to establish the analogous result for χ. Nonetheless, our next two lemmas show that for all bounded sets $A \subset X$ we have

$$\chi(A) = \chi_{\mathcal{U}}((A)_{\mathcal{U}}).$$

LEMMA 2.4. *For all bounded sets $A \subset X$ the following inequality is satisfied:*

$$\chi(A) \leq \chi_{\mathcal{U}}((A)_{\mathcal{U}}).$$

PROOF. Assume that $\chi_{\mathcal{U}}((A)_{\mathcal{U}}) < \chi(A)$ for some bounded set $A \subset X$ and let $\varepsilon_1, \varepsilon_2$ be such that $\chi_{\mathcal{U}}((A)_{\mathcal{U}}) < \varepsilon_1 < \varepsilon_2 < \chi(A)$. Thus, there exists a finite cover of the set $(A)_{\mathcal{U}}$ by closed balls (in the space $(X)_{\mathcal{U}}$) with radii ε_1, that is

$$(A)_{\mathcal{U}} \subset \bigcup_{j=1}^{n} B[\widetilde{(x_i^{(j)})}, \varepsilon_1],$$

where $x_i^{(j)} \in X$ for all $i \in I$, $j \in \{1, 2, \ldots, n\}$. Let us fix $i \in I$. Since $\varepsilon_2 < \chi(A)$, we know that

$$\{B[x_i^{(j)}, \varepsilon_2] \colon j = 1, 2, \ldots, n\}$$

is not a cover of the set A in the space X. Hence, there exists $y_i \in A$ such that

$$y_i \notin \bigcup_{j=1}^{n} B[x_i^{(j)}, \varepsilon_2].$$

Consequently,

$$\|y_i - x_i^{(j)}\| > \varepsilon_2$$

for all $j \in \{1, 2, \ldots, n\}$. Now consider a point $\widetilde{(y_i)} \in (A)_{\mathcal{U}}$. From the above inequality we have

$$\|\widetilde{(y_i)} - \widetilde{(x_i^{(j)})}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|y_i - x_i^{(j)}\| \geq \varepsilon_2$$

for all $j = 1, 2, \ldots, n$. Knowing that $\varepsilon_2 > \varepsilon_1$, we conclude that

$$\widetilde{(y_i)} \notin B[\widetilde{(x_i^{(j)})}, \varepsilon_1]$$

for all $j = 1, 2, \ldots, n$. Therefore,

$$\widetilde{(y_i)} \notin \bigcup_{j=1}^{n} B[\widetilde{(x_i^{(j)})}, \varepsilon_1].$$

On the other hand, we have

$$\widetilde{(y_i)} \in (A)_{\mathcal{U}} \subset \bigcup_{j=1}^{n} B[\widetilde{(x_i^{(j)})}, \varepsilon_1].$$

We have thus obtained a contradiction. □

LEMMA 2.5. *For all bounded sets $A \subset X$ the following inequality is satisfied:*

$$\chi(A) \geq \chi_{\mathcal{U}}((A)_{\mathcal{U}}).$$

PROOF. Let us fix $\varepsilon > \chi(A)$. Thus there exists a finite family of closed balls with radii ε

$$\{B[x^{(j)}, \varepsilon] : j = 1, 2, \ldots, n\} \quad \text{where } x^{(j)} \in X,\ j = 1, 2, \ldots, n,$$

such that

$$A \subset \bigcup_{j=1}^{n} B[x^{(j)}, \varepsilon].$$

It is easy to show that (see the proof of Lemma 2.2)

$$(A)_{\mathcal{U}} \subset \bigcup_{j=1}^{n} B[\widetilde{(x^{(j)})}, \varepsilon].$$

Therefore, $\chi_{\mathcal{U}}((A)_{\mathcal{U}}) \leq \varepsilon$ for all $\varepsilon > \chi(A)$. This leads us to the required inequality. □

COROLLARY 2.6. $\chi(A) = \chi_{\mathcal{U}}((A)_{\mathcal{U}})$ *for all bounded sets $A \subset X$.*

The next lemma follows by an embedding argument similar to that for Lemma 2.1 and the properties observed about $\phi = \beta$.

LEMMA 2.7. *For all bounded sets $A \subset X$ the following inequality is satisfied:*

$$\beta(A) \leq \beta_{\mathcal{U}}((A)_{\mathcal{U}}).$$

However, it is not true that $\beta(A) \geq \beta_{\mathcal{U}}((A)_{\mathcal{U}})$ for all bounded sets $A \subset X$. This can be illustrated by the following example due to S. Prus [4].

EXAMPLE 2.8. For $n \in \mathbb{N}$ let $\ell_n^\infty = (\mathbb{R}^n, \|\cdot\|_\infty)$.

The following Banach space was defined by Day (see [2]):

$$D_\infty = \left\{ x = (x_n) : x_n \in \ell_n^\infty,\ \|x\|_{D_\infty} = \sqrt{\sum_{n=1}^{\infty} \|x_n\|_\infty^2} < \infty \right\}.$$

Let $A = \{x \in D_\infty : \|x\|_{D_\infty} \leq 1\}$. It is known (see [2]) that $\beta(B) = \sqrt{2}$, where B denotes the unit ball in Hilbert space ℓ^2. Using the same method we can easily show

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that $\beta(A) = \sqrt{2}$. Let $I = \mathbb{N}$ and let us fix an ultrafilter $\mathcal{U} \subset 2^{\mathbb{N}}$ containing no finite sets. We show that

$$\beta_{\mathcal{U}}((A)_{\mathcal{U}}) = 2 > \sqrt{2} = \beta(A).$$

Let $\widetilde{((x_i^{(k)}))}$, $k \in \mathbb{N}$, be a sequence of points in the ultrapower of the ball A defined as follows:

$$\begin{aligned}
(x_i^{(1)}) = (\ & ((1), (0,0), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (1,1), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (1,1,1), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (0,0,0), (1,1,1,1), (0,0,0,0,0), \ldots), \ldots);
\end{aligned}$$

$$\begin{aligned}
(x_i^{(2)}) = (\ & ((-1), (0,0), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (-1,1), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (-1,1,1), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (0,0,0), (-1,1,1,1), (0,0,0,0,0), \ldots), \ldots);
\end{aligned}$$

$$\begin{aligned}
(x_i^{(3)}) = (\ & ((-1), (0,0), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (-1,-1), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (-1,-1,1), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (0,0,0), (-1,-1,1,1), (0,0,0,0,0), \ldots), \ldots);
\end{aligned}$$

$$\begin{aligned}
(x_i^{(4)}) = (\ & ((-1), (0,0), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (-1,-1), (0,0,0), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (-1,-1,-1), (0,0,0,0), (0,0,0,0,0), \ldots),\\
& ((0), (0,0), (0,0,0), (-1,-1,-1,1), (0,0,0,0,0), \ldots), \ldots);
\end{aligned}$$

and so on.

More precisely, for all $k \in \mathbb{N}$

$$x_i^{(k)} = (x_{i,n}^{(k)}), \quad n \in \mathbb{N},$$

where

$$\ell_n^\infty \ni x_{i,n}^{(k)} = (x_{i,n,m}^{(k)})$$

and

$$x_{i,i,m}^{(k)} = \begin{cases} -1, & m \leq k-1, \\ 1, & m > k-1, \end{cases}$$

$$x_{i,n}^{(k)} = \vec{0}, \quad n \in \mathbb{N}, i \neq n.$$

Obviously, $\widetilde{(x_i^{(k)})} \in (A)_{\mathcal{U}}$ for all $k \in \mathbb{N}$. It is easy to show that

$$\|\widetilde{(x_i^{(k)})} - \widetilde{(x_i^{(l)})}\|_{\mathcal{U}} = 2$$

for all $k \neq l$. Hence,

$$\beta_{\mathcal{U}}((A)_{\mathcal{U}}) = 2$$

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and

$$\beta_{\mathcal{U}}((A)_{\mathcal{U}}) > \beta(A).$$

Finally, let us observe that in general we always have $\beta_{\mathcal{U}}((A)_{\mathcal{U}}) \leq 2\beta(A)$. However, whether the constant 2 is the best possible is unclear. In the above example we have $\beta_{\mathcal{U}}((A)_{\mathcal{U}}) = \sqrt{2}\beta(A)$.

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WIESŁAWA KACZOR, Instytut Matematyki UMCS, 20-031 Lublin, Poland
e-mail: wieslawa.kaczor@poczta.umcs.lublin.pl

ADAM STACHURA, Institute of Mathematics and Computer Science, John Paul II Catholic University of Lublin, 20-708 Lublin, Poland
e-mail: stachura@kul.lublin.pl

JUSTYNA WALCZUK, Instytut Matematyki PWSZ, 22-400 Zamość, Poland
e-mail: jwalczuk@op.pl

MAGDALENA ZOŁA, Institute of Mathematics and Computer Science, John Paul II Catholic University of Lublin, 20-708 Lublin, Poland
e-mail: mzola@kul.lublin.pl