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REPRESENTABLE DUALITIES BETWEEN FINITELY CLOSED SUBCATEGORIES OF MODULES

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1. Introduction and summary. This paper studies dualities (or contravariant category equivalences) between two categories of R-right and S-left modules which are *finitely closed*; that is, closed under submodules, factor modules and finite direct sums. Omitting the requirement that the categories contain all finitely generated modules from the classical Morita situation provides a generalization which substantially increases the number of such dualities.

We prove that a duality between two finitely closed categories A and B of modules is representable if and only if A and B consist of linearly compact modules. This encompasses work of Mueller ([**7**], [**8**]) for Morita dualities and of Goblot ([**5**], [**6**]). A linearly compact finitely closed category of modules is always an AB5*-category with no infinite direct sums; we demonstrate the converse for certain rings including all commutative ones, thus simplifying our characterization of representable dualities in these cases; we were however unable to obtain this result in general or to find a counterexample.

2. Terminology. Let R, S be two rings with identity and mod-R (respectively S-mod) the category of all R-right (respectively S-left) modules. Recall that an abelian category C is an AB5-*category* if for each $X \in C$ the subobjects of X form a complete lattice and for all subobjects Y of X and all updirected families $(X_i)_{i \in I}$ of subobjects of X, $\bigcup_I (X_i \cap Y) = (\bigcup_I X_i) \cap Y$. The dual of an AB5-category is called an AB5*-*category*. A subcategory A of mod-R is faithful if $\operatorname{ann}_R(A) = \{r \in R| \quad Xr = 0 \quad \text{for all } X \in A\}$ is zero.

A finitely closed subcategory A of mod-R is abelian. Finite limits and finite colimits of A are the same as in mod-R. Clearly A is an AB5-category.

If A is a finitely closed subcategory of mod-R, then there is a right linear topology on R defined as follows: a right ideal I of R is open if and only if $R/I \in A$. We shall refer to this topology as the A-*topology*. For the A-topology on R, dis A is the Grothendieck category of all discrete topological A-modules. It is a full coreflective subcategory of mod-R, hence limits on dis A are the coreflections of limits in mod-R and colimits in dis A are the same as in mod-R. A generates dis A, thus A contains all finitely generated modules of dis A.

Let $F : A \to B$ be a duality between finitely closed subcategories of mod-R and S-mod, respectively. F is automatically an exact additive contravariant

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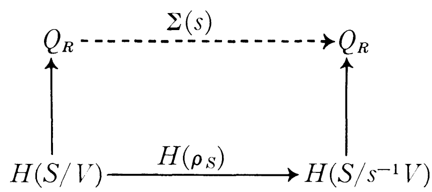
functor. For any $X \in A$ the subobject lattices of X and FX are anti-isomorphic. Since A and B are both AB5-categories, they are also AB5*-categories.

3. Representable dualities. Let the contravariant functors $A \underset{H}{\overset{F}{\rightleftarrows}} B$ be a duality between the finitely closed subcategories A and B of mod-R and S-mod, respectively. We may assume without loss of generality that both A and B are faithful subcategories, since we can always pass to factor rings for which this is true.

There is a linear topology on S, the B-topology, which has as a basis of zero the left ideals V of S such that S/V is an object of B. With respect to the canonical epimorphisms, the S/V form a projective system whose limit is the completion of S in the B-topology. Applying the functor H to this projective system, we obtain a monoinductive system in A. Let $Q_R = \lim_{\rightarrow} H(S/V)$, where the colimit is formed in mod-R. Note that $Q \in \text{dis}\, A$.

LEMMA 1. *There exists a ring homomorphism* $\Sigma : S \rightarrow \text{hom}_R(Q, Q)$, *hence* Q *is an* S-R *bimodule.*

Proof. For $s \in S$, define $\Sigma(s) : Q_R \rightarrow Q_R$ as follows: let V be an open left ideal of S, then $s^{-1}V = \{x \in S \mid xs \in V\}$ is also an open left ideal. Now consider the family of S-homomorphisms, $\rho_s : S/s^{-1}V \rightarrow S/V$, defined by $\rho_s(\bar{l}) = \bar{s}$ for each open V. Applying H we get the following:



where $\Sigma(s)$ is the unique R-homomorphism making the diagram commute for all V. Calculation now shows that Σ is indeed a ring homomorphism.

Lemma 1 allows one to consider $\text{hom}_R(-, Q) : \text{mod-}R \rightarrow S\text{-mod}$ as a functor in the standard way.

LEMMA 2. *Let* $Q = \lim_{\rightarrow} H(S/V)$. *There exists a monic natural transformation* $\mu : F \rightarrow \text{hom}(-, Q)$.

Proof. Since FX is an object of B, $FX = \lim_{\rightarrow} \text{ann}_{FX}(V)$. Secondly, let $J_V : \text{ann}_{FX}(V) \rightarrow \text{hom}_S(S/V, FX)$ be the standard isomorphism, then $J_X = \lim_{\rightarrow} J_V : \lim_{\rightarrow} \text{ann}_{FX}(V) \rightarrow \lim_{\rightarrow} \text{hom}_S(S/V, FX)$ is an isomorphism. Because F, H is a duality, we have an isomorphism $K_V : \text{hom}_S(S/V, FX) \rightarrow \text{hom}_R(X, H(S/V))$; thus $K_X = \lim_{\rightarrow} K_V : \lim_{\rightarrow} \text{hom}_S(S/V, FX) \rightarrow \lim_{\rightarrow} \text{hom}_R(X, H(S/V))$ is an isomorphism. Since $Q_R = \lim_{\rightarrow} H(S/V)$, we have a unique map $t_X : \lim_{\rightarrow} \text{hom}_R(X, H(S/V)) \rightarrow \text{hom}_R(X, {}_SQ_R)$ such that $t_X b_V = \text{hom}_R(X, l_V)$; $l_V : H(S/V) \rightarrow {}_SQ_R$ being the canonical inclusion and b_V being the structure map of the filtered limit. Moreover, t_X is a monomorphism.

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For each $X \in A$, define $\mu_X = t_X K_X J_X$. Thus, it is a monomorphism and is natural as t_X, K_X and J_X are. Note that μ_X was really constructed as a group homomorphism, but simple checking shows that it is actually an S-homomorphism.

Recall that a Hausdorff linearly topological module X is *linearly compact* if any finitely solvable system of congruences $x = x_k (\text{mod } X_k)$, where the X_k are closed submodules of X, is solvable. We call a *subcategory* A of mod-R *linearly compact* if each $X \in A$ is linearly compact with respect to the discrete topology. The basic properties of linearly compact modules are developed in Zelinsky [**10**].

LEMMA 3. *Let* $Q = \lim_{\rightarrow} H(S/V)$. *If* B *is linearly compact, then* $\mu_X : FX \rightarrow \hom_R(X, Q)$ *is an isomorphism for all* $X \in A$.

Proof. If X is a finitely generated module, then t_X is an isomorphism, hence so is μ_X.

Let X be any object of A, then by Lemma 2 $\mu_X : FX \rightarrow \hom_R(X, Q)$ is a monomorphism. Let $\sigma_f : X_f \rightarrow X$ be the inclusion of a finitely generated submodule X_f of X. Consider the following diagram:

$$\begin{array}{ccc} \hom_R(X, Q) & \xleftarrow{\mu_X} & FX \\ \Big\downarrow \hom_R(\sigma_f, Q) & & \Big\downarrow F\sigma_f \\ \hom_R(X_f, Q) & \xleftarrow[\mu_{X_f}]{} & FX_f \end{array}$$

Now $F\sigma_{f'}$, the dual of a monomorphism, is an onto map, and μ_{X_f} is an isomorphism since X_f is a finitely generated module. Let $\phi \in \hom_R(X, Q)$. Then there exists a $b_f \in FX$ such that $\phi | X_f = \mu_{X_f} F\sigma_f(b_f)$ for each finitely generated submodule X_f of X. Consider the congruence $b \equiv b_f(\text{mod Ker } F\sigma_f)$, f running over the finitely generated submodules X_f of X. As the finitely generated submodules are an updirected family, this congruence is clearly finitely solvable. Since $FX \in B$ and B is linearly compact, the congruence is solvable. Let b be a solution; then $\mu_X(b)|X_f = \phi|X_f$ for all finitely generated submodules X_f of X. Thus $\mu_X(b) = \phi$ and μ_X is an isomorphism.

THEOREM 4. *A duality between finitely closed subcategories* A *and* B *of* mod-R *and* S-mod *is representable if and only if* A *and* B *are linearly compact.*

Proof. Assume that the duality between A and B is represented by the bimodule ${}_S Q_R$. Let Y be an object of B and (Y_j) $j \in J$ a downdirected family of subobjects of Y with intersection I. Note that $I = \lim_{\leftarrow B} Y_j = \lim_{\leftarrow S\text{-mod}} Y_j$. By duality Y, I and the Y_j are represented by X, V and modules X_j which form an updirected family of quotients of X with $V = \lim_{\leftarrow A} X_j$. For each $j \in J$, we have the exact sequence:

$$0 \rightarrow K_j \rightarrow X \rightarrow X_j \rightarrow 0$$

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Thus we have the exact sequence:

$$0 \to \lim_{\to} K_j \to X \to \lim_{\to} X_j = V \to 0$$

with colimits taken in mod-R. (Since A is finitely closed, colimits of this type in A coincide with the corresponding mod-R colimits.) As $\hom_R(-, Q)$ is exact on A and takes colimits in mod-R to limits in S-mod, we see that $Y/I = \lim_{\leftarrow S\text{-mod}} Y/Y_j$. Thus for all downdirected families $(Y_j)_{j \in J}$ of submodules of Y the natural map $Y \to \lim_{\leftarrow S\text{-mod}} Y/Y_j$ is onto, hence Y is linearly compact.

In the converse direction we will prove that the bimodule ${}_SQ_R = \lim_{\to} H(S/V)$, as constructed in Lemma 1, represents the duality.

First, F is naturally isomorphic to $\hom_R(-, Q)|A$ by Lemma 3. Secondly, from Lemmas 1, 2 and 3 we known that H is represented by ${}_SW_R = \lim_{\to} \hom_R(R/I, Q)$. Also ${}_SQ = \lim_{\leftarrow} \operatorname{ann}_Q(I)$ since $Q \in \operatorname{dis} A$. Let $f = \lim_{\to} f_I$: ${}_SW \to {}_SQ$ where $f_I : \hom_R(R/I, Q) \to \operatorname{ann}_Q(I)$ is defined by $f_I(\phi) = \phi(l)$. Thus f is an S-isomorphism since all the f_I are S-isomorphisms. We claim that f is also an R-homomorphism. Let ϕ be any element of W, then $\phi \in \hom_R(R/I, Q)$ for some I. By the definition of the R-action on W in Lemma 1, we know $\phi r \in \hom_R(R/r - l_I, Q)$ and $\phi r(x) = \phi(\bar{r}\bar{x})$. Hence $f(\phi r) = \phi r(\bar{l}) = \phi(r\bar{l}) = \phi(\bar{r}) = \phi(\bar{l})r = f(\phi)r$. Thus ${}_SQ_R$ is isomorphic as a bimodule to ${}_SW_R$, and ${}_SQ_R$ represents the duality.

A module X is called *finitely cogenerated* if for all downdirected families $\{X_i\}_{i \in I}$ of subobjects of X with $\bigcap X_i = 0$ there exists an $i \in I$ with $X_i = 0$. This is the case if and only if X is an essential extension of a finite socle (cf. Anderson and Fuller [**1**], Proposition 10.7).

Remark. If the duality between A and B is represented by the bimodule ${}_SQ_R$, then $Q' = \{x \in Q | \quad xR \in A\}$ and $'Q = \{x \in Q | \quad Sx \in B\}$, the coreflections of Q into dis A and dis B respectively, are S-R bimodules. Moreover $'Q = Q'$, and this bimodule represents the duality. Also each representable duality between A and B is represented by a unique (up to isomorphism) bimodule ${}_SQ_R$ with $Q_R \in \operatorname{dis} A$ and/or ${}_SQ \in \operatorname{dis} B$; in particular, we may choose $Q = \lim_{\to} H(S/V)$, the bimodule constructed in Lemma 1. As each $H(S/V)$ is finitely cogenerated (since it is dual to a finitely generated module), clearly Q is essential over its socle. *Henceforth we will assume that the duality between A and B is represented by the bimodule ${}_SQ_R$ with $Q \in$ dis A and $Q \in$ dis B.*

THEOREM 5. *If the duality between A and B is represented by the bimodule ${}_SQ_R$, then Q_R (respectively ${}_SQ$) is an injective cogenerator of* dis A *(respectively* dis B*).*

Proof. We show that Q_R is an injective cogenerator of dis A.

Claim 1. Q_R is A-injective.

Let $0 \to Y \to X$ be an exact sequence in A. Since Q represents the duality, we have the exact sequence $\hom_R(X, Q) \to \hom_R(Y, Q) \to 0$. Thus Q is A-injective.

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Claim 2. Q_R is dis A-injective.
Let

$$\begin{array}{ccccc} 0 & \longrightarrow & X_0 & \hookrightarrow & X' \\ & & \Big\downarrow{\scriptstyle \phi_{X_0}} & & \\ & & Q_R & & \end{array}$$

where $X' \in$ dis A, X_0 is a submodule of X' and $\phi_{X_0} : X \to Q_R$ is any R-homomorphism. Let $T = \{(X, \phi_X) | \quad X_0 \subseteq X \subseteq X' \quad \text{and} \quad \phi_X | X_0 = \phi_{X_0}\}$. Order T as follows: $(X, \phi_X) \leqq (Y, \phi_Y)$ if and only if $X \hookrightarrow Y$ and $\phi_Y | X = \phi_X$. Clearly this ordering is inductive, hence by Zorn's lemma there exists a maximal element (M, ϕ_M). If $M = X'$ we have proved the claim.

Assume $M \neq X'$. Select $a \in X' - M$, then $M \subset (\neq) M + aR$. Consider the following diagram:

$$\begin{array}{ccc} M \cap aR & \hookrightarrow & aR \\ \Big\downarrow{\scriptstyle \phi_M | M \cap aR} & & \\ Q_R & & \end{array}$$

Note that $M \cap aR$ and aR are elements of A. Thus there exists a map $f : aR \to Q_R$ such that $f | M \cap aR = \phi_M | M \cap aR$. Define $h : M + aR \to Q_R$ by $h(M + ar) = \phi_M(M) + f(ar)$. This is a well defined R-homomorphism. Also $(M, \phi_M) < (\neq) (M + aR, h)$, but (M, ϕ_M) was maximal, a contradiction.

Claim 3. Q_R is a cogenerator of dis A.

First, Q_R cogenerates Y if and only if for each $0 \neq y \in Y$ there exists a $f : Y \to Q_R$ such that $f(y) \neq 0$. Secondly, for a bimodule ${}_SQ_R$, the natural map $X \to X^{**} = \hom_S(\hom_R(X, Q), Q)$ is a monomorphism if and only if Q_R cogenerates X. Thus Q_R cogenerates the category A. Let $X \in$ dis A and $0 \neq x \in X$. As $X \in$ dis A, $xR \in A$ and thus there exists a map $\bar{f} : xR \to Q_R$ such that $\bar{f}(x) \neq 0$. By injectivity of Q_R, there is a map $f : X \to Q_R$ such that $f | xR = \bar{f}$; in particular, $f(x) \neq 0$. Thus Q_R cogenerates X.

PROPOSITION 6. *If the duality between A and B is represented by the bimodule ${}_SQ_R$ then* $\operatorname{End}_S Q = \lim_{\leftarrow} R/V$, *the completion of R in the A-topology.*

Proof. A right ideal V of R is open in the A-topology if and only if $R/V \in A$. Consider $\operatorname{ann}_Q(V) = \hom_R(R/V, Q)$. Clearly $\{\operatorname{ann}_Q(v)\}$ is an updirected family of S-submodules and $Q = \lim_{\rightarrow} \operatorname{ann}_Q(V)$ (as $Q \in$ dis A). As $R/V \cong \hom_S(\hom_R(R/V, Q), Q)$ we have $\hom_S(Q, Q) = \hom_S(\lim_{\rightarrow} \operatorname{ann}_Q(V), Q) = \lim_{\leftarrow} \hom_S(\hom_R(R/V, Q), Q) \cong \lim_{\leftarrow} R/V$.

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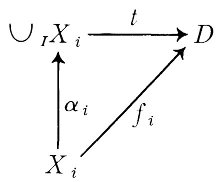
For a linearly topologized ring R, we write dis R for the Grothendieck category of discrete modules and R for the Hausdorff completion of R. If $X \in$ dis R then X can be considered as a $\hat{R}$-module. Hence we may identify dis R and dis $\hat{R}$. Moreover, $X_R \in$ dis R is a linearly compact R-module if and only if $X_R \in$ dis $\hat{R}$ is a linearly compact $\hat{R}$-module (as X_R and $X_{\hat{R}}$ have the same underlying group and the same submodules).

Remark. If there is a duality between the finitely closed subcategories A and B of mod-R and S-mod (represented by the bimodule ${}_SQ_R$) then there is a duality between the finitely closed subcategories A and B of mod-$\hat{R}$ and $\hat{S}$-mod (represented by the bimodule ${}_{\hat{S}}Q_{\hat{R}}$) where $\hat{R}$ (respectively $\hat{S}$) is the completion of R (respectively S) in the A-topology (respectively B-topology). Furthermore, if the duality is represented by Q, from Theorem 4 and Proposition 6 we have $\hat{R} = \text{End}_{\hat{S}}Q$, $\hat{S} = \text{End}\, Q_{\hat{R}}$, $\hat{R}$ linearly compact in the A-topology and $\hat{S}$ is linearly compact in the B-topology.

4. Remarks on limits.

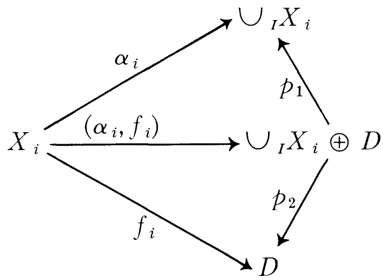
PROPOSITION 7. *Let C be an* AB5*-category and $\{u_i: \ X_i \to X\}_{i \in I}$ an updirected family of subobjects of X; then* $\lim_{\to I} X_i = \bigcup_I X_i$.

Proof. Let $f_i : \{X_i \to D\}_{i \in I}$ be a compatible family. For each $i \in I$, define $\alpha_i : X_i \to \bigcup_I X_i$ as the factorization of $u_i : X_i \to X$ through the union. To prove that $\bigcup_I X_i$ is the colimit we must show that there is a unique map $t : \bigcup_I X_i \to D$ making the diagram



commute for all $i \in I$.

For existence of t we proceed as follows. Let (α_i, f_i) be the unique map defined by the diagram



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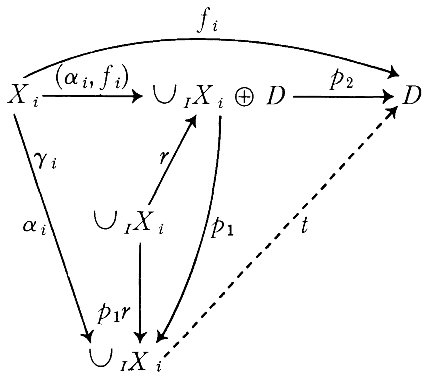
where p_1 and p_2 are the projection maps. Note that (α_i, f_i) is a monomorphism as α_i is, and that the family $\{(\alpha_i, f_i) : \ X_i \rightarrow \bigcup_I X_i \oplus D\}_{i \in I}$ is updirected. Define $\dot{\bigcup} X_i \rightarrow \bigcup_I X_i \oplus D$ as the union of the family $\{(\alpha_i, f_i) : \ X_i \rightarrow \bigcup_I X_i \oplus D\}_{i \in I}$. For each $i \in I$ define $\gamma_i : X_i \rightarrow \dot{\bigcup} X_i$ as the factorization of $(\alpha_i, f_i) : X_i \rightarrow \bigcup_I X_i \oplus D$ through the union $\dot{\bigcup} X_i$.

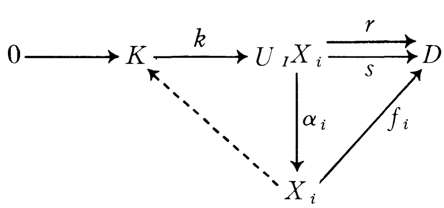
Consider the following commutative diagram for each $i \in I$.



We wish to show that $P_1 r$ is an isomorphism. Each X_i factors through $\text{im}(P_1 r)$, the image of $P_1 r$. As $\text{im}(P_1 r)$ is a subobject of $\bigcup_I X$, it is also a subobject of X. Thus $\text{im}(P_1 r) = \bigcup_I X_i$, as $\bigcup_I X_i$ is the least subobject of X through which X_i factors. Consequently, $P_1 r$ is an epimorphism. Let $K = \ker P_1 r$. Then by AB5, $K = K \cap \dot{\bigcup} X_i = \dot{\bigcup}(K \cap X_i)$. If $K \neq 0$ then $K \cap X_i \neq 0$ for some $i \in I$, but this is a contradiction as α_i is a monomorphism. As C is an abelian category, a map which is a monomorphism and a epimorphism is an isomorphism, thus $P_1 r$ is an isomorphism.

Define $t = P_2 r (P_1 r)^{-1}$. Diagram chasing shows that this is the desired map.

For uniqueness, assume that we have two maps r, s making the following diagram commute for all $i \in I$.



Let (K, k) be the equalizer of r and s. Hence α_i factors through K for each $i \in I$. Note that K is a subobject of X. Thus K must equal $\bigcup_I X_i$ as $\bigcup_I X_i$ is the least subobject of X through which each X_i factors. Thus $r = s$.

Corollary 8. *If C is an* AB5**-category and* $\{u_i: \ X_i \rightarrow X\}_{i \in I}$ *is a down-directed family of subobjects of X, then* $\lim_{\leftarrow I} X/X_i = X/\bigcap X_i$.

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Let A be a finitely closed subcategory of mod-R. We now consider the relationship between limits formed in A, dis A or mod-R.

PROPOSITION 9. *Let R be a ring, A a finitely closed subcategory of* mod-R *and* dis A *the category of modules discrete for the A-topology, then the embedding $A \hookrightarrow$* dis A *commutes with limits existing in A.*

Proof. First, as dis A is a Grothendieck category, the limits in dis A of all diagrams exist. If $D : I \to A$ is a diagram in A with limit (X, Π_i), then it has a limit $(\mathscr{L}, \phi_i)$ in dis A. Now the compatible family (X, Π_i) factor over $(\mathscr{L}, \phi_i)$ by a unique homomorphism $\Pi : X \to \mathscr{L}$. Also as $\mathscr{L}$ is in dis A, $\mathscr{L}$ is the filtered union of subobjects Y_k which lie in A. The restriction of ϕ_i to any of these Y_k yields a compatible family; hence, it factors over (X, Π_i) by a homomorphism $f_k : Y_k \to X$. As $\mathscr{L} = \bigcup_{\to} Y_k$ and the f_k are compatible with the order relation on the Y_k, we obtain a homomorphism $f : \mathscr{L} \to X$. The homomorphism f is easily seen to be the inverse of Π.

Definition. A is a *meager* finitely closed subcategory of mod-R if for all $X \in A$ there exists a $Y \in A$ such that $X \subseteq Y$ and Y is a finitely generated R-module.

PROPOSITION 10. *Let A be a meager finitely closed subcategory of* mod-R. *If the A-topology has a basis of two sided ideals, then A is an* AB5*-*category, if and only if A is linearly compact.*

Proof. Let A be an AB5*-category and let $\{Y_i\}_{i \in I}$ be a downdirected family of submodules of $X \in A$. X is linearly compact if and only if $X/\bigcap Y_i = \lim_{\leftarrow \text{mod-}R} X/Y_i$. Since A is an AB5*-category, we know $\lim_{\leftarrow A} X/Y_i = X/\bigcap Y_i$ (Corollary 8). By Proposition 9 $\lim_{\leftarrow \text{dis } A} X/Y_i = \lim_{\leftarrow A} X/Y_i = X/\bigcap Y_i$. We claim that $\lim_{\leftarrow \text{dis } A} X/Y_i = \lim_{\leftarrow \text{mod-}R} X/Y_i$. As $X \in A$, it is a submodule of a finitely generated module. As the A-topology has a basis of two-sided ideals, there exists an ideal V open in the A-topology such that $XV = 0$. Now $\lim_{\leftarrow \text{mod-}R} X/Y_i$ is a submodule of $\Pi_{\text{mod-}R}\, X/Y_i$. But $(\Pi_{\text{mod-}R}\, X/Y_i)V = 0$; hence $\lim_{\leftarrow \text{mod-}R} X/Y_i$ is an object of dis A. Therefore, $\lim_{\leftarrow \text{mod-}R} X/Y_i = \lim_{\leftarrow \text{dis } A} X/Y_i = X/\bigcap Y_i$, and X is linearly compact.

COROLLARY 11. *If R is commutative, a meager finitely closed* AB5*-*subcategory of* mod-R *is linearly compact.*

5. The Leptin topology. Let τ be a Hausdorff linear topology on a module X. Following Bourbaki [**2**, Chapter III, 2, Exercise 18], we define τ^* to be the linear topology on X with fundamental system of neighbourhoods of zero the filter basis generated by the submodules of X which are open under τ and completely-meet-irreducible. τ^* is sometimes called the *Leptin Topology*.

For a ring with topology τ, dis τ is the Grothendieck category of all discrete topological R-modules.

PROPOSITION 12. *Let τ be a linearly compact Hausdorff topology on R. If $Q_R \in$* dis τ^* *is an injective cogenerator of* dis τ^* *which is essential over its socle,*

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then Q_R is an injective cogenerator of dis τ. *Also if $M \in$* dis τ *is essential over its socle, then $M \in$* dis τ^*.

Proof. Let $M \in$ dis τ be essential over its socle. If $x \in M$, then xR is essential over its socle. As xR is linearly compact, socle(xR) is finite and $\mathrm{ann}_R(x)$ is the finite intersection of completely-meet-irreducible right ideals of R which are open for τ^*. This implies that M is an object of dis τ^*.

As $\tau^* \subseteq \tau$ clearly dis $\tau^* \subseteq$ dis τ. Moreover, dis τ^* and dis τ have the same simple modules. Let $E(Q_R)$ be the injective hull of Q_R in dis τ. Since $E(Q_R)$ is essential over its socle, it is an object of dis τ^*, hence equals Q_R.

Definition. Two Hausdorff linear topologies on X, τ_1 and τ_2, are *Leptin equivalent* if $\tau_1^* = \tau_2^*$.

COROLLARY 13. *Let τ_1 and τ_2 be two Leptin equivalent Hausdorff topologies on R, and let Q_R be essential over its socle. Then Q_R is an injective cogenerator in* dis τ_1 *if and only if it is an injective cogenerator in* dis τ_2.

Remarks. (1) Let A and A' be two finitely closed subcategories of mod-R. The A-topology equals the A'-topology if and only if A and A' contain the same finitely generated modules. Also the Leptin A-topology equals the Leptin A'-topology if and only if A and A' have the same finitely generated finitely cogenerated modules.

(2) A completely-meet-irreducible submodule of a linearly topologized module is closed if and only if it is open. Thus two topologies τ_1 and τ_2 are Leptin equivalent if and only if they have the same submodules closed. If τ_1 and τ_2 are Leptin equivalent and τ_1 is topological linearly compact, then τ_2 is also.

6. Results on rings.

LEMMA 14. (Goblot [**5**], théorème 2, page 1213). *Let A be a finitely closed subcategory of* mod-R, *Q_R an injective cogenerator of* dis A *which is essential over its socle, $S =$* End Q_R *and $B = \hom_R(A, Q)$. If A is an* AB5*-*category with no infinite direct sums, then B is a finitely closed linearly compact subcategory of* S-mod, *and* $\hom_R(-, Q) : A \to B$ *is a duality.*

Definition. For A, a finitely closed subcategory of mod-R let $A_F = \{Y|$ there exists $X \in A$ finitely generated such that $Y \subseteq X$ (as modules)$\}$.

THEOREM 15. *Let A be a finitely closed* AB5*-*subcategory of* mod-R *with no infinite direct sums. If A_F is linearly compact, then A is linearly compact.*

Proof. By the discussion following Proposition 6, we may assume that R is complete and Hausdorff in the A-topology.

Let Q_R be an injective cogenerator in dis A which is essential over its socle, $S =$ End Q_R and $B = \hom_R(A, Q)$. By lemma 14, B is a finitely closed linearly compact subcategory of S-mod and $\hom_R(-, Q)$ is a duality between A and B.

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Note that since Q is the union of submodules in A, the B-topology on S is Hausdorff.

Since B is dual to a finitely closed subcategory and is linearly compact, it is an AB5*-category with no infinite direct sums. If we prove that ${}_SQ$ is an injective cogenerator of dis B which is essential over its socle and that $R = \text{End}_S Q$, then again by Lemma 14 $\text{hom}_S(B, Q)$ consists of linearly compact R-modules. This proves the theorem; for if $X \in A$, then $i : X \to X^{**} = \text{hom}_S(\text{hom}_R(X, Q), Q)$ is a monomorphism, but $X^{**} \in \text{hom}_S(B, Q)$ is linearly compact, thus X is also.

Claim 1. $R = \text{End}_S Q$.

Let $D = \text{hom}_R(A_F, Q)$. As A_F is finitely closed, D is finitely closed. Moreover, D is contained in B. Since A_F is linearly compact, the duality between A_F and D is represented by the bimodule ${}_SQ_R$ (Sandomierski [**9**], Theorem 3.8). As $R/V \in A$ implies $R/V \in A_F$, we see that dis A = dis A_F. Since $Q \in$ dis A_F, ${}_SQ$ is an injective cogenerator of dis D which is essential over its socle (see Theorem 5 and the remark preceding it). By Proposition 6, $R = \text{End}_S Q$.

Claim 2. ${}_SQ$ is an injective cogenerator of dis B which is essential over its socle.

We showed above that ${}_SQ$ is an injective cogenerator of dis D which is essential over its socle. As A_F contains all finitely generated modules of A, D contains all finitely cogenerated modules of B. Thus the B-topology is Leptin equivalent to the D-topology. From Corollary 13, ${}_SQ$ is an injective cogenerator of dis B which is essential over its socle.

COROLLARY 16. *A finitely closed* AB5*-*subcategory A of* mod-R *without infinite sums is linearly compact in each of the following situations:*

(1) *R is topological linearly compact and* $A \subseteq$ dis R.

(2) *R is commutative.*

(3) *R is a right noetherian and right fully bounded ring.*

(4) *R is semi-artinian.*

Proof. (1) $X \in$ dis τ is finitely generated if and only if there exist $V_i \in \tau$, $i = 1, \ldots, n$, such that we have an exact sequence $R/V_i \oplus \ldots \oplus F/V_n \to X \to 0$. Since each R/V_i is linearly compact, X is linearly compact. Thus A_F is linearly compact. Now apply Theorem 15.

(2) A_F is a meager finitely closed AB5-subcategory, thus by Corollary 11, A_F is linearly compact. Now apply Theorem 15.

(3) A right noetherian ring R is right fully bounded if and only if it has condition (H); that is, for every right ideal I there exist $b_1, \ldots, b_n \in R$ such that $\text{ann}_R(R/I) = b_1^{-1}I \cap \ldots \cap b_n^{-1}I$ (Gabriel [**4**], Lemma 2, page 423 and Cauchon [**3**], Corollary 2, page 1156). As $R/I \in A$ implies $R/I \in A_F$, the A-topology on R is also the A_F-topology. If I is open for the A-topology $x^{-1}I$ is open for all $x \in R$. Note that $\text{ann}_R(R/I)$ is the largest two sided ideal contained in I. By condition (H), there exist $b, \ldots, b_n$ such that $\text{ann}_R(R/I) = b_1^{-1}I \cap \ldots$

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$\cap\, b_n^{-1}I$. As finite intersections of open ideals are open, $\text{ann}_R(R/I)$ is open. Thus the A-topology has a basis of two sided ideals. Since A_F is a meager finitely closed AB5*-subcategory of mod-R, A_F is linearly compact (Proposition 10). Now apply Theorem 15.

(4) This result does not depend on Theorem 15. A ring R is semi-artinian if every right R-module is essential over its socle. If $X \in A$, then the socle of X is finite and X is a finitely cogenerated module.

Let Q be an injective cogenerator of dis A which is essential over its socle, $S = \text{End}\, Q_R$ and $B = \text{hom}_R(A, Q)$. By Lemma 14, B is a finitely closed subcategory of S-mod and $\text{hom}_R(-, Q) : A \to B$ is a duality. For $X \in A$, X is complete in the Q-topology since it is finitely cogenerated, and therefore X is Q-reflexive (Mueller [**7**], Lemma 1, page 61). Hence ${}_SQ_R$ represents the duality between A and B and thus A is linearly compact (Theorem 4).

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