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Two inequalities for convex sets in the plane

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Let K be a bounded, closed, convex set in the euclidean plane having diameter d, width w, inradius r, and circumradius R. We show that

$$(w-2r)d \leq 2\sqrt{3}\, r^2$$

and

$$w(2R-d) \leq \sqrt{3}\,(2-\sqrt{3})R^2 ,$$

where both these inequalities are best possible.

Let K be a bounded, closed, convex set in the euclidean plane. We denote the area, perimeter, diameter, (minimal) width, inradius, and circumradius of K by A, p, d, w, r, and R respectively. There are many known inequalities amongst the quantities A, p, d, w, r, and R (see, for example, [1], [2]). The two inequalities established in the present paper appear to be new.

THEOREM 1. $(w-2r)d \leq 2\sqrt{3}\, r^2$, *with equality when and only when* K *is an equilateral triangle of side length* $2\sqrt{3}\, r$.

Proof. We observe that a largest circle inscribed in K must either contain two boundary points of K which are ends of a diameter of the circle, or else it contains three boundary points U, V, W of K which form the vertices of an acute angled triangle (see, for example, [3]).

In the first case, $w = 2r$, and the theorem is trivially true. In

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the second case, the tangents to the circle at U, V, W form a triangle, ΔXYZ say. Since K is convex, K is contained in ΔXYZ. In fact, since we are interested in maximizing the width and diameter of K, we may take K to be ΔXYZ.

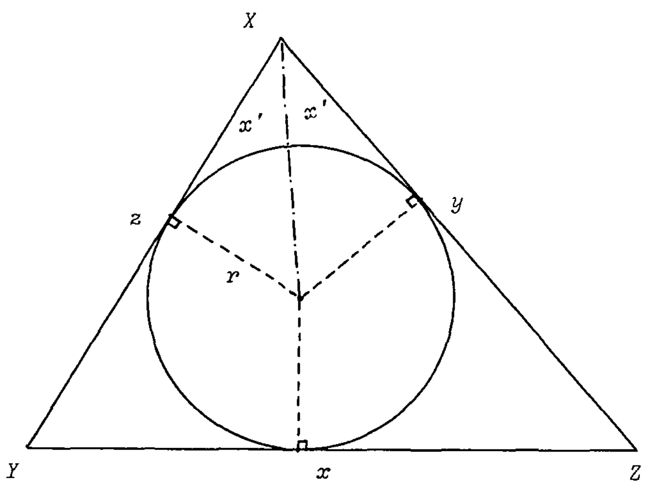
We notice that for a triangle, the diameter is the length of a longest side, and the width is the altitude to that side. Thus

$$wd = 2A = pr$$

and

$$(w-2r)d = r(p-2d) .$$

Hence it is sufficient for us to maximize $p - 2d$ for a fixed value of r.



Using the notation in the diagram, let us assume that $x \geq y \geq z$.

Now

$$\begin{aligned} p - 2d &= (x+y+z) - 2x \\ &= y + z - x \\ &= 2x' . \end{aligned}$$

Since $r = x' \tan(X/2)$, and r is fixed, the maximum value of x' will be assumed when $\angle X$ is as small as possible, subject to the constraint $x \geq y \geq z$. This occurs when $\angle X = \pi/3$ ($= \angle Y = \angle Z$); that is, when and only when ΔXYZ is equilateral. For this equilateral triangle,

$d = 2\sqrt{3}\ r$, $w = 3r$, $(w-2r)d = 2\sqrt{3}\ r^2$.

Hence for any convex set K ,

$$(w-2r)d \le 2\sqrt{3}\ r^2 ,$$

as required.

THEOREM 2. *$w(2R-d) \le \sqrt{3}(2-\sqrt{3})R^2$, with equality when and only when K is a Reuleaux triangle of width $\sqrt{3}\ R$.*

Proof. It is known [3] that if K has circumradius R , then $\sqrt{3}\ R \le d \le 2R$. Also, $w \le d$, so for any d ,

$$(2R-d)w \le (2R-d)d .$$

Now $f(d) = (2R-d)d$ is a decreasing function of d , and so takes its maximum value for $d = \sqrt{3}\ R$. Hence

$$(2R-d)w \le (2-\sqrt{3})\sqrt{3}\ R^2 .$$

For equality here we require a set K having $w = d$; that is, K must be a set of constant width. Finally, it is known, [3], that the only set of constant width which satisfies $d = \sqrt{3}\ R$ is the Reuleaux triangle of width $\sqrt{3}\ R$. This completes the proof.

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