



On Multilinear Fourier Multipliers of Limited Smoothness

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Abstract. In this paper we prove a certain L^2 -estimate for multilinear Fourier multiplier operators with multipliers of limited smoothness. As a consequence, we extend the result of Calderón and Torchinsky in the linear theory to the multilinear case. The sharpness of our results and some related estimates in Hardy spaces are also discussed.

1 Introduction

The area of multilinear harmonic analysis originated in the fundamental work of Coifman and Meyer [4–6]. This area remained unexplored until about the late nineties when certain important advances were made. These advances are too numerous all to be included in this introduction, so we only mention the articles of Bényi and Torres [1], Grafakos and Torres [12], Kenig and Stein [16], and Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [17]. The results contained in these and in other known articles in the area concern multilinear operators whose kernels have an explicit form or satisfy some pointwise estimates (and their derivatives also satisfy analogous pointwise estimates). In this paper, we shall consider multilinear Fourier multiplier operators whose multipliers have limited smoothness described in terms of a function space and not in a pointwise form.

We use the following notations. For Schwartz functions f on \mathbb{R}^d , we define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$$

and the inverse Fourier transform by

$$\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi.$$

For functions m on \mathbb{R}^d and $j \in \mathbb{Z}$, we define

$$(1.1) \quad m_j(\xi) = m(2^j \xi) \Psi(\xi),$$

where we fix a $\Psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$(1.2) \quad \text{supp } \Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}, \quad \sum_{k \in \mathbb{Z}} \Psi(\xi/2^k) = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

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We first recall the linear Fourier multiplier operators. For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator T_m is defined by

$$T_m f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} m(\xi) \widehat{f}(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The Mihlin multiplier theorem says that if m satisfies the differential estimates up to the order “[half of dimension]+1”

$$(1.3) \quad |\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for all } |\alpha| \leq [n/2] + 1,$$

then T_m is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. The Hörmander multiplier theorem [13] states that if $s > n/2$ and $m \in L^\infty(\mathbb{R}^n)$ satisfies

$$(1.4) \quad \sup_{j \in \mathbb{Z}} \|m_j\|_{W^s(\mathbb{R}^n)} < \infty,$$

then T_m is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, where $W^s(\mathbb{R}^n)$ is the Sobolev space (see Section 2 for the definition). The Hörmander multiplier theorem improves the Mihlin multiplier theorem, since condition (1.4) with $n/2 < s < [n/2]+1$ is certainly weaker than (1.3). Calderón and Torchinsky [2] extended the Hörmander multiplier theorem to the case $p \leq 1$; they proved that if $0 < p \leq 1$ and if $m \in L^\infty(\mathbb{R}^n)$ satisfies (1.4) with $s > n(1/p - 1/2)$, then T_m is bounded on the Hardy space $H^p(\mathbb{R}^n)$. The case $p = 1$ is due to Fefferman and Stein [8].

Now we shall consider the multilinear case. Let N be an integer strictly bigger than one. For $m \in L^\infty(\mathbb{R}^{Nn})$, the N -linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix \cdot (\xi_1 + \dots + \xi_N)} m(\xi) \widehat{f}_1(\xi_1) \cdots \widehat{f}_N(\xi_N) d\xi$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $d\xi = d\xi_1 \cdots d\xi_N$. If $\mathcal{F}^{-1}m$, the inverse Fourier transform of m on \mathbb{R}^{Nn} , is an integrable function, then this can also be written as

$$(1.5) \quad T_m(f_1, \dots, f_N)(x) = \int_{\mathbb{R}^{Nn}} \mathcal{F}^{-1}m(x - y_1, \dots, x - y_N) f_1(y_1) \cdots f_N(y_N) dy.$$

This representation of T_m is often valid even when $\mathcal{F}^{-1}m$ is not an integrable function via a principal value integral interpretation.

Coifman and Meyer [5] proved that if $m \in C^L(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies

$$(1.6) \quad |\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \leq C_{\alpha_1, \dots, \alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1| + \dots + |\alpha_N|)}$$

for $|\alpha_1| + \dots + |\alpha_N| \leq L$ with L sufficiently large, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N \leq \infty$ and $1 < p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$. Kenig and Stein [16] and Grafakos and Torres [12] extended the result to the range $p \leq 1$. Finding the best possible L in these results is an important question that arises in applications.

Example It is well known that the Kato–Ponce inequality (see [15]) can be studied via multilinear analysis. Let $\widehat{D^s(f)}(\xi) = \widehat{f}(\xi)|\xi|^s$ with $s > 0$ and $1/p = 1/p_1 + \dots + 1/p_N$. In the study of the N -linear Kato–Ponce type inequality

$$(1.7) \quad \|D^s(f_1 \cdots f_N)\|_{L^p} \leq C \sum_{i=1}^N \|D^s(f_i)\|_{L^{p_i}} \prod_{1 \leq j \neq i \leq N} \|f_j\|_{L^{p_j}}$$

via Littlewood–Paley theory, the following N -linear multiplier arises:

$$m(\xi) = \sum_{j \geq 0} 2^{-sj} \sum_{k \in \mathbb{Z}} \Theta(2^{-k}(\xi_1 + \dots + \xi_N)) \Psi_1(2^{-(j+k)} \xi_1) \cdots \Psi_N(2^{-(j+k)} \xi_N)$$

where Ψ_1, \dots, Ψ_N , and Θ are smooth functions supported in some annulus. It is straightforward to verify that m satisfies condition (1.6) for $|\alpha_1| + \dots + |\alpha_N| < s$ but not for larger $|\alpha_1| + \dots + |\alpha_N|$, and thus the smoothness of m is “limited”. This example plays a motivating role in the theory developed hereby, and its connection with (1.7) is discussed in Appendix B.

The L given in [5] is strictly greater than $2Nn$ and this seems to be too large compared with the case of linear operators. It will be natural to expect that we can take $L = \lceil \text{[half of dimension]} + 1 \rceil = \lceil Nn/2 \rceil + 1$. In fact, Tomita [21] recently proved that if $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$(1.8) \quad \sup_{j \in \mathbb{Z}} \|m_j\|_{W^s(\mathbb{R}^{Nn})} < \infty$$

with $s > Nn/2$, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \dots + 1/p_N = 1/p$. Grafakos and Si [11] extended the result to the case $p \leq 1$ by using the L^r -based Sobolev space, $1 < r \leq 2$.

In this paper, we shall consider multipliers that satisfy (1.8) with the product type Sobolev space $W^{(s_1, \dots, s_N)}(\mathbb{R}^{Nn})$ (for the definition, see Section 2) in place of $W^s(\mathbb{R}^{Nn})$. We shall prove a basic L^2 -estimate and give an extension of the Calderón–Torchinsky multiplier theorem to the multilinear case. We also give extensions and improvements of the results of [11, 21].

The following is the first main result, which gives the basic L^2 -estimate.

Theorem 1.1 Assume that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, \dots, s_N)}(\mathbb{R}^{Nn})} < \infty \quad \text{with } s_1, \dots, s_N > n/2.$$

Then T_m is bounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \times \dots \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

In the case $N = 2$, Theorem 1.1 with $W^{(s_1, s_2)}(\mathbb{R}^{2n})$ replaced by $W^s(\mathbb{R}^{2n})$ with $s > n$ follows from the result of Grafakos and Si [11]. But there is a difference between $N = 2$ and $N \geq 3$, and the argument of [11] cannot be applied to the case $N \geq 3$ (even if the product type Sobolev norm $\|\cdot\|_{W^{(s_1, \dots, s_N)}(\mathbb{R}^{Nn})}$ is replaced by the usual

Sobolev norm $\|\cdot\|_{W^{s_1+\dots+s_N}(\mathbb{R}^{Nn})}$). In the case $N = 2$, it follows from duality that the boundedness of T_m from $L^2 \times L^\infty$ to L^2 is equivalent to that of $T_{m^{*2}}$ from $L^2 \times L^2$ to L^1 , where m^{*2} is the multiplier of the dual operator with respect to the second variable (see Section 8), and, by using m^{*2} instead of m , we do not need to treat L^∞ . However, in the case $N = 3$, the boundedness of T_m from $L^2 \times L^\infty \times L^\infty$ to L^2 is equivalent to that of $T_{m^{*2}}$ (resp. $T_{m^{*3}}$) from $L^2 \times L^2 \times L^\infty$ to L^1 (resp. from $L^2 \times L^\infty \times L^2$ to L^1), and we cannot remove L^∞ . Our proof of Theorem 1.1 does not use this duality and can be applied to all $N \geq 2$. Notice also that in the framework of Sobolev spaces of product type, we cannot use the duality argument (see Section 8).

Using Theorem 1.1, we extend the multiplier theorem of Calderón and Torchinsky [2] to the multilinear case. The following is the second main result.

Theorem 1.2 *Let $0 < p \leq 1$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, \dots, s_N)}} < \infty \quad \text{with } s_1 > n(1/p - 1/2), \quad s_2, \dots, s_N > n/2,$$

then T_m is bounded from $H^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \times \dots \times L^\infty(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

We shall also prove that the numbers $n/2$ and $n(1/p - 1/2)$ in Theorems 1.1 and 1.2 are sharp; see Propositions 7.1 and 7.2. In a recent paper [18], Theorems 1.1 and 1.2 are used as key tools to determine the minimal smoothness conditions on bilinear Fourier multipliers to assure the boundedness of the corresponding operators from $H^{p_1} \times H^{p_2}$ to L^p , $0 < p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$.

From Theorems 1.1 and 1.2, by interpolation, we also obtain the boundedness of multilinear Fourier multiplier operators in

$$H^{p_1} \times H^{p_2} \times \dots \times H^{p_N} \rightarrow L^p, \quad 0 < p_j \leq \infty, \quad 0 < p \leq 2, \quad \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_N} = \frac{1}{p}.$$

The results include some extensions and improvements of the results of [11, 21]. For details, see Theorem 6.1 and Section 8.

It should be mentioned that Grafakos and Kalton [10] considered multilinear Calderón–Zygmund operators, and proved the boundedness of the operators on Hardy spaces. The definition of Calderón–Zygmund operators in [10], however, contains pointwise estimate of kernels, whereas the kernels of the multipliers of our theorems do not have pointwise estimates in general. Hence our results do not follow from the general results in [10].

We explain some ideas of the proofs of the main theorems. In the proof of Theorem 1.1, using a partition of unity with respect to $\xi/|\xi|$ and using the usual dyadic decomposition with respect to $|\xi|$, we reduce the problem to the case when m has appropriate compact support. One of our main tools is a pointwise estimate of $T_m(f_1, f_2, \dots, f_N)(x)$ for compactly supported m , which will be given in Lemma 3.3. Another main tool is a modified version of the Carleson measure estimate related to BMO functions, which will be given in Lemmas 3.1 and 3.2. After Theorem 1.1 is established, Theorem 1.2 can be proved by a rather straightforward generalization of the method used in the case of linear Fourier multiplier operators.

The paper is organized as follows. Sections 2 and 3 contain definitions and preliminary lemmas. In Sections 4 and 5, we prove Theorems 1.1 and 1.2 respectively. In Section 6, we use interpolation to give results on the boundedness from $H^{p_1} \times H^{p_2} \times \dots \times H^{p_N}$ to L^p with $p \leq 1$. In Section 7, sharpness of the conditions of Theorems 1.1 and 1.2 is discussed. In Section 8, we comment on some results for the case $p > 1$.

2 Preliminaries

Throughout this paper, the letter C will denote a constant that may be different in each occasion but is independent of the essential variables. The set of all non-negative integers is denoted by \mathbb{N}_0 . For $1 \leq p \leq \infty$, p' is the conjugate exponent of p , that is, $1/p + 1/p' = 1$. The symbols $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing smooth functions and the space of tempered distributions, respectively. As usual, for a function ψ on \mathbb{R}^n and $t > 0$, we write $\psi_t(x) = t^{-n}\psi(x/t)$.

The Sobolev space $W^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W^s} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We also use the Sobolev space of product type $W^{(s_1, \dots, s_N)}(\mathbb{R}^{Nn})$, $(s_1, \dots, s_N) \in \mathbb{R} \times \dots \times \mathbb{R}$, which is defined by the norm

$$\|F\|_{W^{(s_1, \dots, s_N)}} = \left(\int_{\mathbb{R}^{Nn}} \langle \xi_1 \rangle^{2s_1} \dots \langle \xi_N \rangle^{2s_N} |\widehat{F}(\xi)|^2 d\xi \right)^{1/2},$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $d\xi = d\xi_1 \dots d\xi_N$. For $s \in \mathbb{R}$, we set

$$w_s(x) = \langle x_1 \rangle^s \dots \langle x_N \rangle^s, \quad x = (x_1, \dots, x_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n.$$

The weighted Lebesgue space $L^q(w_s)$ consists of all measurable functions F on \mathbb{R}^{Nn} such that

$$\|F\|_{L^q(w_s)} = \left(\int_{\mathbb{R}^{Nn}} |F(x)|^q \langle x_1 \rangle^s \dots \langle x_N \rangle^s dx \right)^{1/q} < \infty.$$

We recall the definition and some properties of Hardy spaces on \mathbb{R}^n (see [19, Chapter 3]). Let $0 < p \leq \infty$, and let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \Phi(x) dx \neq 0$. Then the Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p} = \left\| \sup_{0 < t < \infty} |\Phi_t * f| \right\|_{L^p} < \infty.$$

It is known that $H^p(\mathbb{R}^n)$ does not depend on the choice of the function Φ ([19, Chapter 3, Theorem 1]). If $1 < p \leq \infty$, then $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ ([19, Chapter 3, Section 1.2]). For $0 < p \leq 1$, a function a on \mathbb{R}^n is called an H^p -atom if there exists a cube $Q = Q_a$ such that

$$\text{supp } a \subset Q, \quad \|a\|_{L^\infty} \leq |Q|^{-1/p}, \quad \int_{\mathbb{R}^n} x^\alpha a(x) dx = 0 \text{ for } |\alpha| \leq [n(1/p - 1)],$$

where $|Q|$ is the Lebesgue measure of Q and $[n(1/p - 1)]$ is the integer part of $n(1/p - 1)$. It is known that every $f \in H^p(\mathbb{R}^n)$ can be written as an infinite sum $f = \sum_{i=1}^\infty \lambda_i a_i$ convergent in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_i\}$ is a collection of H^p -atoms and $\{\lambda_i\}$ is a sequence of complex numbers with $\sum_{i=1}^\infty |\lambda_i|^p < \infty$. Moreover,

$$C^{-1} \inf \left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p} \leq \|f\|_{H^p} \leq C \inf \left(\sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p},$$

where the infimum is taken over all representations of f ([19, Chapter 3, Theorem 2]).

We denote by $BMO(\mathbb{R}^n)$ the space of all locally integrable functions f on \mathbb{R}^n that satisfy

$$\|f\|_{BMO} = \sup \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where f_Q is the average of f over Q and the supremum is taken over all cubes Q in \mathbb{R}^n . A positive measure ν on $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ is said to be a *Carleson measure* if there exists a constant $A > 0$ such that

$$\nu(Q \times (0, \ell(Q))) \leq A|Q| \quad \text{for all cubes } Q \text{ in } \mathbb{R}^n,$$

where $\ell(Q)$ is the side length of Q . The infimum of the possible values of the constant A is called the *Carleson constant* of ν and is denoted by $\|\nu\|$.

We end this section by quoting the following facts, which will be used in the sequel.

Lemma 2.1 *Let $2 \leq q < \infty$, $r > 0$, and $s \geq 0$. Then there exists a constant $C > 0$ such that*

$$\|\widehat{F}\|_{L^q(w_{sq})} \leq C \|\widehat{F}\|_{L^2(w_{2s})} = C \|F\|_{W^{(s,\dots,s)}}$$

for all $F \in W^{(s,\dots,s)}(\mathbb{R}^{Nn})$ with $\text{supp } F \subset \{|x| \leq r\}$.

Lemma 2.1 is a simple case of [9, Lemma A.1], but we shall give a proof for the reader's convenience in Appendix A.

Proposition 2.2 *If $s_j > n/2$ for $1 \leq j \leq N$, then $W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$ is an algebra under pointwise multiplication.*

The proof of Proposition 2.2 is also given in Appendix A for the reader's convenience.

3 Lemmas

In this section, we prepare the lemmas that will be used in the proof of Theorem 1.1.

Lemma 3.1 *Let $b \in BMO(\mathbb{R}^n)$, $\zeta(x) = (1 + |x|)^{-(n+\epsilon)}$ with $\epsilon > 0$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \psi(x) dx = 0$. Then the measure ν defined by*

$$d\nu = (\zeta_t * |\psi_t * b|^2)(x) \frac{dx dt}{t}$$

is a Carleson measure with Carleson constant $\|\nu\| \leq C \|b\|_{BMO}^2$.

Proof Since $\|b(\cdot + x_0)\|_{\text{BMO}} = \|b(r\cdot)\|_{\text{BMO}} = \|b\|_{\text{BMO}}$, and since

$$\int_{x_0+Q} \int_0^{\ell(x_0+Q)} (\zeta_t * |\psi_t * b|^2)(x) \frac{dt dx}{t} = \int_Q \int_0^{\ell(Q)} (\zeta_t * |\psi_t * [b(\cdot + x_0)]|^2)(x) \frac{dt dx}{t}$$

and

$$\begin{aligned} & \int_{[-r,r]^n} \int_0^{2r} (\zeta_t * |\psi_t * b|^2)(x) \frac{dt dx}{t} \\ &= r^n \int_{[-1,1]^n} \int_0^{2r} (\zeta_{t/r} * |\psi_{t/r} * [b(r\cdot)]|^2)(x) \frac{dt dx}{t} \\ &= 2^{-n} |[-r, r]^n| \int_{[-1,1]^n} \int_0^2 (\zeta_t * |\psi_t * [b(r\cdot)]|^2)(x) \frac{dt dx}{t}, \end{aligned}$$

it is enough to prove

$$\int_{[-1,1]^n} \int_0^2 (\zeta_t * |\psi_t * b|^2)(x) \frac{dt dx}{t} \leq C \|b\|_{\text{BMO}}^2.$$

We recall the fact that if μ is a Carleson measure, then

$$(3.1) \quad \int_{\mathbb{R}_+^{n+1}} |F(x, t)| d\mu(x, t) \leq C \|\mu\| \int_{\mathbb{R}^n} F^*(x) dx,$$

where F^* is the nontangential maximal function of F , which is defined by

$$F^*(x) = \sup_{|x-y|<t} |F(y, t)|$$

(see, e.g., [19, Theorem 2, p. 59] or [7, Proof of Theorem 9.5]). On the other hand, it is well known that $|\psi_t * b(x)|^2 \frac{dx dt}{t}$ is a Carleson measure with Carleson constant dominated by $C \|b\|_{\text{BMO}}^2$ (see, e.g., [7, Theorem 9.6]). Hence,

$$\begin{aligned} & \int_{[-1,1]^n} \int_0^2 (\zeta_t * |\psi_t * b|^2)(x) \frac{dt dx}{t} \\ &= \int_{\mathbb{R}_+^{n+1}} \chi_{[-1,1]^n \times [0,2]}(x, t) \left(\int_{\mathbb{R}^n} \zeta_t(x-y) |\psi_t * b(y)|^2 dy \right) \frac{dt dx}{t} \\ &= \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}^n} \zeta_t(x-y) \chi_{[-1,1]^n \times [0,2]}(x, t) dx \right) |\psi_t * b(y)|^2 \frac{dt dy}{t} \\ &\leq C \|b\|_{\text{BMO}}^2 \int_{\mathbb{R}^n} G^*(x) dx, \end{aligned}$$

where $\chi_{[-1,1]^n \times [0,2]}$ is the characteristic function of $[-1, 1]^n \times [0, 2]$ and $G^*(x)$ is the nontangential maximal function of

$$G(y, t) = \int_{\mathbb{R}^n} \zeta_t(z - y) \chi_{[-1,1]^n \times [0,2]}(z, t) dz.$$

Thus it is sufficient to show that G^* is integrable on \mathbb{R}^n . Note that $G(y, t) = 0$ for $t > 2$, and $G(y, t) \leq \|\zeta\|_{L^1}$ for $y \in \mathbb{R}^n$ and $t \leq 2$. Moreover, since $|y - z| \geq |y|/2$ for $|y| \geq 2\sqrt{n}$ and $z \in [-1, 1]^n$, we have $G(y, t) \leq 2^n t^{-n} (|y|/2t)^{-(n+\epsilon)} \leq C|y|^{-(n+\epsilon)}$ for $|y| \geq 2\sqrt{n}$ and $t \leq 2$. Therefore $G^*(x) \leq C(1 + |x|)^{-(n+\epsilon)}$, and consequently G^* is integrable on \mathbb{R}^n . The proof is complete. ■

Lemma 3.2 *Let $1 < q < 2$, $\zeta(x) = (1 + |x|)^{-(n+\epsilon)}$ with $\epsilon > 0$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \psi(x) dx = 0$. Then*

$$(3.2) \quad \int_{\mathbb{R}^{n+1}} |\psi_t * f(x)|^2 \frac{dxdt}{t} \leq C \|f\|_{L^2}^2,$$

$$(3.3) \quad \int_{\mathbb{R}^{n+1}} (\zeta_t * |\psi_t * f|^q)(x)^{2/q} \frac{dxdt}{t} \leq C \|f\|_{L^2}^2,$$

$$(3.4) \quad \int_{\mathbb{R}^{n+1}} (\zeta_t * |f|^q)(x)^{2/q} (\zeta_t * |\psi_t * g|^q)(x)^{2/q} \frac{dxdt}{t} \leq C \|f\|_{L^2}^2 \|g\|_{\text{BMO}}^2,$$

$$(3.5) \quad \int_{\mathbb{R}^{n+1}} (\zeta_t * |f|)(x)^2 (\zeta_t * |\psi_t * g|^q)(x)^{2/q} \frac{dxdt}{t} \leq C \|f\|_{L^2}^2 \|g\|_{\text{BMO}}^2.$$

Proof The inequality (3.2) is well known and is easily proved by an application of Plancherel’s theorem; see, e.g., [5, p. 148].

To prove (3.3), observe that Young’s inequality gives

$$\int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f|^q)(x)^{2/q} dx \leq (\|\zeta_t\|_{L^1} \|\psi_t * f^q\|_{L^{2/q}})^{2/q} = C \int_{\mathbb{R}^n} |\psi_t * f(x)|^2 dx.$$

Integrating over $0 < t < \infty$ and using (3.2), we obtain (3.3).

To prove (3.4), observe that Hölder’s inequality gives

$$(\zeta_t * |\psi_t * g|^q)(x)^{2/q} \leq (\|\zeta_t\|_{L^1}^{1-q/2} (\zeta_t * |\psi_t * g|^2)(x)^{q/2})^{2/q} = C (\zeta_t * |\psi_t * g|^2)(x).$$

Thus, by Lemma 3.1, the measure

$$d\nu = (\zeta_t * |\psi_t * g|^q)(x)^{2/q} \frac{dxdt}{t}$$

is a Carleson measure with Carleson constant $\leq C \|g\|_{\text{BMO}}^2$. Hence, by (3.1), the left-hand side of (3.4) is majorized by

$$C \|g\|_{\text{BMO}}^2 \int_{\mathbb{R}^n} \sup_{|z-x|<t} ((\zeta_t * |f|^q)(x))^{2/q} dz = U.$$

But the integrand of the above integral is majorized by $CM(|f|^q)(z)^{2/q}$ with M denoting the Hardy-Littlewood maximal operator (see, e.g., [19, Chapter 1, Section 2.1] or [7, Chapter 2, Section 8.7]). Hence, since $q < 2$, the maximal theorem gives

$$U \leq C \|g\|_{\text{BMO}}^2 \| |f|^q \|_{L^{2/q}}^{2/q} = C \|g\|_{\text{BMO}}^2 \|f\|_{L^2}^2.$$

Finally, (3.5) follows from (3.4), since Hölder’s inequality gives

$$(\zeta_t * |f|)(x)^2 \leq (\|\zeta_t\|_{L^1}^{1-1/q} (\zeta_t * |f|^q)(x)^{1/q})^2 = C (\zeta_t * |f|^q)(x)^{2/q}.$$

The proof of Lemma 3.2 is complete. ■

Using the idea given in [21], we prove the following lemma.

Lemma 3.3 *Let $s > n/2$, $\max\{1, n/s\} < q < 2$ and $\zeta(x) = (1 + |x|)^{-sq}$. Suppose $m \in W^{(s, \dots, s)}(\mathbb{R}^{Nn})$, $t > 0$ and $\text{supp } m \subset \{|\xi| \leq 2/t\}$. Then there exists a constant $C > 0$ depending only on N, n, s and q such that*

$$|T_m(f_1, \dots, f_N)(x)| \leq C \|m(\cdot/t)\|_{W^{(s, \dots, s)}} (\zeta_t * |f_1|^q)(x)^{1/q} \dots (\zeta_t * |f_N|^q)(x)^{1/q}$$

for all $x \in \mathbb{R}^n$.

Proof We write $m(\xi/t) = \tilde{m}(\xi)$. By (1.5) and by Hölder’s inequality,

$$\begin{aligned} & |T_m(f_1, \dots, f_N)(x)| \\ & \leq \int_{\mathbb{R}^{Nn}} t^{-Nn} |\mathcal{F}^{-1} \tilde{m}((x - y_1)/t, \dots, (x - y_N)/t) f_1(y_1) \dots f_N(y_N)| dy \\ & = t^{-Nn} \int_{\mathbb{R}^{Nn}} (1 + |x - y_1|/t)^s \dots (1 + |x - y_N|/t)^s |\mathcal{F}^{-1} \tilde{m}((x - y_1)/t, \dots, (x - y_N)/t)| \\ & \quad \times (1 + |x - y_1|/t)^{-s} \dots (1 + |x - y_N|/t)^{-s} |f_1(y_1) \dots f_N(y_N)| dy \\ & \leq \left(\int_{\mathbb{R}^{Nn}} (1 + |z_1|)^{sq'} \dots (1 + |z_N|)^{sq'} |\mathcal{F}^{-1} \tilde{m}(z_1, \dots, z_N)|^{q'} dz \right)^{1/q'} \\ & \quad \times \left(\frac{1}{t^{Nn}} \int_{\mathbb{R}^{Nn}} \frac{|f_1(y_1) \dots f_N(y_N)|^q}{(1 + |x - y_1|/t)^{sq} \dots (1 + |x - y_N|/t)^{sq}} dy \right)^{1/q} \\ & \approx \|\mathcal{F}^{-1} \tilde{m}\|_{L^{q'}(w_{sq'})} (\zeta_t * |f_1|^q)(x)^{1/q} \dots (\zeta_t * |f_N|^q)(x)^{1/q}. \end{aligned}$$

Since $\text{supp } \tilde{m} \subset \{|\xi| \leq 2\}$ and $2 < q' < \infty$, Lemma 2.1 gives $\|\mathcal{F}^{-1} \tilde{m}\|_{L^{q'}(w_{sq'})} \leq C \|\tilde{m}\|_{W^{(s, \dots, s)}}$, which, combined with the above inequality, implies the desired estimate. The proof is complete. ■

Lemma 3.4 *Let $s_1, \dots, s_N > n/2$, and let $\tilde{\Psi} \in \mathcal{S}(\mathbb{R}^{Nn})$ be such that $\text{supp } \tilde{\Psi}$ is compact and does not contain the origin. Assume that $\Phi \in C^\infty(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \Phi(\xi_1, \dots, \xi_N)| \leq C_{\alpha_1, \dots, \alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1| + \dots + |\alpha_N|)}$$

for all $\alpha_1, \dots, \alpha_N \in \mathbb{N}_0^n$. Then there exists a constant $C > 0$ such that

$$\sup_{t>0} \|m(t \cdot) \Phi(t \cdot) \tilde{\Psi}\|_{W^{(s_1, \dots, s_N)}} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, \dots, s_N)}}$$

for all $m \in L^\infty(\mathbb{R}^{Nn})$ satisfying $\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, \dots, s_N)}} < \infty$, where m_j is defined by (1.1).

Proof We may assume that $\text{supp } \tilde{\Psi} \subset \{1/2^{j_0} \leq |\xi| \leq 2^{j_0}\}$ for some $j_0 \in \mathbb{N}$. Given $t > 0$, take $j \in \mathbb{Z}$ satisfying $2^{j-1} \leq t < 2^j$. Then, since $1 < 2^j/t \leq 2$, by a change of variables,

$$\|m(t \cdot) \Phi(t \cdot) \tilde{\Psi}\|_{W^{(s_1, \dots, s_N)}} \leq C \|m(2^j \cdot) \Phi(2^j \cdot) \tilde{\Psi}(2^j t^{-1} \cdot)\|_{W^{(s_1, \dots, s_N)}}.$$

Let $\Psi \in \mathcal{S}(\mathbb{R}^{Nn})$ be as in (1.2) with $d = Nn$, and note that

$$\text{supp } \Psi(\cdot/2^k) \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}.$$

Using $\text{supp } \tilde{\Psi}(2^j t^{-1} \cdot) \subset \{1/2^{j_0+1} \leq |\xi| \leq 2^{j_0}\}$, we have, by Proposition 2.2,

$$\begin{aligned} & \|m(2^j \cdot) \Phi(2^j \cdot) \tilde{\Psi}(2^j t^{-1} \cdot)\|_{W^{(s_1, \dots, s_N)}} \\ & \leq \sum_{k=-(j_0+1)}^{j_0} \|m(2^j \cdot) \Phi(2^j \cdot) \tilde{\Psi}(2^j t^{-1} \cdot) \Psi(\cdot/2^k)\|_{W^{(s_1, \dots, s_N)}} \\ & \leq C \sum_{k=-(j_0+1)}^{j_0} \|m(2^j \cdot) \Psi(\cdot/2^k)\|_{W^{(s_1, \dots, s_N)}} \|\Phi(2^j \cdot) \tilde{\Psi}(2^j t^{-1} \cdot)\|_{W^{(s_1, \dots, s_N)}} \\ & \leq C \sum_{k=-(j_0+1)}^{j_0} \|m(2^{j+k} \cdot) \Psi\|_{W^{(s_1, \dots, s_N)}} \|\Phi(t \cdot) \tilde{\Psi}\|_{W^{(s_1, \dots, s_N)}} \\ & \leq C \left(\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, \dots, s_N)}} \right) \left(\sup_{t>0} \|\Phi(t \cdot) \tilde{\Psi}\|_{W^{(s_1, \dots, s_N)}} \right). \end{aligned}$$

Since $|\partial_\xi^\alpha \Phi(t\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$ and $\text{supp } \tilde{\Psi}$ does not contain the origin,

$$\begin{aligned} & \sup_{t>0} \|\Phi(t \cdot) \tilde{\Psi}\|_{W^{(s_1, \dots, s_N)}} \leq \\ & C \sup_{t>0} \left(\sum_{|\alpha_1| \leq [s_1]+1} \dots \sum_{|\alpha_N| \leq [s_N]+1} \|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} (\Phi(t \cdot) \tilde{\Psi})\|_{L^2} \right) < \infty. \end{aligned}$$

The proof is complete. ■

4 The Boundedness from $L^2 \times L^\infty \times \dots \times L^\infty$ to L^2

Proof of Theorem 1.1 We shall give a proof in which the case $N \geq 3$ and the case $N = 2$ are treated in a parallel way (cf. the comments given in the paragraph just below Theorem 1.1). To provide clarity in the exposition, we give the proof only in the typical case $N = 3$. It will be obvious that our argument can be generalized to every $N \geq 2$ with trivial modifications.

If we set $s = \min\{s_1, s_2, s_3\}$, then $W^{(s_1, s_2, s_3)}(\mathbb{R}^{3n}) \hookrightarrow W^{(s, s, s)}(\mathbb{R}^{3n})$. Hence, in the proof of Theorem 1.1, it is sufficient to consider the case $s_1 = s_2 = s_3 > n/2$. Thus we assume $s > n/2$ and consider m that satisfies $\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s, s, s)}} < \infty$. We use the following notations: \mathcal{A}_0 denotes the set of even functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\text{supp } \widehat{\varphi}$ is compact; \mathcal{A}_1 denotes the set of even functions $\psi \in \mathcal{S}(\mathbb{R}^n)$ for which $\text{supp } \widehat{\psi}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$. Notice that the boundedness of T_m from $L^2 \times L^\infty \times L^\infty$ to L^2 is equivalent to the estimate

$$\left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x)dx \right| \leq C \|f_1\|_{L^2} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \|g\|_{L^2}$$

for all $f_i, g \in \mathcal{S}$. We shall use the following identity:

$$(4.1) \quad \theta * T_m(f_1, f_2, f_3)(x) = \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{ix \cdot (\xi_1 + \xi_2 + \xi_3)} \widehat{\theta}(\xi_1 + \xi_2 + \xi_3) m(\xi) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi.$$

We first decompose m into a finite number of multipliers each of which is supported on a cone in \mathbb{R}^{3n} . To do this, consider the unit sphere

$$\Sigma = \{\eta \in \mathbb{R}^{3n} \mid |\eta| = 1\}$$

and notice the following simple fact: for each $\eta \in \Sigma$, at least two of the four \mathbb{R}^n -vectors η_1, η_2, η_3 , and $\eta_1 + \eta_2 + \eta_3$ are not equal to 0. Hence there exists a constant $c_0 > 0$ such that the compact set Σ is covered by the following six open subsets:

$$\begin{aligned} V_1 &= \{\eta \in \Sigma \mid |\eta_1| > c_0, |\eta_1 + \eta_2 + \eta_3| > c_0\}, \\ V_2 &= \{\eta \in \Sigma \mid |\eta_2| > c_0, |\eta_1 + \eta_2 + \eta_3| > c_0\}, \\ V_3 &= \{\eta \in \Sigma \mid |\eta_3| > c_0, |\eta_1 + \eta_2 + \eta_3| > c_0\}, \\ V_4 &= \{\eta \in \Sigma \mid |\eta_1| > c_0, |\eta_2| > c_0\}, \\ V_5 &= \{\eta \in \Sigma \mid |\eta_1| > c_0, |\eta_3| > c_0\}, \\ V_6 &= \{\eta \in \Sigma \mid |\eta_2| > c_0, |\eta_3| > c_0\}. \end{aligned}$$

We write

$$\Gamma(V_i) = \{\xi \in \mathbb{R}^{3n} \setminus \{0\} \mid \xi/|\xi| \in V_i\}.$$

We take functions $\Phi_i, i = 1, 2, \dots, 6$, on \mathbb{R}^{3n} such that each Φ_i is homogeneous of degree 0, smooth away from the origin, $\text{supp } \Phi_i \subset \Gamma(V_i)$, and $\sum_{i=1}^6 \Phi_i(\xi) = 1$ for all $\xi \neq 0$. We decompose m as

$$m(\xi) = \sum_{i=1}^6 m(\xi)\Phi_i(\xi).$$

It is sufficient to prove the boundedness of each $T_{m\Phi_i}$. By Lemma 3.4, we see that each $m\Phi_i$ satisfies

$$\sup_{j \in \mathbb{Z}} \|(m\Phi_i)_j\|_{W^{(s,s)}} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s,s)}}.$$

Thus, in the rest of the proof, writing simply m instead of $m\Phi_i$, we shall assume that the support of our multiplier m is included in one of $\Gamma(V_i)$. By symmetry of the situation, the cases $i = 2$ and $i = 3$ are treated in the same way, and $i = 4$ and $i = 5$ are also treated in the same way. Therefore, we shall only consider the four cases $i = 1, 2, 4, 6$.

We make another decomposition of m . We take a function $\Theta \in C^\infty(\mathbb{R}^{3n})$ such that $\text{supp } \Theta \subset \{1/2 \leq |\xi| \leq 2\}$ and

$$\int_0^\infty \Theta(t\xi) \frac{dt}{t} = 1 \quad \text{for all } \xi \in \mathbb{R}^{3n} \setminus \{0\},$$

and write m as

$$m(\xi) = \int_0^\infty m(\xi)\Theta(t\xi) \frac{dt}{t} = \int_0^\infty m_t(\xi) \frac{dt}{t},$$

where $m_t(\xi) = m(\xi)\Theta(t\xi)$. Thus,

$$\int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x) dx = \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) \frac{dxdt}{t}.$$

For the operator $T_{m_t}, t > 0$, we use Lemmas 3.3 and 3.4 to obtain the following pointwise estimate:

$$(4.2) \quad |T_{m_t}(f_1, f_2, f_3)(x)| \leq C(\zeta_t * |f_1|^q)(x)^{1/q}(\zeta_t * |f_2|^q)(x)^{1/q}(\zeta_t * |f_3|^q)(x)^{1/q},$$

where q is a number satisfying $\max\{1, n/s\} < q < 2$ and $\zeta(x) = (1 + |x|)^{-sq}$. In the rest of the proof, we shall consider the four cases separately.

The case $\text{supp } m \subset \Gamma(V_1)$: In this case, for $(\xi_1, \xi_2, \xi_3) \in \text{supp } m_t$, we have $|\xi_1 + \xi_2 + \xi_3| \approx |\xi_1| \approx |\xi| \approx 1/t$. Hence we can find a function $\psi \in \mathcal{A}_1$ such that $\widehat{\psi}(t(\xi_1 + \xi_2 + \xi_3))\widehat{\psi}(t\xi_1) = 1$ on the support of m_t . Thus, using (4.1), we can write

$$\begin{aligned} & T_{m_t}(f_1, f_2, f_3)(x) \\ &= \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{ix \cdot (\xi_1 + \xi_2 + \xi_3)} \widehat{\psi}(t(\xi_1 + \xi_2 + \xi_3)) m_t(\xi) \widehat{\psi}(t\xi_1) \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \\ &= \psi_t * T_{m_t}(\psi_t * f_1, f_2, f_3)(x), \end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx = \int_{\mathbb{R}^n} T_{m_t}(\psi_t * f_1, f_2, f_3)(x)(\psi_t * g)(x) dx.$$

From this expression and from (4.2), we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx \right| \\ & \leq \int_{\mathbb{R}^n} |T_{m_t}(\psi_t * f_1, f_2, f_3)(x)| |\psi_t * g(x)| dx \\ & \leq C \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |f_2|^q)(x)^{1/q} (\zeta_t * |f_3|^q)(x)^{1/q} |\psi_t * g(x)| dx \\ & \leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} |\psi_t * g(x)| dx. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x) dx \right| \\ & = \left| \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) \frac{dxdt}{t} \right| \\ & \leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} |\psi_t * g(x)| \frac{dxdt}{t} \\ & \leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{2/q} \frac{dxdt}{t} \right)^{1/2} \\ & \quad \times \left(\int_0^\infty \int_{\mathbb{R}^n} |\psi_t * g(x)|^2 \frac{dxdt}{t} \right)^{1/2} \end{aligned}$$

by Schwarz’s inequality. By Lemma 3.2, (3.3), and (3.2), the last quantity is majorized by $C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \|f_1\|_{L^2} \|g\|_{L^2}$ as desired.

The case $\text{supp } m \subset \Gamma(V_2)$: In this case, for $(\xi_1, \xi_2, \xi_3) \in \text{supp } m_t$, we have $|\xi_1 + \xi_2 + \xi_3| \approx |\xi_2| \approx |\xi_3| \approx 1/t$. Hence we can find a $\psi \in \mathcal{A}_1$ such that $\widehat{\psi}(t(\xi_1 + \xi_2 + \xi_3))\widehat{\psi}(t\xi_2) = 1$ on the support of m_t . Thus, in the same way as in the first case, we can write

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx = \int_{\mathbb{R}^n} T_{m_t}(f_1, \psi_t * f_2, f_3)(x)(\psi_t * g)(x) dx.$$

From this expression and (4.2), we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx \right| \\ & \leq \int_{\mathbb{R}^n} |T_{m_t}(f_1, \psi_t * f_2, f_3)(x)| |\psi_t * g(x)| dx \\ & \leq C \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} (\zeta_t * |f_3|^q)(x)^{1/q} |\psi_t * g(x)| dx \\ & \leq C \|f_3\|_{L^\infty} \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\psi_t * g(x)| dx. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x) dx \right| \\ & = \left| \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) \frac{dxdt}{t} \right| \\ & \leq C \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\psi_t * g(x)| \frac{dxdt}{t} \\ & \leq C \|f_3\|_{L^\infty} \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{2/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{2/q} \frac{dxdt}{t} \right)^{1/2} \\ & \quad \times \left(\int_0^\infty \int_{\mathbb{R}^n} |\psi_t * g(x)|^2 \frac{dxdt}{t} \right)^{1/2}. \end{aligned}$$

By Lemma 3.2, (3.4), and (3.2), the last quantity is majorized by

$$C \|f_3\|_{L^\infty} \|f_2\|_{\text{BMO}} \|f_1\|_{L^2} \|g\|_{L^2} \leq C \|f_3\|_{L^\infty} \|f_2\|_{L^\infty} \|f_1\|_{L^2} \|g\|_{L^2}.$$

The case $\text{supp } m \subset \Gamma(V_4)$: In this case, for $(\xi_1, \xi_2, \xi_3) \in \text{supp } m_t$, we have $|\xi_1| \approx |\xi_2| \approx |\xi_3| \approx 1/t$. Hence we can find functions $\psi \in \mathcal{A}_1$ and $\varphi \in \mathcal{A}_0$ such that $\widehat{\varphi}(t(\xi_1 + \xi_2 + \xi_3)) \widehat{\psi}(t\xi_1) \widehat{\psi}(t\xi_2) = 1$ on the support of m_t . Thus we can write

$$\begin{aligned} & T_{m_t}(f_1, f_2, f_3)(x) \\ & = \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{ix \cdot (\xi_1 + \xi_2 + \xi_3)} \widehat{\varphi}(t(\xi_1 + \xi_2 + \xi_3)) m_t(\xi) \widehat{\psi}(t\xi_1) \widehat{f}_1(\xi_1) \widehat{\psi}(t\xi_2) \widehat{f}_2(\xi_2) \widehat{f}_3(\xi_3) d\xi \\ & = \varphi_t * T_{m_t}(\psi_t * f_1, \psi_t * f_2, f_3)(x), \end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx = \int_{\mathbb{R}^n} T_{m_t}(\psi_t * f_1, \psi_t * f_2, f_3)(x)(\varphi_t * g)(x) dx.$$

From this expression and (4.2), we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx \right| \\ & \leq \int_{\mathbb{R}^n} |T_{m_t}(\psi_t * f_1, \psi_t * f_2, f_3)(x)| |\varphi_t * g(x)| dx \\ & \leq C \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} (\zeta_t * |f_3|^q)(x)^{1/q} |\varphi_t * g(x)| dx \\ & \leq C \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\varphi_t * g(x)| dx. \end{aligned}$$

Thus

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x) dx \right| \\ & = \left| \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) \frac{dxdt}{t} \right| \\ & \leq C \|f_3\|_{L^\infty} \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} |\varphi_t * g(x)| \frac{dxdt}{t} \\ & \leq C \|f_3\|_{L^\infty} \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_1|^q)(x)^{2/q} \frac{dxdt}{t} \right)^{1/2} \\ & \quad \times \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_2|^q)(x)^{2/q} |\varphi_t * g(x)|^2 \frac{dxdt}{t} \right)^{1/2}. \end{aligned}$$

By Lemma 3.2, (3.3), and (3.5), the last quantity is majorized by

$$C \|f_3\|_{L^\infty} \|f_1\|_{L^2} \|f_2\|_{\text{BMO}} \|g\|_{L^2} \leq C \|f_3\|_{L^\infty} \|f_1\|_{L^2} \|f_2\|_{L^\infty} \|g\|_{L^2}.$$

The case $\text{supp } m \subset \Gamma(V_6)$: In this case, for $(\xi_1, \xi_2, \xi_3) \in \text{supp } m_t$, we have $|\xi_2| \approx |\xi_3| \approx |\xi| \approx 1/t$. Hence we can find functions $\psi \in \mathcal{A}_1$ and $\varphi \in \mathcal{A}_0$ such that $\widehat{\varphi}(t(\xi_1 + \xi_2 + \xi_3))\widehat{\psi}(t\xi_2)\widehat{\psi}(t\xi_3) = 1$ on the support of m_t . Thus, in the same way as in the third case, we can write

$$\int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx = \int_{\mathbb{R}^n} T_{m_t}(f_1, \psi_t * f_2, \psi_t * f_3)(x)(\varphi_t * g)(x) dx.$$

From this expression and (4.2), we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) dx \right| \\ & \leq \int_{\mathbb{R}^n} |T_{m_t}(f_1, \psi_t * f_2, \psi_t * f_3)(x)| |\varphi_t * g(x)| dx \\ & \leq C \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_3|^q)(x)^{1/q} |\varphi_t * g(x)| dx. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_m(f_1, f_2, f_3)(x)g(x) dx \right| \\ &= \left| \int_0^\infty \int_{\mathbb{R}^n} T_{m_t}(f_1, f_2, f_3)(x)g(x) \frac{dxdt}{t} \right| \\ &\leq C \int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{1/q} (\zeta_t * |\psi_t * f_3|^q)(x)^{1/q} |\varphi_t * g(x)| \frac{dxdt}{t} \\ &\leq C \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |f_1|^q)(x)^{2/q} (\zeta_t * |\psi_t * f_2|^q)(x)^{2/q} \frac{dxdt}{t} \right)^{1/2} \\ &\quad \times \left(\int_0^\infty \int_{\mathbb{R}^n} (\zeta_t * |\psi_t * f_3|^q)(x)^{2/q} |\varphi_t * g(x)|^2 \frac{dxdt}{t} \right)^{1/2}. \end{aligned}$$

By Lemma 3.2 (3.4) and (3.5), the last quantity is majorized by

$$C \|f_1\|_{L^2} \|f_2\|_{\text{BMO}} \|f_3\|_{\text{BMO}} \|g\|_{L^2} \leq C \|f_1\|_{L^2} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \|g\|_{L^2}.$$

This completes the proof of Theorem 1.1 ■

5 The Boundedness from $H^p \times L^\infty \times \dots \times L^\infty$ to L^p with $p \leq 1$

Proof of Theorem 1.2 We shall give the proof for the case $N = 3$. The general case $N \geq 2$ can be proved in a similar way.

Let p, s_1, s_2, s_3 , and m satisfy the assumptions of Theorem 1.2 with $N = 3$. Without loss of generality, we may assume $\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2, s_3)}} = 1$. We write $L = [n(1/p - 1)]$. It is sufficient to consider the case

$$(5.1) \quad n(1/p - 1/2) < s_1 < L + n/2 + 1.$$

We first observe that the desired boundedness of T_m follows if we prove the estimate

$$(5.2) \quad \|T_m(f_1, f_2, f_3)\|_{L^p} \leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}$$

for all f_1 such that

$$(5.3) \quad \text{supp } f_1 \subset \{x \in \mathbb{R}^n : |x| \leq r\}, \quad \|f_1\|_{L^\infty} \leq r^{-n/p}, \quad \int_{\mathbb{R}^n} x^\alpha f_1(x) dx = 0 \quad (|\alpha| \leq L)$$

with some $r > 0$ (where r depends on f_1). Indeed, since the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^\infty}$ are translation invariant and since the operator T_m is also translation invariant in the sense that

$$T_m(f_1, f_2, f_3)(x + x_0) = T_m(f_1(\cdot + x_0), f_2(\cdot + x_0), f_3(\cdot + x_0))(x),$$

if (5.2) holds for all f_1 satisfying (5.3), then it follows that (5.2) holds for all H^p -atoms f_1 . Hence, by considering the linear operator $f_1 \mapsto T_m(f_1, f_2, f_3)$ and applying the usual argument of using the atomic decomposition, we obtain the desired estimate

$$\|T_m(f_1, f_2, f_3)\|_{L^p} \leq C \|f_1\|_{H^p} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}.$$

(Notice that here we benefited from the particular combination of the exponents (p, ∞, ∞) ; the atomic decomposition could not be directly used to prove the $H^{p_1} \rightarrow L^p$ estimate if $1 \geq p_1 > p$.)

In the rest of the proof, we assume that f_1 is a function satisfying (5.3).

We first prove

$$\|T_m(f_1, f_2, f_3)(x)\|_{L^p(|x| \leq 2r)} \leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}.$$

Since $s_1 > n(1/p - 1/2) \geq n/2$ and $s_2, s_3 > n/2$, it follows from Theorem 1.1 that T_m is bounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Note that $\|f_1\|_{L^2} \leq Cr^{-n(1/p-1/2)}$. Thus, by Hölder’s inequality, we obtain

$$\begin{aligned} \|T_m(f_1, f_2, f_3)(x)\|_{L^p(|x| \leq 2r)} &\leq C(2r)^{n(1/p-1/2)} \|T_m(f_1, f_2, f_3)\|_{L^2} \\ &\leq Cr^{n(1/p-1/2)} \|f_1\|_{L^2} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \\ &\leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}. \end{aligned}$$

Thus, what is left is to prove the estimate

$$(5.4) \quad \|T_m(f_1, f_2, f_3)(x)\|_{L^p(|x| > 2r)} \leq C \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}.$$

To prove this, we take a function Ψ satisfying (1.2) with $d = 3n$ and decompose m as

$$m(\xi) = \sum_{j=-\infty}^{\infty} m(\xi)\Psi(\xi/2^j).$$

We define

$$K_j = \mathcal{F}^{-1}[m(\cdot)\Psi(\cdot/2^j)] = \mathcal{F}^{-1}[m_j(\cdot/2^j)].$$

If we write $\tilde{K}_j = \mathcal{F}^{-1}[m_j]$, then $K_j(x) = 2^{3jn}\tilde{K}_j(2^jx)$. The function $T_m(f_1, f_2, f_3)$ can be written as

$$\begin{aligned} T_m(f_1, f_2, f_3)(x) &= \sum_{j \in \mathbb{Z}} T_{m(\cdot)\Psi(\cdot/2^j)}(f_1, f_2, f_3)(x) \\ &= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^{3n}} K_j(x - y_1, x - y_2, x - y_3) f_1(y_1) f_2(y_2) f_3(y_3) dy \\ &= \sum_{j \in \mathbb{Z}} F_j(x), \quad \text{say.} \end{aligned}$$

By the subadditivity of the p -th power of the L^p -norm, $p \leq 1$, and by Hölder's inequality, we have

$$\begin{aligned}
 (5.5) \quad & \|T_m(f_1, f_2, f_3)(x)\|_{L^p(|x|>2r)}^p \\
 & \leq \sum_{j \in \mathbb{Z}} \|F_j(x)\|_{L^p(|x|>2r)}^p \\
 & \leq \sum_{j \in \mathbb{Z}} \left(\int_{|x|>2r} |x|^{-s_1/(1/p-1/2)} dx \right)^{1-p/2} \left(\int_{|x|>2r} |x|^{2s_1} |F_j(x)|^2 dx \right)^{p/2} \\
 & = C \sum_{j \in \mathbb{Z}} \left\{ r^{-s_1+n/p-n/2} \| |x|^{s_1} F_j(x) \|_{L^2(|x|>2r)} \right\}^p,
 \end{aligned}$$

where we have used the assumption $s_1 > n(1/p - 1/2)$ to obtain the last equality.

We shall estimate the function $F_j(x)$. Writing $\partial_1^\alpha K_j(y_1, y_2, y_3) = \partial_{y_1}^\alpha K_j(y_1, y_2, y_3)$ and using the moment condition on f_1 , we have

$$\begin{aligned}
 F_j(x) &= \int_{\mathbb{R}^{3n}} \left\{ K_j(x - y_1, x - y_2, x - y_3) \right. \\
 & \quad \left. - \sum_{|\alpha| \leq L} \frac{(-y_1)^\alpha}{\alpha!} \partial_1^\alpha K_j(x, x - y_2, x - y_3) \right\} f_1(y_1) f_2(y_2) f_3(y_3) dy \\
 &= (L + 1) \sum_{|\alpha|=L+1} \int_{y \in \mathbb{R}^{3n}} \int_{0 < t < 1} \frac{(-y_1)^\alpha}{\alpha!} (1 - t)^L \partial_1^\alpha K_j(x - ty_1, x - y_2, x - y_3) \\
 & \quad \times f_1(y_1) f_2(y_2) f_3(y_3) dt dy \\
 &= (L + 1) \sum_{|\alpha|=L+1} \int_{y \in \mathbb{R}^{3n}} \int_{0 < t < 1} \frac{(-y_1)^\alpha}{\alpha!} (1 - t)^L \partial_1^\alpha K_j(x - ty_1, y_2, y_3) \\
 & \quad \times f_1(y_1) f_2(x - y_2) f_3(x - y_3) dt dy.
 \end{aligned}$$

Thus using the support condition and the L^∞ -norm condition on f_1 , we have

$$\begin{aligned}
 (5.6) \quad & |F_j(x)| \leq Cr^{L+1-n/p} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \\
 & \quad \times \sum_{|\alpha|=L+1} \int_{\substack{|y_1| \leq r \\ y_2, y_3 \in \mathbb{R}^n}} \int_{0 < t < 1} |\partial_1^\alpha K_j(x - ty_1, y_2, y_3)| dt dy.
 \end{aligned}$$

In the same way, without using the moment condition of f_1 , we obtain

$$(5.7) \quad |F_j(x)| \leq Cr^{-n/p} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty} \int_{\substack{|y_1| \leq r \\ y_2, y_3 \in \mathbb{R}^n}} |K_j(x - y_1, y_2, y_3)| dy.$$

We shall estimate the weighted L^2 -norm $\| |x|^{s_1} F_j(x) \|_{L^2(|x|>2r)}$ using (5.6) and (5.7). First consider the integral appearing in (5.6). Notice that $|x|/2 \leq |x - ty_1| \leq 3|x|/2$ for $|x| > 2r$, $|y_1| \leq r$, and $0 \leq t \leq 1$. Hence, using Minkowski's inequality for integrals, we have

$$\begin{aligned} & \left\| |x|^{s_1} \int_{\substack{|y_1| \leq r \\ y_2, y_3 \in \mathbb{R}^n}} \int_{0 < t < 1} |\partial_1^\alpha K_j(x - ty_1, y_2, y_3)| dt dy \right\|_{L^2(|x|>2r)} \\ & \leq C \int_{\substack{|y_1| \leq r \\ y_2, y_3 \in \mathbb{R}^n}} \int_{0 < t < 1} \left\| |x - ty_1|^{s_1} \partial_1^\alpha K_j(x - ty_1, y_2, y_3) \right\|_{L^2(|x|>2r)} dt dy \\ & \leq Cr^n \int_{y_2, y_3 \in \mathbb{R}^n} \left\| |x|^{s_1} \partial_1^\alpha K_j(x, y_2, y_3) \right\|_{L^2(\mathbb{R}_x^n)} dy_2 dy_3 \\ & = (*). \end{aligned}$$

In terms of \tilde{K}_j and $\partial_1^\alpha \tilde{K}_j(z_1, z_2, z_3) = \partial_{z_1}^\alpha \tilde{K}_j(z_1, z_2, z_3)$, the last term can be written as

$$\begin{aligned} (*) & = Cr^n \int_{y_2, y_3 \in \mathbb{R}^n} 2^{j(-s_1+3n+|\alpha|)} \left\| |2^j x|^{s_1} \partial_1^\alpha \tilde{K}_j(2^j x, 2^j y_2, 2^j y_3) \right\|_{L^2(\mathbb{R}_x^n)} dy_2 dy_3 \\ & = Cr^n 2^{j(-s_1+n/2+|\alpha|)} \int_{z_2, z_3 \in \mathbb{R}^n} \left\| |z_1|^{s_1} \partial_1^\alpha \tilde{K}_j(z_1, z_2, z_3) \right\|_{L^2(\mathbb{R}_{z_1}^n)} dz_2 dz_3. \end{aligned}$$

Since $s_2, s_3 > n/2$, Schwarz's inequality gives

$$\begin{aligned} (*) & \leq Cr^n 2^{j(-s_1+n/2+|\alpha|)} \left(\int_{\mathbb{R}^{2n}} \left\| |z_1|^{s_1} \langle z_2 \rangle^{s_2} \langle z_3 \rangle^{s_3} \partial_1^\alpha \tilde{K}_j(z_1, z_2, z_3) \right\|_{L^2(\mathbb{R}_{z_1}^n)}^2 dz_2 dz_3 \right)^{1/2} \\ & \leq Cr^n 2^{j(-s_1+n/2+|\alpha|)} \left\| \langle z_1 \rangle^{s_1} \langle z_2 \rangle^{s_2} \langle z_3 \rangle^{s_3} \partial_1^\alpha \tilde{K}_j(z_1, z_2, z_3) \right\|_{L^2(\mathbb{R}^{3n})} \\ & = Cr^n 2^{j(-s_1+n/2+|\alpha|)} \left\| \xi_1^\alpha m_j(\xi_1, \xi_2, \xi_3) \right\|_{W^{(s_1, s_2, s_3)}}. \end{aligned}$$

We shall see that the last $\| \dots \|_{W^{(s_1, s_2, s_3)}}$ is majorized by C . In fact, if we take a function $\tilde{\Psi} \in \mathcal{S}(\mathbb{R}^{3n})$ such that $\tilde{\Psi} = 1$ on $\{1/2 \leq |\xi| \leq 2\}$, then, by Proposition 2.2,

$$\begin{aligned} \left\| \xi_1^\alpha m_j \right\|_{W^{(s_1, s_2, s_3)}} & = \left\| \xi_1^\alpha m(2^j \xi) \Psi(\xi) \right\|_{W^{(s_1, s_2, s_3)}} = \left\| \xi_1^\alpha m(2^j \xi) \Psi(\xi) \tilde{\Psi}(\xi) \right\|_{W^{(s_1, s_2, s_3)}} \\ & \leq C \left\| m(2^j \xi) \Psi(\xi) \right\|_{W^{(s_1, s_2, s_3)}} \left\| \xi_1^\alpha \tilde{\Psi}(\xi) \right\|_{W^{(s_1, s_2, s_3)}} = C \left\| m_j \right\|_{W^{(s_1, s_2, s_3)}} \leq C. \end{aligned}$$

Thus we have

$$\left\| |x|^{s_1} \int_{\substack{|y_1| \leq r \\ y_2, y_3 \in \mathbb{R}^n}} \int_{0 < t < 1} |\partial_1^\alpha K_j(x - ty_1, y_2, y_3)| dt dy \right\|_{L^2(|x|>2r)} \leq Cr^n 2^{j(-s_1+n/2+|\alpha|)}.$$

Combining this with (5.6), we obtain

$$(5.8) \quad r^{-s_1+n/p-n/2} \| |x|^{s_1} F_j(x) \|_{L^2(|x|>2r)} \leq C(2^j r)^{-s_1+n/2+L+1} \| f_2 \|_{L^\infty} \| f_3 \|_{L^\infty}.$$

In the same way, using (5.7) in place of (5.6), we obtain

$$(5.9) \quad r^{-s_1+n/p-n/2} \| |x|^{s_1} F_j(x) \|_{L^2(|x|>2r)} \leq C(2^j r)^{-s_1+n/2} \|f_2\|_{L^\infty} \|f_3\|_{L^\infty}.$$

Now using (5.5), (5.8), and (5.9), we obtain

$$\begin{aligned} & \|T_m(f_1, f_2, f_3)\|_{L^p(|x|\geq 2r)}^p \\ & \leq C(\|f_2\|_{L^\infty} \|f_3\|_{L^\infty})^p \left(\sum_{2^j r \leq 1} (2^j r)^{p(-s_1+n/2+L+1)} + \sum_{2^j r > 1} (2^j r)^{p(-s_1+n/2)} \right) \\ & \leq C(\|f_2\|_{L^\infty} \|f_3\|_{L^\infty})^p, \end{aligned}$$

where we have used (5.1). Thus we proved (5.4). The proof of Theorem 1.2 is complete. ■

6 The Boundedness from $H^{p_1} \times H^{p_2} \times \dots \times H^{p_N}$ to L^p with $p \leq 1$

In this section, using interpolation, we prove the following theorem.

Theorem 6.1 *Let $0 < p_1, \dots, p_N \leq \infty$, $0 < p \leq 1$, and $1/p_1 + \dots + 1/p_N = 1/p$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, \dots, s_N)}(\mathbb{R}^{Nn})} < \infty \quad \text{with } s_j > \frac{n}{p_j}(1-p) + \frac{n}{2}, \quad j = 1, \dots, N,$$

then T_m is bounded from $H^{p_1}(\mathbb{R}^n) \times \dots \times H^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof We give the proof for the case $N = 3$. The argument can be easily extended to the case $N \geq 2$. We shall divide the proof into two steps.

Step 1

Let $0 < \theta < 1$, $0 < p_j, p_{j,k} \leq \infty$, and $s_{j,k} > n/2$ ($j \in \{0, 1\}, k \in \{1, 2, 3\}$). Set $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/p_k = (1-\theta)/p_{0,k} + \theta/p_{1,k}$, and $s_k = (1-\theta)s_{0,k} + \theta s_{1,k}$. In this step, we prove that if

$$\|T_m\|_{H^{p_{0,1}} \times H^{p_{0,2}} \times H^{p_{0,3}} \rightarrow L^{p_0}} \leq C \sup_{k \in \mathbb{Z}} \|m_k\|_{W^{(s_{0,1}, s_{0,2}, s_{0,3})}}$$

and

$$\|T_m\|_{H^{p_{1,1}} \times H^{p_{1,2}} \times H^{p_{1,3}} \rightarrow L^{p_1}} \leq C \sup_{k \in \mathbb{Z}} \|m_k\|_{W^{(s_{1,1}, s_{1,2}, s_{1,3})}},$$

then

$$(6.1) \quad \|T_m\|_{H^{p_1} \times H^{p_2} \times H^{p_3} \rightarrow L^p} \leq C \sup_{k \in \mathbb{Z}} \|m_k\|_{W^{(s_1, s_2, s_3)}}.$$

To do this, we construct a family of multilinear Fourier multipliers m_z as follows:

$$m_z(\xi) = \sum_{j \in \mathbb{Z}} m_{z,j}(\xi/2^j) \Phi(\xi/2^j), \quad z \in \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\},$$

where $\Phi \in \mathcal{S}(\mathbb{R}^{3n})$ satisfies $\Phi = 1$ on $\{1/2 \leq |\xi| \leq 2\}$ and $\operatorname{supp} \Phi \subset \{1/4 \leq |\xi| \leq 4\}$, and

$$\begin{aligned} m_{z,j}(\xi) &= \langle D_1 \rangle^{(s_{0,1}-s_{1,1})(z-\theta)} \langle D_2 \rangle^{(s_{0,2}-s_{1,2})(z-\theta)} \langle D_3 \rangle^{(s_{0,3}-s_{1,3})(z-\theta)} m_j(\xi) \\ &= \frac{1}{(2\pi)^{3n}} \int_{\mathbb{R}^{3n}} e^{i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2 + x_3 \cdot \xi_3)} \\ &\quad \times \langle x_1 \rangle^{(s_{0,1}-s_{1,1})(z-\theta)} \langle x_2 \rangle^{(s_{0,2}-s_{1,2})(z-\theta)} \langle x_3 \rangle^{(s_{0,3}-s_{1,3})(z-\theta)} \widehat{m}_j(x) \, dx. \end{aligned}$$

Since $m_{\theta,j}(\xi) = m_j(\xi)$ and $\Phi = 1$ on $\operatorname{supp} \Psi$,

$$\begin{aligned} m_{\theta}(\xi) &= \sum_{j \in \mathbb{Z}} m(\xi) \Psi(\xi/2^j) \Phi(\xi/2^j) \\ &= \sum_{j \in \mathbb{Z}} m(\xi) \Psi(\xi/2^j) = m(\xi). \end{aligned}$$

Then it follows from the interpolation theorem for analytic families of operators ([14, 20]) that

$$\begin{aligned} (6.2) \quad \|T_m\|_{H^{p_1} \times H^{p_2} \times H^{p_3} \rightarrow L^p} &\leq \left(\sup_{t \in \mathbb{R}} \|T_{m_{it}}\|_{H^{p_{0,1}} \times H^{p_{0,2}} \times H^{p_{0,3}} \rightarrow L^{p_0}} \right)^{1-\theta} \\ &\quad \times \left(\sup_{t \in \mathbb{R}} \|T_{m_{1+it}}\|_{H^{p_{1,1}} \times H^{p_{1,2}} \times H^{p_{1,3}} \rightarrow L^{p_1}} \right)^{\theta}. \end{aligned}$$

Using $\operatorname{supp} \Psi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\operatorname{supp} \Phi(2^{k-j} \cdot) \subset \{2^{j-k-2} \leq |\xi| \leq 2^{j-k+2}\}$, we have

$$\begin{aligned} (m_{it})_k(\xi) &= m_{it}(2^k \xi) \Psi(\xi) \\ &= \left(\sum_{j \in \mathbb{Z}} m_{it,j}(2^{k-j} \xi) \Phi(2^{k-j} \xi) \right) \Psi(\xi) \\ &= \sum_{j=k-2}^{k+2} m_{it,j}(2^{k-j} \xi) \Phi(2^{k-j} \xi) \Psi(\xi). \end{aligned}$$

Recall that $s_{0,1}, s_{0,2}, s_{0,3} > n/2$. Then, by Proposition 2.2 and a change of variables,

$$\begin{aligned} \|(m_{it})_k\|_{W^{(s_{0,1}, s_{0,2}, s_{0,3})}} &\leq C \sum_{j=k-2}^{k+2} \|m_{it,j}(2^{k-j}\cdot)\|_{W^{(s_{0,1}, s_{0,2}, s_{0,3})}} \|\Phi(2^{k-j}\cdot)\Psi\|_{W^{(s_{0,1}, s_{0,2}, s_{0,3})}} \\ &\leq C \sum_{j=k-2}^{k+2} \|m_{it,j}\|_{W^{(s_{0,1}, s_{0,2}, s_{0,3})}} \\ &\leq C \sup_{j \in \mathbb{Z}} \|\langle D_1 \rangle^{s_{0,1}} \langle D_2 \rangle^{s_{0,2}} \langle D_3 \rangle^{s_{0,3}} m_{it,j}\|_{L^2} \\ &= C \sup_{j \in \mathbb{Z}} \left(\int_{\mathbb{R}^{3n}} |\langle x_1 \rangle^{s_{0,1} + (s_{0,1} - s_{1,1})(it - \theta)}|^2 |\langle x_2 \rangle^{s_{0,2} + (s_{0,2} - s_{1,2})(it - \theta)}|^2 \right. \\ &\quad \left. \times |\langle x_3 \rangle^{s_{0,3} + (s_{0,3} - s_{1,3})(it - \theta)}|^2 |\widehat{m}_j(x)|^2 dx \right)^{1/2} \\ &= C \sup_{j \in \mathbb{Z}} \left(\int_{\mathbb{R}^{3n}} \langle x_1 \rangle^{2s_1} \langle x_2 \rangle^{2s_2} \langle x_3 \rangle^{2s_3} |\widehat{m}_j(x)|^2 dx \right)^{1/2} \\ &= C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2, s_3)}}. \end{aligned}$$

Hence, our assumption implies

$$(6.3) \quad \|T_{m_{it}}\|_{H^{p_0,1} \times H^{p_0,2} \times H^{p_0,3} \rightarrow L^{p_0}} \leq C \sup_{k \in \mathbb{Z}} \|(m_{it})_k\|_{W^{(s_{0,1}, s_{0,2}, s_{0,3})}} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2, s_3)}}.$$

Similarly we have

$$(6.4) \quad \begin{aligned} \|T_{m_{1+it}}\|_{H^{p_1,1} \times H^{p_1,2} \times H^{p_1,3} \rightarrow L^{p_1}} &\leq C \sup_{k \in \mathbb{Z}} \|(m_{1+it})_k\|_{W^{(s_{1,1}, s_{1,2}, s_{1,3})}} \\ &\leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2, s_3)}}. \end{aligned}$$

The estimate (6.1) now follows from (6.2)–(6.4).

Step 2

Let $0 < p \leq 1$ and $\epsilon > 0$. By interchanging the role of p_1 and p_2 or p_3 , we have by Theorem 1.2

$$(6.5) \quad \|T_m\|_{H^p \times L^\infty \times L^\infty \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n(1/p-1/2)+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$

$$(6.6) \quad \|T_m\|_{L^\infty \times H^p \times L^\infty \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n(1/p-1/2)+\epsilon, n/2+\epsilon)}},$$

$$(6.7) \quad \|T_m\|_{L^\infty \times L^\infty \times H^p \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n(1/p-1/2)+\epsilon)}}.$$

Then, it follows from Step 1 that (6.5) and (6.6) give

$$(6.8) \quad \|T_m\|_{H^{p_1} \times H^{p_2} \times L^\infty \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n(1-p)/p_1+n/2+\epsilon, n(1-p)/p_2+n/2+\epsilon, n/2+\epsilon)}},$$

where $p \leq p_1, p_2 \leq \infty$ and $1/p_1 + 1/p_2 = 1/p$. Furthermore, (6.7) and (6.8) give

$$\|T_m\|_{H^{p_1} \times H^{p_2} \times H^{p_3} \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n(1-p)/p_1+n/2+\epsilon, n(1-p)/p_2+n/2+\epsilon, n(1-p)/p_3+n/2+\epsilon)}},$$

where $p \leq p_1, p_2, p_3 \leq \infty$ and $1/p_1 + 1/p_2 + 1/p_3 = 1/p$. The proof of Theorem 6.1 is complete. ■

7 Sharpness of the Conditions of Theorems 1.1 and 1.2

In this section, we consider the sharpness of Theorems 1.1 and 1.2.

Proposition 7.1 *The estimate*

$$(7.1) \quad \|T_m(f_1, f_2, \dots, f_N)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2, \dots, s_N)}(\mathbb{R}^{Nn})} \|f_1\|_{L^2(\mathbb{R}^n)} \|f_2\|_{L^\infty(\mathbb{R}^n)} \dots \|f_N\|_{L^\infty(\mathbb{R}^n)}$$

holds only if $s_1, s_2, \dots, s_N \geq n/2$.

Proof We give the proof in the case $N = 2$. Generalization to $N \geq 3$ will be obvious. We take functions ψ and φ such that

$$\begin{aligned} \psi &\in \mathcal{S}(\mathbb{R}^n), \quad \psi \neq 0, \quad \text{supp } \widehat{\psi} \subset \{\xi \in \mathbb{R}^n \mid 9/10 \leq |\xi| \leq 11/10\}, \\ \varphi &\in \mathcal{S}(\mathbb{R}^n), \quad \widehat{\varphi}(0) \neq 0, \quad \text{supp } \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq 1\}. \end{aligned}$$

To prove the necessity of the condition $s_2 \geq n/2$, we set, for sufficiently small $\epsilon > 0$,

$$(7.2) \quad m(\xi_1, \xi_2) = \widehat{\psi}(\xi_1) \widehat{\varphi}(\xi_2/\epsilon).$$

For this m , we have

$$T_m(f_1, f_2)(x) = \mathcal{F}^{-1}[\widehat{\psi} \widehat{f_1}](x) \mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon) \widehat{f_2}(\cdot)](x),$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform on \mathbb{R}^n . To estimate the norm $\|m_j\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})}$, we choose the function $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$, which appeared in the definition of m_j , so that we have

$$\begin{aligned} \text{supp } \Psi &\subset \{\xi \in \mathbb{R}^{2n} \mid 2^{-1/2-\alpha} \leq |\xi| \leq 2^{1/2+\alpha}\}, \\ \sum_{k \in \mathbb{Z}} \Psi(2^{-k}\xi) &= 1 \quad \text{for all } \xi \neq 0, \\ \Psi(\xi) &= 1 \quad \text{if } 2^{-1/2+\alpha} \leq |\xi| \leq 2^{1/2-\alpha}, \end{aligned}$$

where $\alpha > 0$ is a sufficiently small number. Then, for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned} \text{supp } m &\subset \{(\xi_1, \xi_2) \in \mathbb{R}^{2n} \mid 9/10 \leq |\xi_1| \leq 11/10, |\xi_2| \leq \epsilon\} \\ &\subset \{\xi \in \mathbb{R}^{2n} \mid 2^{-1/2+\alpha} \leq |\xi| \leq 2^{1/2-\alpha}\}. \end{aligned}$$

Hence,

$$(7.3) \quad m_j(\xi) = m(2^j \xi) \Psi(\xi) = \begin{cases} m(\xi) & (j = 0) \\ 0 & (j \neq 0) \end{cases}$$

and

$$\begin{aligned} \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} &= \|m\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} = \|\widehat{\psi}(\xi_1) \widehat{\varphi}(\xi_2/\epsilon)\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\psi}\|_{W^{s_1}(\mathbb{R}^n)} \|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_2}(\mathbb{R}^n)}. \end{aligned}$$

Thus the inequality (7.1) for m of (7.2) is equivalent to

$$(7.4) \quad \|\mathcal{F}^{-1}[\widehat{\psi} \widehat{f}_1](x) \mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon) \widehat{f}_2(\cdot)](x)\|_{L^2} \leq C \|\widehat{\psi}\|_{W^{s_1}} \|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_2}} \|f_1\|_{L^2} \|f_2\|_{L^\infty}.$$

We have $\|\widehat{\psi}\|_{W^{s_1}} = C$ and

$$\begin{aligned} &\|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_2}} \\ &= \|\epsilon^n \varphi(\epsilon x) \langle x \rangle^{s_2}\|_{L^2} \\ &\lesssim \epsilon^n \left(\int_{\mathbb{R}^n} (1 + |x|)^{2s_2} (1 + \epsilon|x|)^{-2N} dx \right)^{1/2} \quad (N > 0 \text{ large}) \\ &\approx \epsilon^n \left(\int_{|x| \leq 1} dx + \int_{1 < |x| \leq 1/\epsilon} |x|^{2s_2} dx + \int_{1/\epsilon < |x| < \infty} |x|^{2s_2} (\epsilon|x|)^{-2N} dx \right)^{1/2} \\ &\approx \epsilon^{-s_2+n/2}. \end{aligned}$$

Hence (7.4) implies

$$(7.5) \quad \|\mathcal{F}^{-1}[\widehat{\psi} \widehat{f}_1](x) \mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon) \widehat{f}_2(\cdot)](x)\|_{L^2} \leq C \epsilon^{-s_2+n/2} \|f_1\|_{L^2} \|f_2\|_{L^\infty}.$$

We test (7.5) for $f_1(x) = \psi(x)$, $f_2(x) = 1$. Then

$$\text{(the left-hand side of (7.5))} = \|\mathcal{F}^{-1}[\widehat{\psi}^2](x) \widehat{\varphi}(0)\|_{L^2} = C,$$

$$\text{(the right-hand side of (7.5))} = C \epsilon^{-s_2+n/2} \|\psi\|_{L^2} = C \epsilon^{-s_2+n/2}.$$

Thus (7.5) holds only if $s_2 \geq n/2$.

To prove the necessity of the condition $s_1 \geq n/2$, we set, for sufficiently small $\epsilon > 0$,

$$(7.6) \quad m(\xi_1, \xi_2) = \widehat{\varphi}(\xi_1/\epsilon)\widehat{\psi}(\xi_2).$$

By the same reason as above, we have

$$\begin{aligned} \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} &= \|m\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} = \|\widehat{\varphi}(\xi_1/\epsilon)\widehat{\psi}(\xi_2)\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\varphi}(\cdot/\epsilon)\|_{W^{s_1}(\mathbb{R}^n)} \|\widehat{\psi}\|_{W^{s_2}(\mathbb{R}^n)} \leq C\epsilon^{-s_1+n/2} \end{aligned}$$

and the inequality (7.1) with m of (7.6) implies

$$(7.7) \quad \|\mathcal{F}^{-1}[\widehat{\varphi}(\cdot/\epsilon)\widehat{f}_1(\cdot)](x)\mathcal{F}^{-1}[\widehat{\psi}\widehat{f}_2](x)\|_{L^2} \leq C\epsilon^{-s_1+n/2}\|f_1\|_{L^2}\|f_2\|_{L^\infty}.$$

We test (7.7) for $\widehat{f}_1(\xi_1) = \epsilon^{-n/2}\widehat{\varphi}(\xi_1/\epsilon)$, $f_2(x) = e^{i\eta^\circ \cdot x}$, where we choose η° so that $\widehat{\psi}(\eta^\circ) \neq 0$. Then

$$\begin{aligned} \text{(the left-hand side of (7.7))} &= \|\mathcal{F}^{-1}[\epsilon^{-n/2}\widehat{\varphi}(\cdot/\epsilon)^2]\|_{L^2} |\widehat{\psi}(\eta^\circ)| \\ &= \|\widehat{\varphi}^2\|_{L^2} |\widehat{\psi}(\eta^\circ)| = C, \\ \text{(the right-hand side of (7.7))} &= C\epsilon^{-s_1+n/2}\|\mathcal{F}^{-1}[\epsilon^{-n/2}\widehat{\varphi}(\cdot/\epsilon)]\|_{L^2} \\ &= C\epsilon^{-s_1+n/2}\|\widehat{\varphi}\|_{L^2} = C\epsilon^{-s_1+n/2}. \end{aligned}$$

Thus (7.7) holds only if $s_1 \geq n/2$. Proposition 7.1 is proved. ■

Proposition 7.2 *Let $0 < p \leq 1$. Then the estimate*

$$(7.8) \quad \|T_m(f_1, f_2, \dots, f_N)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2, \dots, s_N)}(\mathbb{R}^{Nn})} \|f_1\|_{HP(\mathbb{R}^n)} \|f_2\|_{L^\infty(\mathbb{R}^n)} \cdots \|f_N\|_{L^\infty(\mathbb{R}^n)}$$

holds only if $s_1 \geq n(1/p - 1/2)$ and $s_2, \dots, s_N \geq n/2$.

Proof We give the proof in the case $N = 2$. Generalization to $N \geq 3$ will be obvious. The necessity of the condition $s_2 \geq n/2$ can be proved in the same way as in the first part of the proof of Proposition 7.1.

To prove the necessity of the condition $s_1 \geq n(1/p - 1/2)$, we take the functions ψ and φ as in the proof of Proposition 7.1, and we take a $\zeta^\circ \in \mathbb{R}^n$ such that $|\zeta^\circ| = 1/10$. We set

$$(7.9) \quad m(\xi_1, \xi_2) = \widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon)\widehat{\psi}(\xi_2).$$

For sufficiently small ϵ ,

$$\begin{aligned} \text{supp } m &\subset \{(\xi_1, \xi_2) \in \mathbb{R}^{2n} \mid |\xi_1 - \zeta^\circ| \leq \epsilon, 9/10 \leq |\xi_2| \leq 11/10\} \\ &\subset \{\xi \in \mathbb{R}^{2n} \mid 2^{-1/2+\alpha} \leq |\xi| \leq 2^{1/2-\alpha}\}, \end{aligned}$$

and hence (7.3) holds again and we have

$$\begin{aligned} \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} &= \|m\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon) \widehat{\psi}(\xi_2)\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})} \\ &= \|\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\|_{W^{s_1}(\mathbb{R}^n)} \|\widehat{\psi}\|_{W^{s_2}(\mathbb{R}^n)} \\ &\leq C\epsilon^{-s_1+n/2}. \end{aligned}$$

Thus the inequality (7.8) for m of (7.9) implies

$$(7.10) \quad \left\| \mathcal{F}^{-1}[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon) \widehat{f}_1(\cdot)](x) \mathcal{F}^{-1}[\widehat{\psi} \widehat{f}_2](x) \right\|_{L^p} \leq C\epsilon^{-s_1+n/2} \|f_1\|_{H^p} \|f_2\|_{L^\infty}.$$

We test (7.10) for $f_1(x) = \psi'(x)$, $f_2(x) = e^{i\eta^\circ \cdot x}$, where ψ' and η° are chosen so that $\psi' \in \mathcal{S}(\mathbb{R}^n)$, $\text{supp } \widehat{\psi}'$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$, $\widehat{\psi}'(\xi_1) = 1$ in a neighborhood of ζ° , $\eta^\circ \in \mathbb{R}^n$, and $\widehat{\psi}(\eta^\circ) \neq 0$. Then

$$\begin{aligned} (\text{the left-hand side of (7.10)}) &= \|\mathcal{F}^{-1}[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)](x)\|_{L^p} |\widehat{\psi}(\eta^\circ)| \\ &= \epsilon^{n-n/p} \|\varphi\|_{L^p} |\widehat{\psi}(\eta^\circ)| = C\epsilon^{n-n/p}, \end{aligned}$$

$$(\text{the right-hand side of (7.10)}) = C\epsilon^{-s_1+n/2} \|\psi'\|_{H^p} = C\epsilon^{-s_1+n/2}.$$

Thus (7.10) holds only if $s_1 \geq n/p - n/2$. Proposition 7.2 is proved. ■

8 Related Results and Comments

As a corollary of Theorems 1.1 and 1.2, we can also prove the boundedness of T_m from $L^{p_1} \times L^{p_2} \times L^{p_3}$ to L^p for $1 < p \leq 2$ and for m satisfying the product type estimate. (We treated the case $0 < p \leq 1$ in Theorem 6.1). In fact, by Theorems 1.1 and 1.2,

$$\|T_m\|_{L^2 \times L^\infty \times L^\infty \rightarrow L^2} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$

$$\|T_m\|_{H^1 \times L^\infty \times L^\infty \rightarrow L^1} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}}.$$

Then it follows from Step 1 in Section 6 that

$$(8.1) \quad \|T_m\|_{L^p \times L^\infty \times L^\infty \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$

where $1 < p \leq 2$. By interchanging the role of p_1 and p_2 or p_3 , we have

$$\|T_m\|_{L^\infty \times L^p \times L^\infty \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$

$$\|T_m\|_{L^\infty \times L^\infty \times L^p \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}}.$$

Hence, by the same argument as in Section 6, Step 2, we obtain

$$(8.2) \quad \|T_m\|_{L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}},$$

where $1/p_1 + 1/p_2 + 1/p_3 = 1/p$, $1 < p_j \leq \infty$, and $1 < p \leq 2$.

We remark that (8.2) also holds for all $2 < p_1, p_2, p_3, p < \infty$ satisfying $1/p_1 + 1/p_2 + 1/p_3 = 1/p$ (see [9, Theorem 6.2]). But at present, it is unclear to the authors whether one or more of indices p_1, p_2, p_3 can be equal to ∞ in the case $2 < p < \infty$.

We end this section by commenting on duality. For $m \in L^\infty(\mathbb{R}^{Nn})$ and $1 \leq k \leq N$, we set

$$m^{*k}(\xi) = m(\xi_1, \dots, \xi_{k-1}, -(\xi_1 + \dots + \xi_N), \xi_{k+1}, \dots, \xi_N),$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$. Then

$$(8.3) \quad \int_{\mathbb{R}^n} T_m(f_1, \dots, f_N)g \, dx = \int_{\mathbb{R}^n} T_{m^{*k}}(f_1, \dots, f_{k-1}, g, f_{k+1}, \dots, f_N) f_k \, dx$$

for all $f_1, \dots, f_N, g \in \mathcal{S}(\mathbb{R}^n)$. The formula (8.3) says that the boundedness of T_m from $L^p \times L^\infty \times L^\infty$ to L^p is equivalent to that of $T_{m^{*1}}$ from $L^{p'} \times L^\infty \times L^\infty$ to $L^{p'}$. However, in the framework of Sobolev spaces of product type we cannot use the duality argument, because Sobolev spaces of product type are not invariant under the map $m \mapsto m^{*1}$. More precisely, the following inequality does not hold:

$$(8.4) \quad \sup_{j \in \mathbb{Z}} \|m(-2^j(\xi_1 + \xi_2 + \xi_3), 2^j\xi_2, 2^j\xi_3) \Psi(\xi)\|_{W^{(s_1, s_2, s_3)}} \leq C \sup_{j \in \mathbb{Z}} \|m(2^j\xi)\Psi(\xi)\|_{W^{(s_1, s_2, s_3)}},$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ and Ψ is as in (1.2) with $d = 3n$.

It should be pointed out that (8.4) holds if we replace $W^{(s_1, s_2, s_3)}$ by the (usual) Sobolev space W^s (see [11, 21]). Then, since $W^{3(n/2+\epsilon)} \hookrightarrow W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}$, it follows from (8.1) and duality that

$$\begin{aligned} \|T_m\|_{L^{p'} \times L^\infty \times L^\infty \rightarrow L^{p'}} &= \|T_{m^{*1}}\|_{L^p \times L^\infty \times L^\infty \rightarrow L^p} \leq C \sup_{j \in \mathbb{Z}} \|(m^{*1})_j\|_{W^{(n/2+\epsilon, n/2+\epsilon, n/2+\epsilon)}} \\ &\leq C \sup_{j \in \mathbb{Z}} \|(m^{*1})_j\|_{W^{3(n/2+\epsilon)}} \leq C \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{3(n/2+\epsilon)}}, \end{aligned}$$

where $(m^{*1})_j(\xi) = m^{*1}(2^j\xi)\Psi(\xi)$ and $1 < p < 2$. Therefore, if $\sup_{j \in \mathbb{Z}} \|m_j\|_{W^s} < \infty$ with $s > 3n/2$, then T_m is bounded from $L^p \times L^\infty \times L^\infty$ to L^p for $2 < p < \infty$.

A Appendix

We shall give proofs of Lemma 2.1 and Proposition 2.2.

Proof of Lemma 2.1 We only consider the case $N = 3$; the argument can be immediately extended to the case $N \geq 2$. Suppose $F \in W^{(s,s,s)}(\mathbb{R}^{3n})$ and $\text{supp } F \subset \{|x| \leq r\}$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Take a $\Phi \in \mathcal{S}(\mathbb{R}^{3n})$ such that $\Phi = 1$ on $\{|x| \leq r\}$ and $\text{supp } \Phi \subset \{|x| \leq 2r\}$. Then $F(x) = \Phi(x)F(x)$. Hence, by Schwarz's inequality and Young's inequality,

$$\begin{aligned} \|\widehat{F}\|_{L^q(w_{sq})}^q &= \frac{1}{(2\pi)^{3nq}} \int_{\mathbb{R}^{3n}} \langle \xi_1 \rangle^{sq} \langle \xi_2 \rangle^{sq} \langle \xi_3 \rangle^{sq} |\widehat{\Phi} * \widehat{F}(\xi)|^q d\xi \\ &\leq C \int_{\mathbb{R}^{3n}} \left(\int_{\mathbb{R}^{3n}} \langle \xi_1 - \eta_1 \rangle^s \langle \xi_2 - \eta_2 \rangle^s \langle \xi_3 - \eta_3 \rangle^s |\widehat{\Phi}(\xi - \eta)| \langle \eta_1 \rangle^s \langle \eta_2 \rangle^s \langle \eta_3 \rangle^s |\widehat{F}(\eta)| d\eta \right)^q d\xi \\ &\leq C \left(\sup_{\xi \in \mathbb{R}^{3n}} \int_{\mathbb{R}^{3n}} \langle \xi_1 - \eta_1 \rangle^s \langle \xi_2 - \eta_2 \rangle^s \langle \xi_3 - \eta_3 \rangle^s |\widehat{\Phi}(\xi - \eta)| \langle \eta_1 \rangle^s \langle \eta_2 \rangle^s \langle \eta_3 \rangle^s |\widehat{F}(\eta)| d\eta \right)^{q-2} \\ &\quad \times \left\{ \int_{\mathbb{R}^{3n}} \left(\int_{\mathbb{R}^{3n}} \langle \xi_1 - \eta_1 \rangle^s \langle \xi_2 - \eta_2 \rangle^s \langle \xi_3 - \eta_3 \rangle^s |\widehat{\Phi}(\xi - \eta)| \langle \eta_1 \rangle^s \langle \eta_2 \rangle^s \langle \eta_3 \rangle^s |\widehat{F}(\eta)| d\eta \right)^2 d\xi \right\} \\ &\leq C \left(\|\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \widehat{\Phi}\|_{L^2} \|\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \widehat{F}\|_{L^2} \right)^{q-2} \\ &\quad \times \left(\|\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \widehat{\Phi}\|_{L^1} \|\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \widehat{F}\|_{L^2} \right)^2 \\ &= C \|\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \widehat{F}\|_{L^2}^q = C \|F\|_{W^{(s,s,s)}}^q. \end{aligned}$$

The proof is complete. ■

Proof of Proposition 2.2 We only consider the case $N = 3$; the argument is easily extended to the case of a general N . Since

$$\langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \langle \xi_3 \rangle^{s_3} \leq C \left(\langle \xi_1 - \eta_1 \rangle^{s_1} + \langle \eta_1 \rangle^{s_1} \right) \left(\langle \xi_2 - \eta_2 \rangle^{s_2} + \langle \eta_2 \rangle^{s_2} \right) \left(\langle \xi_3 - \eta_3 \rangle^{s_3} + \langle \eta_3 \rangle^{s_3} \right),$$

we see that

$$\begin{aligned} \|FG\|_{W^{(s_1,s_2,s_3)}} &= \frac{1}{(2\pi)^{3n}} \left(\int_{\mathbb{R}^{3n}} \langle \xi_1 \rangle^{2s_1} \langle \xi_2 \rangle^{2s_2} \langle \xi_3 \rangle^{2s_3} \left| \int_{\mathbb{R}^{3n}} \widehat{F}(\xi - \eta) \widehat{G}(\eta) d\eta \right|^2 d\xi \right)^{1/2} \\ &\leq C \sum_{i_1, i_2, i_3=0}^1 \|\widehat{F}_{(i_1, i_2, i_3)} * \widehat{G}_{(1-i_1, 1-i_2, 1-i_3)}\|_{L^2}, \end{aligned}$$

where

$$\begin{aligned} \widehat{F}_{(i_1, i_2, i_3)}(\xi) &= \langle \xi_1 \rangle^{i_1 s_1} \langle \xi_2 \rangle^{i_2 s_2} \langle \xi_3 \rangle^{i_3 s_3} |\widehat{F}(\xi)|, \\ \widehat{G}_{(1-i_1, 1-i_2, 1-i_3)}(\xi) &= \langle \xi_1 \rangle^{(1-i_1)s_1} \langle \xi_2 \rangle^{(1-i_2)s_2} \langle \xi_3 \rangle^{(1-i_3)s_3} |\widehat{G}(\xi)|. \end{aligned}$$

It is not difficult to estimate $\widehat{F}_{(1,1,1)} * \widehat{G}_{(0,0,0)}$ and $\widehat{F}_{(0,0,0)} * \widehat{G}_{(1,1,1)}$. In fact, since $s_1, s_2, s_3 > n/2$, by Young's inequality and Schwarz's inequality, we have

$$\begin{aligned} \|\widehat{F}_{(1,1,1)} * \widehat{G}_{(0,0,0)}\|_{L^2} &\leq \|\widehat{F}_{(1,1,1)}\|_{L^2} \|\widehat{G}_{(0,0,0)}\|_{L^1} = \|F\|_{W^{(s_1,s_2,s_3)}} \|\widehat{G}\|_{L^1} \\ &\leq C \|F\|_{W^{(s_1,s_2,s_3)}} \|\langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \langle \xi_3 \rangle^{s_3} \widehat{G}\|_{L^2} \\ &= C \|F\|_{W^{(s_1,s_2,s_3)}} \|G\|_{W^{(s_1,s_2,s_3)}} \end{aligned}$$

and the similar estimate for $\widehat{F}_{(0,0,0)} * \widehat{G}_{(1,1,1)}$.

As an example of the remaining terms, let us consider $\widehat{F}_{(1,0,1)} * \widehat{G}_{(0,1,0)}$. By Minkowski's inequality for integrals and Young's inequality,

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{3n}} \widehat{F}_{(1,0,1)}(\xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \widehat{G}_{(0,1,0)}(\eta_1, \eta_2, \eta_3) d\eta_1 d\eta_2 d\eta_3 \right\|_{L^2_{\xi_1, \xi_2, \xi_3}} \\ & \leq \left\| \int_{\mathbb{R}^{2n}} \left\| \int_{\mathbb{R}^n} \widehat{F}_{(1,0,1)}(\xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \widehat{G}_{(0,1,0)}(\eta_1, \eta_2, \eta_3) d\eta_1 \right\|_{L^2_{\xi_1}} d\eta_2 d\eta_3 \right\|_{L^2_{\xi_2, \xi_3}} \\ & \leq \left\| \int_{\mathbb{R}^{2n}} \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \right\|_{L^2_{\xi_1}} \left\| \widehat{G}_{(0,1,0)}(\xi_1, \eta_2, \eta_3) \right\|_{L^1_{\xi_1}} d\eta_2 d\eta_3 \right\|_{L^2_{\xi_2, \xi_3}}. \end{aligned}$$

Repeating this argument, we have

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{2n}} \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \right\|_{L^2_{\xi_1}} \left\| \widehat{G}_{(0,1,0)}(\xi_1, \eta_2, \eta_3) \right\|_{L^1_{\xi_1}} d\eta_2 d\eta_3 \right\|_{L^2_{\xi_2, \xi_3}} \\ & \leq \left\| \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2 - \eta_2, \xi_3 - \eta_3) \right\|_{L^2_{\xi_1}} \left\| \widehat{G}_{(0,1,0)}(\xi_1, \eta_2, \eta_3) \right\|_{L^1_{\xi_1}} d\eta_2 \right\|_{L^2_{\xi_2}} d\eta_3 \right\|_{L^2_{\xi_3}} \\ & \vdots \\ & \leq \left\| \left\| \left\| \widehat{F}_{(1,0,1)}(\xi_1, \xi_2, \xi_3) \right\|_{L^2_{\xi_1}} \left\| \left\| \left\| \widehat{G}_{(0,1,0)}(\xi_1, \xi_2, \xi_3) \right\|_{L^1_{\xi_1}} \right\|_{L^2_{\xi_2}} \right\|_{L^1_{\xi_3}} \right\|. \end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned} & \left[\int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{F}_{(1,0,1)}(\xi_1, \xi_2, \xi_3)^2 d\xi_1 \right)^{1/2} d\xi_2 \right\}^2 d\xi_3 \right]^{1/2} \\ & \leq C \left(\int_{\mathbb{R}^{3n}} \langle \xi_2 \rangle^{2s_2} \widehat{F}_{(1,0,1)}(\xi_1, \xi_2, \xi_3)^2 d\xi_1 d\xi_2 d\xi_3 \right)^{1/2} = C \|F\|_{W^{(s_1, s_2, s_3)}} \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{G}_{(0,1,0)}(\xi_1, \xi_2, \xi_3) d\xi_1 \right)^2 d\xi_2 \right\}^{1/2} d\xi_3 \\ & \leq C \left(\int_{\mathbb{R}^{3n}} \langle \xi_1 \rangle^{2s_1} \langle \xi_3 \rangle^{2s_3} \widehat{G}_{(0,1,0)}(\xi_1, \xi_2, \xi_3)^2 d\xi_1 d\xi_2 d\xi_3 \right)^{1/2} = C \|G\|_{W^{(s_1, s_2, s_3)}}. \end{aligned}$$

Thus $\|\widehat{F}_{(1,0,1)} * \widehat{G}_{(0,1,0)}\|_{L^2} \leq C \|F\|_{W^{(s_1, s_2, s_3)}} \|G\|_{W^{(s_1, s_2, s_3)}}$. We can estimate the other terms in the same way. ■

B Appendix

We explain the connection of the Kato–Ponce inequality (1.7) and multilinear multipliers of limited smoothness.

For matters of simplicity in the presentation we take $N = 2$ and we work with Schwartz functions f and g . Introduce a smooth bump Ψ that is supported in the annulus $6/7 < |\xi| < 2$ and is equal to one on $1 < |\xi| < 12/7$ and such that

$$\sum_j \Psi(2^{-j}\xi) = 1$$

for all $\xi \neq 0$. Let Δ_j be the associated Littlewood–Paley operator given by $\widehat{\Delta_j(f)}(\xi) = \Psi(2^{-j}\xi)\widehat{f}(\xi)$. Then we have

$$fg = \sum_{j,k} \Delta_j(f)\Delta_k(g),$$

and this identity holds for every $x \in \mathbb{R}^n$. We introduce the operator $S_k = \sum_{j \leq k} \Delta_j$ and we note that it is given by multiplication on the Fourier transform by a function $\Phi(2^{-k}\xi)$, which is equal to one on the ball $|\xi| \leq 2^k$.

We write the product fg as a sum of three terms:

$$\Pi_1(f, g) = \sum_{j < k-1} \Delta_j(f)\Delta_k(g),$$

$$\Pi_2(f, g) = \sum_{k < j-1} \Delta_j(f)\Delta_k(g),$$

$$\Pi_3(f, g) = \sum_{|j-k| \leq 1} \Delta_j(f)\Delta_k(g).$$

Let $s > 0$. Then

$$D^s(\Pi_1(f, g))(x) = \frac{1}{(2\pi)^{2n}} \sum_k \int \int e^{ix \cdot (\xi+\eta)} S_{k-2}(f)(\xi) \widehat{\Delta_k(g)}(\eta) |\xi + \eta|^s d\xi d\eta,$$

which equals

$$D^s(\Pi_1(f, g))(x) = \frac{1}{(2\pi)^{2n}} \int \int \widehat{f}(\xi) |\eta|^s \widehat{g}(\eta) e^{ix \cdot (\xi+\eta)} \left[\sum_k \Phi(2^{-k+2}\xi) \Psi(2^{-k}\eta) \frac{|\xi + \eta|^s}{|\eta|^s} \right] d\xi d\eta,$$

and the expression inside the square brackets is a bilinear Coifman–Meyer multiplier, hence boundedness holds. A similar argument applies for Π_2 .

Now we look at Π_3 . For simplicity let us only consider the term where $j = k$, i.e.,

$$\sum_j \Delta_j(f)\Delta_j(g).$$

Then we write

$$\begin{aligned} D^s(\Pi_3(f, g)) &= \sum_k \Delta_k D^s \left(\sum_j \Delta_j(f)\Delta_j(g) \right) \\ &= \sum_k D^s \Delta_k \left(\sum_{j \geq k-2} \Delta_j(f)\Delta_j(g) \right) \\ &= \sum_k 2^{ks} \tilde{\Delta}_k \left(\sum_{j \geq k-2} \Delta_j(f)\Delta_j(g) \right) \\ &= \sum_k 2^{ks} \tilde{\Delta}_k \left(\sum_{j' \geq -2} \Delta_{j'+k}(f)\Delta_{j'+k}(g) \right) \\ &= \sum_{j' \geq -2} 2^{-sj'} \sum_k \tilde{\Delta}_k \left(2^{(j'+k)s} \Delta_{j'+k}(f)\Delta_{j'+k}(g) \right) \\ &= \sum_{j \geq -2} 2^{-sj} \sum_k \tilde{\Delta}_k \left(\Delta'_{j+k}(D^s f)\Delta_{j+k}(g) \right), \end{aligned}$$

where

$$\begin{aligned} \tilde{\Delta}_k f &= \mathcal{F}^{-1} \left(\widehat{f}(\xi) |2^{-k}\xi|^s \Psi(2^{-k}\xi) \right), \\ \Delta'_k f &= \mathcal{F}^{-1} \left(\widehat{f}(\xi) |2^{-k}\xi|^{-s} \Psi(2^{-k}\xi) \right). \end{aligned}$$

The symbol of the preceding bilinear operator is

$$\sum_{j \geq -2} 2^{-sj} \sum_{k \in \mathbb{Z}} \Theta(2^{-k}(\xi + \eta)) \Psi_1(2^{-(j+k)}\xi) \Psi(2^{-(j+k)}\eta)$$

with $\Theta(\xi) = |\xi|^s \Psi(\xi)$ and $\Psi_1(\xi) = |\xi|^{-s} \Psi(\xi)$. This is a type of multiplier of limited smoothness, which is studied in this work. The study of the Kato–Ponce inequality via this approach is motivated by the work of Christ and Weinstein [3].

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