



Uniform distribution of the Steinitz invariants of quadratic and cubic extensions

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ABSTRACT

It is shown that the Steinitz invariants of the cubic extensions of a number field are uniformly distributed in the class group when the cubic extensions are ordered by the ideal norm of their relative discriminants. This remains true even if the extensions are restricted by specifying their splitting type at a finite number of places. The same statement is also proved for quadratic extensions.

1. Introduction

Algebraic number theory began with a focus on the arithmetic of number fields viewed as extensions of the rational numbers. As the subject developed, its internal logic forced a change in perspective: any number field k might serve as the domain of rationality and extensions of k were to be studied as such. As the ring of integers, \mathcal{O}_k , of k is not generally a PID, new phenomena appear in this relative setting. For example, there might be finitely-generated torsion-free \mathcal{O}_k -modules that are not free. There is an invariant of such \mathcal{O}_k -modules, the Steinitz invariant, which takes its values in the ideal class group, $\text{Cl}(k)$, of k and quantifies this phenomenon. The rings of integers of finite extensions of k are an abundant and important source of finitely-generated torsion-free \mathcal{O}_k -modules and it is natural to inquire about the properties of their Steinitz invariants. There is substantial and growing literature that studies this question by algebraic methods [Art65, Car98, Fro60, God02, GS03, KP00, Sov02]. Here we use analytic methods to establish an equidistribution result for the Steinitz invariant of the ring of integers of a quadratic or cubic extension of a fixed number field. We may state a simplified version of our main result, Corollary 5.6, as follows.

THEOREM 1.1. *Let k be a number field of class number h and fix $\mathcal{C} \in \text{Cl}(k)$. If M is a finitely-generated torsion-free \mathcal{O}_k -module then let $S(M)$ denote its Steinitz invariant. If K is a finite extension of k then denote by $\Delta_{K/k}$ the relative discriminant of K over k and by $\mathcal{N}(\Delta_{K/k})$ the absolute ideal norm of $\Delta_{K/k}$. For $d = 2$ or 3 we have*

$$\lim_{B \rightarrow \infty} \frac{|\{K \mid [K : k] = d, \mathcal{N}(\Delta_{K/k}) \leq B, S(\mathcal{O}_K) = \mathcal{C}\}|}{|\{K \mid [K : k] = d, \mathcal{N}(\Delta_{K/k}) \leq B\}|} = \frac{1}{h},$$

where the vertical bars stand for cardinality.

As mentioned in the abstract, the full version of this result allows the extension K to be restricted by specifying its completion, up to isomorphism, at each of finitely-many places of k . Also, for $d = 3$, the contribution to the numerator and denominator from cyclic cubic extensions is negligible and so they may be excluded. The conclusion remains the same.

Received 6 April 2004, accepted in final form 27 April 2005.

2000 Mathematics Subject Classification 11R45 (primary), 11R11, 11R16, 11S90, 11M41 (secondary).

Keywords: Steinitz invariant, cubic extension, equidistribution, Shintani zeta function.

The first-named author was partially supported by NSF grant DMS-0244741.

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In [Fro60], Fröhlich proved that every class in $\text{Cl}(k)$ is the Steinitz invariant of \mathcal{O}_K for some quadratic extension K of k . For $d = 2$, our result is a refinement of Fröhlich’s. For $d = 3$, it has a corollary that every class in $\text{Cl}(k)$ may be realized as the Steinitz invariant of a non-normal cubic extension of k .

We now describe the ingredients that go into the proof of the theorem. The first is a general construction of a $\text{Cl}(k)$ -valued invariant associated with certain group actions. Specifically, suppose that we have an affine algebraic group G , an affine space V on which G acts rationally, a non-zero polynomial P on V , and a character ω of G such that $P(gx) = \omega(g)^m P(x)$ for some $m \geq 1$. Suppose also that all of these objects are defined over the ring \mathcal{O}_k . In this setting we obtain a map $S : G_k \backslash V'_k \rightarrow \text{Cl}(k)$, where V' is the complement of the hypersurface $P = 0$. The construction of S is described in § 2.

If we take $G = \text{GL}(2)$ and V to be the space of binary cubic forms, with the associated discriminant polynomial and character, then it is known that the set $G_k \backslash V'_k$ is in one-to-one correspondence with the set of isomorphism classes of separable cubic k -algebras. We write A_x for the cubic k -algebra associated with $x \in V'_k$. The correspondence $x \leftrightarrow A_x$ was constructed by Delone and Fadeev [DF64] by giving an explicit model for the algebra A_x in terms of the coefficients of x . The key observation that makes our result possible is that the Steinitz invariant of \mathcal{O}_{A_x} is equal to the invariant $S(x)$ described in the previous paragraph. This identification is made in § 3.

In order to exploit this equality, we turn to the zeta function of the space of binary cubic forms. This remarkable meromorphic function was introduced by Shintani [Shi72] when $k = \mathbb{Q}$ as a specific example of the general theory of the zeta functions of prehomogeneous vector spaces [SS74, Yuk93]. It was studied over general numbers fields by Datskovsky and Wright in a series of papers [DW86, DW88, Wri85], and further progress has recently been made by Yukié [Yuk03]. By using this body of theory we are able to obtain the analytic properties of a family of Dirichlet series that should be

$$\xi_d(s, \chi) = \sum_{[K:k]=d} \frac{\chi(S(\mathcal{O}_K))}{\mathcal{N}(\Delta_{K/k})^s},$$

where χ is a character of the class group of k . In reality, the series we obtain are more complicated than this: each term has an additional weight factor that is itself a Dirichlet series of a simpler kind. For conceptual purposes, this complication may be ignored. With the analytic properties of $\xi_d(s, \chi)$ in hand, standard methods of analytic number theory allow us to extract the equidistribution theorem that is our goal. The argument we use originated in Dirichlet’s proof of the infinitude of primes in an arithmetic progression. The theory of the zeta function of the space of binary cubic forms is recalled in § 4 and the complications due to the extra weight factors are dealt with in § 5, thus completing the proof. We would like to emphasize that Yukié’s results in [Yuk03] are essential for our success. A forthcoming paper of Taniguchi will include still more precise information on the analytic properties of the zeta function. This work should appear in due course, and can serve as an alternate reference whenever we require results from [Yuk03].

In response to a question raised by the referee, we offer some remarks on error terms and related issues. Denote by $F_3(B)$ the set of cubic extensions K of k with $\mathcal{N}(\Delta_{K/k}) \leq B$, and by $F_3(\mathcal{C}, B)$ the subset of $F_3(B)$ consisting of fields with $S(\mathcal{O}_K) = \mathcal{C}$. It is known that there is a constant $\alpha > 0$ such that $|F_3(B)| = \alpha B + o(B)$ as $B \rightarrow \infty$. In unpublished work, the second-named author improved this to $|F_3(B)| = \alpha B + O(B/\log(B))$. Our main result here states that $|F_3(\mathcal{C}, B)| = (\alpha/h)B + o(B)$, and this can be improved in the same way. However, it is expected that this is very far from the true error term. In fact, as explained in [Rob01], one expects at least that there are constants $\beta < 0$ and $r < 5/6$ such that $|F_3(B)| = \alpha B + \beta B^{5/6} + O(B^r)$. One can modify the heuristics leading to this proposal to take account of the Steinitz class and the result is very interesting. It is

that $|F_3(\mathcal{C}, B)| = (\alpha/h)B + \beta(\mathcal{C})B^{5/6} + O(B^r)$, where the constant $\beta(\mathcal{C})$ depends in a non-trivial way on \mathcal{C} . That is, while all Steinitz classes are asymptotically equally populated to first order, when the second-order terms are taken into account, some classes are systematically favored over others. The precise formula for $\beta(\mathcal{C})$ will be presented in a later paper. In recent work, Belabas *et al.* [BBP04] obtained the error estimate $|F_3(B)| = \alpha B + O(B^{19/20+\epsilon})$ for any $\epsilon > 0$ when $k = \mathbb{Q}$. A first step in pursuing their method in the current setting would be to extend the Delone–Fadeev correspondence in a suitable manner to all cubic algebras over a Dedekind domain.

Next let us discuss possible extensions of our results. There are prehomogeneous vector spaces that play the same role for quartic and quintic extensions of k that the space of binary cubic forms plays for cubic extensions [WY92]. The invariant $S : G_k \backslash V'_k \rightarrow \text{Cl}(k)$ attached to these spaces again coincides with the Steinitz invariant of the ring of integers of the field parameterized by an orbit and so we may hope to extend our result to $d = 4$ and $d = 5$. For quartic extensions, the analytic theory of the associated zeta function has nearly reached the point where this will be possible, thanks again to the efforts of Yukié. For quintic extensions it will probably be some time before the same can be said. However, the standard conjectures on the analytic behavior of the zeta function of a prehomogeneous vector space also imply our theorem in that case. It seems unlikely that there is anything special about $d \leq 5$ and so we venture to suggest the following conjecture. By a *symmetric extension of degree d* we mean an extension K/k of degree d such that the Galois group of the normal closure of K over k is isomorphic to \mathfrak{S}_d .

CONJECTURE 1.2. For any $d \geq 2$ the Steinitz invariant of \mathcal{O}_K is equidistributed in $\text{Cl}(k)$ as K varies over the symmetric extensions of k of degree d , ordered by $\mathcal{N}(\Delta_{K/k})$. This remains true even if K is restricted by specifying its splitting type at a finite number of places.

Beyond this there are many other quadruples (G, V, P, ω) whose orbit space $G_k \backslash V'_k$ is in one-to-one correspondence with some set of objects of arithmetic interest. For prehomogeneous examples the reader may consult [Yuk00]. There are also non-prehomogeneous examples, such as the space of binary n -tics for $n \geq 4$. It is likely that the invariant $S(x)$ will be equidistributed (with suitable weights) in $\text{Cl}(k)$ as x varies over $G_k \backslash V'_k$, provided that this set is infinite. One challenge in these cases is to identify the invariant S in more familiar terms and to elucidate its significance.

2. A general construction

In this section we define the ideal class invariant that lies at the heart of our work. To begin with we require notation for some of the standard objects of algebraic number theory. Let k be a number field, \mathcal{O}_k be its ring of integers, $\text{Cl}(k)$ be its ideal class group, and $\mathfrak{M}(k)$, $\mathfrak{M}_f(k)$ and $\mathfrak{M}_\infty(k)$ be the set of places, finite places and infinite places of k , respectively. For $v \in \mathfrak{M}(k)$, k_v will denote the completion of k at v and $|\cdot|_v$ the normalized absolute value on k_v , as in [Wei74]. If $v \in \mathfrak{M}_f(k)$ then \mathcal{O}_v will be the ring of integers in k_v , \mathcal{P}_v the maximal ideal, q_v the cardinality of $\mathcal{O}_v/\mathcal{P}_v$, \mathcal{O}_v^\times the units of \mathcal{O}_v and $\varpi_v \in \mathcal{P}_v$ a fixed uniformizer. If \mathfrak{a} is a fractional ideal of k whose completion at a finite place v is \mathcal{P}_v^i then we shall write $\text{ord}_v(\mathfrak{a}) = i$; we extend this notation to elements $\alpha \in k^\times$ by confusing α with the fractional ideal it generates. The completion of \mathfrak{a} at v is denoted by \mathfrak{a}_v , so that $\mathfrak{a}_v = \mathcal{P}_v^{\text{ord}_v(\mathfrak{a})}$. Each finite place v of k is associated with a prime ideal $\mathfrak{p}_v = \mathcal{O}_k \cap \mathcal{P}_v$ of \mathcal{O}_k . There is a unique homomorphism \mathcal{N} from the group of fractional ideals of k to \mathbb{Q}^\times such that $\mathcal{N}(\mathfrak{p}_v) = q_v$ for all $v \in \mathfrak{M}_f(k)$; we call \mathcal{N} the *ideal norm*. If \mathfrak{a} is a non-zero ideal of \mathcal{O}_k then $\mathcal{N}(\mathfrak{a})$ is the cardinality of the finite ring $\mathcal{O}_k/\mathfrak{a}$.

The adèles of k will be denoted by \mathbb{A}_k , the ideles by \mathbb{A}_k^\times and the idele norm by $|\cdot|$. We put

$$\mathbb{A}_{k,\infty}^\times = \prod_{v \in \mathfrak{M}_\infty(k)} k_v^\times$$

and regard $\mathbb{A}_{k,\infty}^\times$ as a subgroup of \mathbb{A}_k^\times . The class group of k may be identified with the quotient of \mathbb{A}_k^\times by the subgroup

$$J_k = k^\times \cdot \mathbb{A}_{k,\infty}^\times \cdot \prod_{v \in \mathfrak{M}_f(k)} \mathcal{O}_v^\times.$$

This identification is made by sending the class of the fractional ideal \mathfrak{a} of k to the class of the idele $\iota(\mathfrak{a})$ whose components are given by

$$\iota(\mathfrak{a})_v = \begin{cases} 1 & \text{if } v \in \mathfrak{M}_\infty(k), \\ \varpi_v^{\text{ord}_v(\mathfrak{a})} & \text{if } v \in \mathfrak{M}_f(k). \end{cases}$$

We shall henceforth identify $\text{Cl}(k)$ with \mathbb{A}_k^\times/J_k via ι .

Let G be an algebraic group, V a rational representation of G , ω a rational character of G and P a non-zero polynomial on V that satisfies the transformation law $P(gx) = \omega(g)^m P(x)$ for some fixed $m \geq 1$. Denote by V' the complement of the hypersurface $P = 0$ in V . Assume that the quadruple (G, V, ω, P) is defined over \mathcal{O}_k . In addition, assume that if $v \in \mathfrak{M}_f(k)$ and $x \in V'_{k_v}$ then $G_{k_v}x \cap V_{\mathcal{O}_v} \neq \emptyset$. This hypothesis will be satisfied, for example, if the center of G contains a split torus that acts on V by a non-trivial rational character. In this situation we define

$$\Delta_v(x) = \max\{|P(y)|_v \mid y \in G_{k_v}x \cap V_{\mathcal{O}_v}\}$$

for $v \in \mathfrak{M}_f(k)$ and $x \in V'_{k_v}$. Note that $|P(y)|_v$ is bounded above for $y \in V_{\mathcal{O}_v}$ and that zero is the only limit point of the set $|k_v^\times|_v$; thus the indicated maximum is achieved, and $\Delta_v(x)$ is well defined. It is clear that $\Delta_v(x)$ depends only on the G_{k_v} -orbit of x . Furthermore, $\Delta_v(x) \leq 1$ for all $v \in \mathfrak{M}_f(k)$ and $x \in V'_{k_v}$.

Now let $x \in V'_k$. For each finite place v of k we may choose $g_{x,v} \in G_{k_v}$ such that $|P(g_{x,v}x)|_v = \Delta_v(x)$. Let $a(x)$ be the idele whose components are

$$a(x)_v = \begin{cases} 1 & \text{if } v \in \mathfrak{M}_\infty(k), \\ \omega(g_{x,v}) & \text{if } v \in \mathfrak{M}_f(k), \end{cases}$$

and $S(x)$ be the class of $a(x)$ in $\text{Cl}(k)$. Note that $x \in V_{\mathcal{O}_v}$ and $|P(x)|_v = 1$ for all but finitely many $v \in \mathfrak{M}_f(k)$. Thus $\omega(g_{x,v}) \in \mathcal{O}_v^\times$ for all but finitely many $v \in \mathfrak{M}_f(k)$, so that $a(x)$ is well defined.

LEMMA 2.1. *The class $S(x) \in \text{Cl}(k)$ does not depend on the choices made in its definition. If $x \in V'_k$ and $g \in G_k$ then $S(gx) = S(x)$, so that S may be regarded as a map from $G_k \backslash V'_k$ to $\text{Cl}(k)$.*

Proof. The only choices made in the definition of $S(x)$ are those of $g_{x,v}$ for $v \in \mathfrak{M}_f(k)$. Suppose that $(g_{x,v})_v$ is one possible choice and $(g'_{x,v})_v$ is another. Then

$$|\omega(g_{x,v})|_v = (|P(g_{x,v}x)|_v / |P(x)|_v)^{1/m} = (\Delta_v(x) / |P(x)|_v)^{1/m}$$

and similarly with $g'_{x,v}$ in place of $g_{x,v}$. It follows that $|\omega(g_{x,v})|_v = |\omega(g'_{x,v})|_v$ and so $\omega(g'_{x,v})\omega(g_{x,v})^{-1} \in \mathcal{O}_v^\times$. If $a(x)$ is the idele defined with $(g_{x,v})_v$ and $a'(x)$ is the idele defined with $(g'_{x,v})_v$ then we conclude that $a'(x)a(x)^{-1} \in J_k$ and so $[a(x)] = [a'(x)]$.

Take $x \in V'_k$ and $g \in G_k$ and suppose that $(g_{x,v})_v$ is as above. As $\Delta_v(gx) = \Delta_v(x)$, the family $(g_{x,v}g^{-1})_v$ has the defining property with respect to gx and so $a(gx) = a(x)\omega(g)^{-1}$. However, $\omega(g) \in k^\times$ and so $a(gx)a(x)^{-1} \in J_k$. Thus $[a(gx)] = [a(x)]$. \square

The construction of the map S is rather formal and the reader might wonder what its significance is. In the next section we throw some light on this question by identifying the map S associated with the space of binary cubic forms with the Steinitz invariant of the ring of integers of a cubic extension of a number field.

3. Binary cubic forms and the Steinitz invariant

For the reader’s convenience, we recall certain facts from the theory of modules over Dedekind domains. This theory is developed at length in [Bou65, ch. 7] and is also described in [Coh00, ch. 1] and in [FT91, ch. 2]. Let R be a Dedekind domain with field of fractions F and M be a finitely-generated R -module. Then M is projective if and only if it is torsion-free. If M is torsion-free then of rank m then there is a fractional ideal \mathfrak{a} of F such that $M \cong R^{m-1} \oplus \mathfrak{a}$ and the ideal class of \mathfrak{a} depends only on the isomorphism class of M . This ideal class is called the *Steinitz invariant* of M and we shall denote it by $S(M)$. Note that in the Picard group of \mathcal{O}_k we have $S(M) = \wedge^m M$; this interpretation of the Steinitz invariant has recently been emphasized by Khare and Prasad [KP00]. The theory we have just recalled applies in particular to the ring of integers of a number field. In this case, we have the following result [Art65].

LEMMA 3.1 (Artin [Art65]). *Let k be a number field and K a finite extension of k . Let $\delta_{K/k} \in k^\times$ be the discriminant of the trace form with respect to some k -basis of K . There is a fractional ideal \mathfrak{a} of k such that $\Delta_{K/k} = (\delta_{K/k})\mathfrak{a}^2$ and $S(\mathcal{O}_K) = [\mathfrak{a}]$.*

Our next task is to describe the space of binary cubic forms and its relation with the set of based cubic algebras. This subject has received detailed treatments in [DF64, WY92] and so, apart from establishing essential notation, we shall be brief. The new feature of our discussion is Proposition 3.3, which identifies the Steinitz invariant of a cubic ring in this picture.

Working initially over \mathbb{Z} , let V be affine four-space and denote by x_1, \dots, x_4 the canonical coordinates on V . With a point $x = (x_1, \dots, x_4)$ in V we associate the binary cubic form

$$F_x(u_1, u_2) = x_1u_1^3 + x_2u_1^2u_2 + x_3u_1u_2^2 + x_4u_2^3.$$

The group $G = \text{GL}(2)$ acts on V in such a way that

$$F_{gx}(u_1, u_2) = \det(g)^{-1}F_x((u_1, u_2)g)$$

and the quartic polynomial

$$P(x) = x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_2^3x_4 - 4x_1x_3^3 - 27x_1^2x_4^2$$

on V satisfies the transformation law

$$P(gx) = \det(g)^2P(x)$$

with respect to this action. With $\omega(g) = \det(g)$ and $m = 2$, this places us in the situation of § 2 and we shall employ the notation introduced there. When k is a number field we obtain a map $S : G_k \backslash V'_k \rightarrow \text{Cl}(k)$ associated with the quadruple (G, V, ω, P) .

Let R be a commutative ring with 1 and A a free R -module of rank three with a fixed ordered basis e, ω_1, ω_2 . If $x \in V_R$ then we may introduce an R -bilinear multiplication on A in such a way that it becomes a commutative, associative R -algebra with multiplicative identity e by defining the basic products as

$$\begin{aligned} \omega_1^2 &= -x_1x_3e - x_2\omega_1 + x_1\omega_2, \\ \omega_2^2 &= -x_2x_4e - x_4\omega_1 + x_3\omega_2, \\ \omega_1\omega_2 &= -x_1x_4e. \end{aligned}$$

We denote the resulting R -algebra by A_x . The map $x \mapsto A_x$ was discovered by Delone and Fadeev [DF64] and hence we call it the Delone–Fadeev map. For brevity’s sake, we define a *cubic R -algebra* to be a commutative, associative R -algebra with identity that is free of rank three as an R -module. The following result may be extracted from the discussion in [DF64, § 15].

LEMMA 3.2 (Delone–Fadeev [DF64]). *The isomorphism class of A_x depends only on the G_R -orbit of x in V_R . The discriminant of the trace form of A_x with respect to the basis e, ω_1, ω_2 is $P(x)$. The set $G_R \backslash V'_R$ is in one-to-one correspondence with the set of isomorphism classes of separable cubic R -algebras via the Delone–Fadeev map.*

Let k be a number field. If K is a separable cubic k -algebra then K is isomorphic to a direct sum of number fields and so we may make sense of the notation \mathcal{O}_K as the direct sum of the rings of integers in each of these fields. Then \mathcal{O}_K is a torsion-free \mathcal{O}_k -module and so the Steinitz invariant $S(\mathcal{O}_K)$ is defined. Note that if $K \cong k^3$ then $S(\mathcal{O}_K)$ is the identity in $\text{Cl}(k)$ and if $K \cong k \oplus E$, with E a quadratic extension of k , then $S(\mathcal{O}_K) = S(\mathcal{O}_E)$. Observe that Lemma 3.1 extends without change to this setting.

Take $v \in \mathfrak{M}(k)$. As the set of extensions of k_v of fixed degree is finite, $G_{k_v} \backslash V'_{k_v}$ is finite. It will be convenient to choose specific representatives for each of the orbits in this set. Let $n(v)$ be the cardinality of $G_{k_v} \backslash V'_{k_v}$. If $v \in \mathfrak{M}_\infty(k)$ then choose representatives $w_{i,v}, 1 \leq i \leq n(v)$, for the orbits in $G_{k_v} \backslash V'_{k_v}$ such that $|P(w_{i,v})|_v = 1$. Note that $n(v) = 1$ if v is a complex place and $n(v) = 2$ if v is a real place. Suppose now that $v \in \mathfrak{M}_f(k)$, take $x \in G_{k_v} \backslash V'_{k_v}$ and let A_x be the associated separable cubic k_v -algebra. Then A_x is isomorphic to a direct sum of local fields and there is unique maximal compact subring R_x in A_x . The ring R_x is in fact a cubic \mathcal{O}_v -algebra and there is an orbit $G_{\mathcal{O}_v} w$ in $G_{\mathcal{O}_v} \backslash V'_{\mathcal{O}_v}$ that corresponds to R_x under the Delone–Fadeev map for \mathcal{O}_v . As every \mathcal{O}_v -subalgebra of A_x is contained in R_x , $G_{\mathcal{O}_v} w$ is the unique $G_{\mathcal{O}_v}$ -orbit on which $|P(y)|_v$ achieves its maximum value for $y \in G_{k_v} x \cap V_{\mathcal{O}_v}$. In the notation of § 2 this means that $\Delta_v(x) = |P(w)|_v$ and, moreover, that this equality determines the $G_{\mathcal{O}_v}$ -orbit of w . We now choose representatives $w_{i,v}, 1 \leq i \leq n(v)$, for the orbits in $G_{k_v} \backslash V'_{k_v}$ that satisfy $\Delta_v(w_{i,v}) = |P(w_{i,v})|_v$. With orbital representatives fixed at all $v \in \mathfrak{M}(k)$, we introduce one more piece of notation: if $x \in V'_{k_v}$ then let $i_v(x)$ be the index such that $x \in G_{k_v} w_{i_v(x),v}$.

PROPOSITION 3.3. *Let k be a number field, $x \in V'_k$ and K the separable cubic k -algebra corresponding to x . Then $S(\mathcal{O}_K) = S(x)$.*

Proof. For each $v \in \mathfrak{M}_f(k)$ choose $g_{x,v} \in G_{k_v}$ such that $g_{x,v}x = w_{i_v(x),v}$. Let $a(x)$ be the idele whose infinite components are all 1 and whose component at a finite place v is $\det(g_{x,v})$. The choice of orbital representatives $w_{i,v}$ made above is consistent with the definition of S and so $S(x) = [a(x)]$. The cubic \mathcal{O}_v -algebra $R_v = \mathcal{O}_v \otimes_{\mathcal{O}_k} \mathcal{O}_K$ is the maximal compact subring of the cubic k_v -algebra $k_v \otimes_k K$. As such, R_v corresponds to $w_{i_v(x),v}$ under the Delone–Fadeev map, and it follows from Lemma 3.2 that the discriminant ideal of R_v over \mathcal{O}_v is generated by $P(w_{i_v(x),v})$. This ideal coincides with the completion of $\Delta_{K/k}$ at v . We have $P(w_{i_v(x),v}) = \det(g_{x,v})^2 P(x)$ and so if $\mathfrak{a}(x)$ is the fractional ideal of k corresponding to the idele $a(x)$ then $\Delta_{K/k} = (P(x))\mathfrak{a}(x)^2$. By Lemma 3.2, $P(x)$ is the discriminant of the trace form with respect to some k -basis of K . It follows from Lemma 3.1, or its extension to separable algebras, that $S(\mathcal{O}_K) = [\mathfrak{a}(x)]$. □

4. The zeta function

In this section we describe the zeta function of the space of binary cubic forms and its connection to the problem considered here. As usual, we must begin by introducing some further notation and recalling various standard facts.

Let k be a number field. A continuous homomorphism $\omega : \mathbb{A}_k^\times / k^\times \rightarrow \mathbb{C}^\times$ will be referred to as an *idele class character* of k . The group of all idele class characters of k will be denoted by $\Omega(k)$. Included in this group are the principal idele class characters ω_s for $s \in \mathbb{C}$, defined by $\omega_s(a) = |a|^s$. Let $\mathbb{A}_k^1 = \{a \in \mathbb{A}_k^\times \mid |a| = 1\}$ be the kernel of the idele norm. Associated with the idele norm there

is a short exact sequence

$$\{1\} \longrightarrow \mathbb{A}_k^1 \longrightarrow \mathbb{A}_k^\times \longrightarrow \mathbb{R}_+ \longrightarrow \{1\},$$

where \mathbb{R}_+ denotes the multiplicative group of positive real numbers. This sequence is split and we may choose a specific splitting $\lambda \mapsto \underline{\lambda}$ such that $\underline{\lambda}_v = 1$ for all $v \in \mathfrak{M}_f(k)$. Once this splitting is chosen we obtain an isomorphism

$$\mathbb{A}_k^\times \cong \mathbb{R}_+ \times \mathbb{A}_k^1. \tag{1}$$

If $a \in \mathbb{A}_k$ then let $(|a|, \tilde{a}) \in \mathbb{R}_+ \times \mathbb{A}_k^1$ be the pair corresponding to a under the isomorphism (1). We also have an isomorphism

$$\mathbb{A}_k^\times/k^\times \cong \mathbb{R}_+ \times \mathbb{A}_k^1/k^\times \tag{2}$$

induced by (1).

Let $\Omega^1(k)$ be the group of continuous homomorphisms from \mathbb{A}_k^1/k^\times to \mathbb{C}^\times ; note that, since \mathbb{A}_k^1/k^\times is compact, the image of any such homomorphism actually lies in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Corresponding to the isomorphism (2) we obtain an isomorphism $\Omega(k) \cong \mathbb{C} \times \Omega^1(k)$. The character $\omega \in \Omega(k)$ corresponds under this isomorphism to the pair $(s(\omega), \omega|_{\mathbb{A}_k^1/k^\times})$, where $s(\omega)$ is the unique complex number such that $\omega(\underline{\lambda}) = \lambda^{s(\omega)}$. If $\omega \in \Omega^1(k)$ then let $\tilde{\omega} \in \Omega(k)$ be the character defined by $\tilde{\omega}(a) = \omega(\tilde{a})$. Note that $s(\tilde{\omega}) = 0$ for all $\omega \in \Omega^1(k)$. As J_k contains the image of $\lambda \mapsto \underline{\lambda}$, the inclusion $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k$ induces an isomorphism

$$\text{Cl}(k) \cong \mathbb{A}_k^1/J_k^1,$$

where $J_k^1 = J_k \cap \mathbb{A}_k^1$. Thus, we may identify the characters of the finite group $\text{Cl}(k)$ with the characters of \mathbb{A}_k^1 that are trivial on J_k^1 .

Denote by $G_{\mathbb{A}}$, respectively $V_{\mathbb{A}}$, be the adèle points of G , respectively V , as a group over k . Let $U_v \subset G_{k_v}$ be the orthogonal group if v is a real place of k , the unitary group if v is a complex place of k , and $G_{\mathcal{O}_v}$ if v is a finite place of k . Then

$$U = \prod_{v \in \mathfrak{M}(k)} U_v \subset G_{\mathbb{A}}$$

is a maximal compact subgroup of $G_{\mathbb{A}}$. The image of the determinant map restricted to U is $\mathbb{A}_k^0 = \{a \in \mathbb{A}_k \mid |a_v|_v = 1 \ \forall v \in \mathfrak{M}(k)\}$. Let $\mathcal{S}(V_{\mathbb{A}})$ be the space of Schwartz–Bruhat functions on $V_{\mathbb{A}}$ and $\mathcal{S}_0(V_{\mathbb{A}}) \subset \mathcal{S}(V_{\mathbb{A}})$ be the subspace consisting of U -invariant functions.

In [DW86, Wri85], specific Haar measures were fixed on \mathbb{A}_k , \mathbb{A}_k^\times and $G_{\mathbb{A}}$, as well as on the local factors of each of these groups. We employ the same measures here, albeit with a simplified notation that should be clear in context. From these basic choices, we also obtain Haar measures on $V_{\mathbb{A}}$ and its local factors and on quotient spaces such as $G_{\mathbb{A}}/G_k$.

Let $X \subset V'_k$ be a G_k -invariant set, $\omega \in \Omega(k)$ and $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$. We define the associated zeta function to be

$$Z(\omega, \Phi; X) = \int_{G_{\mathbb{A}}/G_k} \omega(\det(g)) \sum_{x \in X} \Phi(gx) dg.$$

Note that $Z(\omega, \Phi; X) \equiv 0$ unless ω is trivial on $\mathbb{A}_k^0 \cdot k^\times/k^\times$. If $x \in V'_k$ then let A_x be the cubic k -algebra corresponding to x under the Delone–Fadeev map. We have to consider the G_k -invariant sets

$$\begin{aligned} X_2 &= \{x \in V'_k \mid A_x \cong k \oplus E \text{ with } E \text{ a field}\}, \\ X_3 &= \{x \in V'_k \mid A_x \text{ is a field}\}. \end{aligned}$$

Define functionals Σ_2 and Σ_3 on $\mathcal{S}_0(V_{\mathbb{A}})$ by

$$\Sigma_2(\Phi) = \int_{\mathbb{A}_k^\times} \int_{\mathbb{A}_k} \int_{\mathbb{A}_k} |t|^2 \Phi(0, t, y_1, y_2) dy_1 dy_2 d^\times t$$

and

$$\Sigma_3(\Phi) = \int_{V_{\mathbb{A}}} \Phi(x) dx.$$

The following result summarizes some of the analytic properties of the zeta functions $Z(\omega, \Phi; X_2)$ and $Z(\omega, \Phi; X_3)$ in a form convenient for our applications. When discussing the convergence of integrals we use the phrase *normally convergent* to mean absolutely and locally uniformly convergent.

PROPOSITION 4.1. *Let $\chi \in \Omega^1(k)$, $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$ and $s \in \mathbb{C}$ and put*

$$Z_d(s, \chi, \Phi) = Z(\omega_s \tilde{\chi}, \Phi; X_d)$$

for $d = 2, 3$. *The integral defining $Z_d(s, \chi, \Phi)$ is normally convergent in the region $\operatorname{re}(s) > 2$. The resulting holomorphic function has a meromorphic continuation to the region $\operatorname{re}(s) > 5/3$. If χ is non-trivial then this continuation is, in fact, holomorphic. If $\chi = \chi_0$ is trivial then the only possible singularity of $Z_d(s, \chi_0, \Phi)$ in the region $\operatorname{re}(s) > 5/3$ is a simple pole at $s = 2$ with residue $C_d \Sigma_d(\Phi)$, where C_2 and C_3 are positive constants depending only on k .*

Proof. The analytic properties of $Z(\omega, \Phi; V'_k)$ were studied in detail in [Wri85]. If $x_0 \in V'_k$ is a point corresponding to the cubic k -algebra k^3 then we have $V'_k = X_3 \cup X_2 \cup G_k x_0$ and $Z(\omega, \Phi; G_k x_0)$ is easily understood. Thus, if one can obtain the analytic properties of either $Z(\omega, \Phi; X_2)$ or $Z(\omega, \Phi; X_3)$ then those of the other will follow. A direct analysis of $Z(\omega_s, \Phi; X_2)$ is achieved in [Yuk03]. Although this work does not consider $Z(\omega_s \tilde{\chi}, \Phi; X_2)$ for non-trivial χ , the proof turns on the estimation of the region of absolute convergence of certain integrals connected with $Z(\omega_s, \Phi; X_2)$ and, as $|\tilde{\chi}(a)| = 1$ for all $a \in \mathbb{A}_k^\times$, it is a straightforward matter to modify the arguments to cover the more general case. \square

Every $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$ is a finite sum of functions that factor as a product over $v \in \mathfrak{M}(k)$ of local Schwartz–Bruhat functions. We assume henceforth that all Φ that we consider have a factorization $\Phi = \bigotimes_v \Phi_v$. Note that Φ_v is invariant under the group U_v . If $v \in \mathfrak{M}_f(k)$ then denote by $\Phi_{0,v}$ the characteristic function of the compact open set V_{O_v} . If $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$ then $\Phi_v = \Phi_{0,v}$ for all but finitely many places v . If $\omega \in \Omega(k)$ then the pullback of ω to \mathbb{A}_k^\times has a factorization $\omega = \bigotimes_v \omega_v$. The local character ω_v is unramified for all but finitely many v . We have $|\omega_v(x)| = |x|_v^{\operatorname{re}(s(\omega))}$ for all $v \in \mathfrak{M}(k)$ and all $x \in k_v^\times$. When we are discussing a single place of k we may denote the exponent in this identity by $\operatorname{re}(\omega_v)$, which is defined without regard to a global character of which ω_v is a factor.

Recall that in § 3 we chose standard local orbital representatives $w_{i,v} \in V'_{k_v}$ subject to certain conditions. If $x \in V'_k$ then, for all $v \in \mathfrak{M}(k)$, $x \in G_{k_v} w_{i_v(x),v}$. For $x \in V'_k$, $\omega \in \Omega(k)$ and $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$ we define

$$\Xi_x(\omega, \Phi) = \prod_{v \in \mathfrak{M}(k)} \Xi_{x,v}(\omega_v, \Phi_v),$$

where

$$\Xi_{x,v}(\omega_v, \Phi_v) = \int_{G_{k_v}} \omega_v(\det(g_v)) \Phi_v(g_v w_{i_v(x),v}) dg_v.$$

Note that $\Xi_x(\omega, \Phi)$ depends on x only through its G_k -orbit. In order to justify these definitions, we give a preliminary convergence result.

PROPOSITION 4.2. *The integral defining $\Xi_{x,v}(\omega_v, \Phi_v)$ is normally convergent in the region $\operatorname{re}(\omega_v) > 1/3$. After possibly discarding a finite number of vanishing factors, the product that defines $\Xi_x(\omega, \Phi)$ is absolutely convergent in the region $\operatorname{re}(s(\omega)) > 1$.*

Proof. The statements are a consequence of [DW86, Corollaries 3.1 and 3.2 and Theorem 3.1], together with the observations recorded above. \square

The following result establishes the essential connection between $Z_d(s, \chi, \Phi)$ and the distribution of the Steinitz invariant of degree d extensions of k in the class group. For each $x \in V'_k$ let $o(x)$ be the cardinality of the isotropy group of x in G_k . The number $o(x)$ is also the cardinality of the group of automorphisms of the cubic k -algebra A_x (see, for example, [DW86]); in particular, $1 \leq o(x) \leq 6$.

PROPOSITION 4.3. *Let χ be a character of $\text{Cl}(k)$ and regard χ equivalently as an element of $\Omega^1(k)$. Let $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$ and $s \in \mathbb{C}$ with $\text{re}(s) > 2$. There is a positive constant C_0 , depending only on k , such that*

$$Z_d(s, \chi, \Phi) = C_0 \sum_{x \in G_k \backslash X_d} o(x)^{-1} \chi(S(\mathcal{O}_{A_x})) \Xi_x(\omega_s \tilde{\chi}, \Phi) \mathcal{N}(\Delta_{A_x/k})^{-s/2}$$

for $d = 2, 3$.

Proof. By unfolding the integral against the sum in the definition of $Z_d(s, \chi, \Phi)$ we obtain

$$Z_d(s, \chi, \Phi) = \sum_{x \in G_k \backslash X_d} o(x)^{-1} \int_{G_{\mathbb{A}}} |\det(g)|^s \tilde{\chi}(\det(g)) \Phi(gx) dg.$$

Each term in this series may be written as a product of local factors. The Haar measure on $G_{\mathbb{A}}$ is necessarily proportional to the product of the Haar measures on G_{k_v} and so

$$Z_d(s, \chi, \Phi) = C_0 \sum_{x \in G_k \backslash X_d} o(x)^{-1} \prod_{v \in \mathfrak{M}(k)} \int_{G_{k_v}} |\det(g_v)|_v^s \tilde{\chi}_v(\det(g_v)) \Phi_v(g_v x) dg_v,$$

where C_0 depends only on k . For each $x \in X_d$ and $v \in \mathfrak{M}(k)$ we may choose $g_{x,v} \in G_{k_v}$ such that $g_{x,v} x = w_{i_v(x),v}$. Let $a(x)$ be the idele whose component at v is $\det(g_{x,v})$. After making the change of variable $g_v \mapsto g_v g_{x,v}$ in each of the local integrals, we arrive at

$$Z_d(s, \chi, \Phi) = C_0 \sum_{x \in G_k \backslash X_d} o(x)^{-1} |a(x)|^s \tilde{\chi}(a(x)) \Xi_x(\omega_s \tilde{\chi}, \Phi).$$

As $\mathbb{A}_{k,\infty}^\times \subset J_k$, it follows from the definition that $a(x)$ lies in the class $S(x)$ in $\mathbb{A}_k^\times / J_k$. By Proposition 3.3 this is also the class $S(\mathcal{O}_{A_x})$ and so $\tilde{\chi}(a(x)) = \chi(S(\mathcal{O}_{A_x}))$. Furthermore, we have $P(w_{i_v(x),v}) = \det(g_{x,v})^2 P(x)$ and so

$$|a(x)| = \prod_{v \in \mathfrak{M}(k)} |P(w_{i_v(x),v})|_v^{1/2} |P(x)|_v^{-1/2} = \prod_{v \in \mathfrak{M}(k)} |P(w_{i_v(x),v})|_v^{1/2},$$

where we have applied the Artin product formula to $P(x) \in k^\times$ to obtain the second equality. The representatives $w_{i_v(x),v}$ were chosen so that $|P(w_{i_v(x),v})|_v = 1$ if $v \in \mathfrak{M}_\infty(k)$ and $w_{i_v(x),v}$ corresponds under the Delone–Fadeev map to the maximal order $R_{x,v} = \mathcal{O}_v \otimes_{\mathcal{O}_k} \mathcal{O}_{A_x}$ if $v \in \mathfrak{M}_f(k)$. If $v \in \mathfrak{M}_f(k)$ then it follows from Lemma 3.2 that $P(w_{i_v(x),v})\mathcal{O}_v$ is the discriminant ideal of $R_{x,v}$. Thus

$$|a(x)| = \prod_{v \in \mathfrak{M}_f(k)} |P(w_{i_v(x),v})|_v^{1/2} = \mathcal{N}(\Delta_{A_x/k})^{-1/2}$$

and we obtain the required identity. \square

5. The filtration process

If $\Xi_x(\omega_s \tilde{\chi}, \Phi)$ were absent from the expression for $Z_d(s, \chi, \Phi)$ given in Proposition 4.3 then our main result would follow from Proposition 4.1, a Tauberian theorem for Dirichlet series and the

orthogonality relations for the characters of $\text{Cl}(k)$. The purpose of this section is to make the technical modifications to this simple argument necessitated by the presence of $\Xi_x(\omega_s \tilde{\chi}, \Phi)$. We refer to these modifications as the filtration process. This process, first described in [DW88], has previously been used to obtain density results from series such as $Z_d(s, \chi_0, \Phi)$. As we must deal with $Z_d(s, \chi, \Phi)$, with χ not necessarily trivial, we require a minor variant of the usual argument.

We have to consider the set of extensions of k whose completion at a finite number of places is fixed. To this end, let $T \subset \mathfrak{M}(k)$ be a finite set. Recall that $n(v)$ denotes the cardinality of the set $G_{k_v} \backslash V'_{k_v}$ and that we have indexed the G_{k_v} -orbits by choosing representatives $w_{i,v}$, $1 \leq i \leq n(v)$, for them. A T -type α will be a map $\alpha : T \rightarrow \mathbb{N}$ such that $1 \leq \alpha(v) \leq n(v)$ for each $v \in T$. If α is a T -type and $x \in V'_k$ then we write $x \approx \alpha$ if $i_v(x) = \alpha(v)$ for all $v \in T$. Thus, $x \approx \alpha$ if the completion of the cubic k -algebra A_x at each $v \in T$ is the cubic k_v -algebra corresponding to the orbit indexed by $\alpha(v)$. If $T \subset T'$, α is a T -type and α' is a T' -type then we write $\alpha'|T = \alpha$ if $\alpha'(v) = \alpha(v)$ for all $v \in T$.

If $T \subset \mathfrak{M}(k)$ is a finite set, α is a T -type, $\mathcal{C} \in \text{Cl}(k)$ and $B > 0$ then the sets

$$\begin{aligned} X_d(\alpha) &= \{x \in X_d \mid x \approx \alpha\}, \\ X_d(\alpha, B) &= \{x \in X_d \mid x \approx \alpha, \mathcal{N}(\Delta_{A_x/k}) \leq B\}, \\ X_d(\alpha, \mathcal{C}) &= \{x \in X_d \mid x \approx \alpha, \mathcal{S}(\mathcal{O}_{A_x}) = \mathcal{C}\}, \\ X_d(\alpha, \mathcal{C}, B) &= X_d(\alpha, B) \cap X_d(\alpha, \mathcal{C}) \end{aligned}$$

are G_k -invariant. The quotient sets $G_k \backslash X_d(\alpha, B)$ and $G_k \backslash X_d(\alpha, \mathcal{C}, B)$ are finite because of the well-known theorem on the finiteness of the set of number fields with bounded discriminant.

For $v \in \mathfrak{M}(k)$, $1 \leq i \leq n(v)$, ω_v a character of k_v^\times and $\Phi_v \in \mathcal{S}(V_{k_v})$ we let

$$Z_{i,v}(\omega_v, \Phi_v) = \int_{G_{k_v}} \omega_v(\det(g_v)) \Phi_v(g_v w_{i,v}) dg_v.$$

Of course, $\Xi_{x,v}(\omega_v, \Phi_v) = Z_{i_v(x),v}(\omega_v, \Phi_v)$, but it will be convenient to discuss these functions without reference to any point $x \in V'_k$.

LEMMA 5.1. *Let $v \in \mathfrak{M}(k)$, $1 \leq i \leq n(v)$, and ω_v^* be an unramified character of k_v^\times . There exists a U_v -invariant function $\Phi_v \in \mathcal{S}(V_{k_v})$ such that the support of Φ_v is contained in $G_{k_v} w_{i,v}$, $Z_{i,v}(\omega_v, \Phi_v)$ is an entire function of ω_v and $Z_{i,v}(\omega_v^*, \Phi_v) \neq 0$.*

Proof. Start with a positive Schwartz function of sufficiently small compact support in the orbit and then average it over U_v . □

For $v \in \mathfrak{M}(k)$ and $\Phi_v \in \mathcal{S}(V_{k_v})$ a U_v -invariant function let

$$\Sigma_{2,v}(\Phi_v) = \int_{k_v^\times} \int_{k_v} \int_{k_v} |t_v|_v^2 \Phi_v(0, t_v, y_{1,v}, y_{2,v}) dy_{1,v} dy_{2,v} d^\times t_v$$

and

$$\Sigma_{3,v}(\Phi_v) = \int_{V_{k_v}} \Phi_v(x_v) dx_v.$$

There are positive constants C'_2 and C'_3 such that if $\Phi \in \mathcal{S}_0(V_{\mathbb{A}})$ and $\Phi = \otimes_v \Phi_v$ then

$$\Sigma_d(\Phi) = C'_d \prod_{v \in \mathfrak{M}(k)} \Sigma_{d,v}(\Phi_v)$$

for $d = 2, 3$. The constant C'_d is the constant of proportionality between the global measures and the product of the local measures.

In order to state the next lemma, it will be convenient to introduce an assumption on the standard representatives $w_{i,v}$ which has not been needed so far. We assume that the list $w_{1,v}, \dots, w_{n(v),v}$

is so arranged that the elements corresponding to cubic k_v -algebras that are fields come at the end. Thus, there is a number $n_2(v) \leq n(v)$ such that $w_{i,v}$ corresponds to a field if and only if $i > n_2(v)$. The point of this is that if $x \in X_2$ then $1 \leq i_v(x) \leq n_2(v)$. If $T \subset \mathfrak{M}(k)$ is a finite set and α is a T -type then we say that α is a *quadratic T -type* if $\alpha(v) \leq n_2(v)$ for all $v \in T$.

LEMMA 5.2. *Let $v \in \mathfrak{M}(k)$. There are positive constants $b_{i,v}$, for $1 \leq i \leq n(v)$, and $c_{i,v}$, for $1 \leq i \leq n_2(v)$, such that*

$$Z_{i,v}(| \cdot |^2_v, \Phi_v) = b_{i,v} \Sigma_{3,v}(\Phi_v)$$

and

$$Z_{i,v}(| \cdot |^2_v, \Phi_v) = c_{i,v} \Sigma_{2,v}(\Phi_v)$$

whenever $\Phi_v \in \mathcal{S}(V_{k_v})$ is a U_v -invariant function whose support is contained in $G_{k_v} w_{i,v}$.

Proof. The first identity is [DW86, (2.4)]. The second follows from Theorem 5.2 of the same reference. □

For $v \in \mathfrak{M}(k)$ we define

$$E_{2,v} = \sum_{i=1}^{n_2(v)} c_{i,v}^{-1}$$

and

$$E_{3,v} = \sum_{i=1}^{n(v)} b_{i,v}^{-1}.$$

LEMMA 5.3. *Let $v \in \mathfrak{M}_f(k)$, $1 \leq i \leq n(v)$ and ω_v be an unramified character of k_v^\times that takes its values in S^1 . As before, let $\Phi_{0,v}$ denote the characteristic function of $V_{\mathcal{O}_v}$. There are constants $\{a_{i,v,n}\}_{n=0}^\infty$ such that*

$$Z_{i,v}(\omega_v | \cdot |^s_v, \Phi_{0,v}) = \sum_{n=0}^\infty a_{i,v,n} \omega_v(\varpi_v)^n q_v^{-ns}$$

for $\text{re}(s) > 1/3$. We have $a_{i,v,0} = 1$ and $a_{i,v,n} \geq 0$ for all n .

Proof. The local integral defining $Z_{i,v}(\omega_v | \cdot |^s_v, \Phi_{0,v})$ was evaluated for all i and $v \in \mathfrak{M}_f(k)$ in [DW86, Theorem 3.1]. The results are conveniently summarized in [DW88, § 2]. The claims follow by inspection. In [Yuk03, § 6] the same result is proved by a simpler method when v is neither dyadic nor triadic. As we shall remark in the proof of Theorem 5.5, this would in fact suffice for our application. □

LEMMA 5.4. *For each $v \in \mathfrak{M}_f(k)$ there is a series*

$$L_v(s) = \sum_{n=0}^\infty l_{v,n} q_v^{-ns},$$

convergent for $\text{re}(s) > 1/3$, such that $l_{v,0} = 1$ and $a_{i,v,n} \leq l_{v,n}$ for all $1 \leq i \leq n(v)$. These series may be chosen so that the product $\prod_{v \in \mathfrak{M}_f(k)} L_v(s)$ is absolutely convergent in the region $\text{re}(s) > 1$.

Proof. It follows from the discussion in [DW88, § 2] that we may take $L_v(s)$ to be $Z_{i,v}(| \cdot |^s_v, \Phi_{0,v})$ for that index i corresponding to the cubic k_v -algebra k_v^3 . A similar result is proved in [Yuk03, § 6] under the restriction that v be neither dyadic nor triadic, and again this would suffice for our application. □

THEOREM 5.5. Let h be the class number of k and $\mathcal{C} \in \text{Cl}(k)$. Choose a finite set $T_0 \subset \mathfrak{M}(k)$ and a T_0 -type α . Let $d = 2$ or 3 and assume that α is a quadratic T_0 -type if $d = 2$. We have

$$\lim_{B \rightarrow \infty} B^{-1} \sum_{x \in G_k \setminus X_d(\alpha, \mathcal{C}, B)} o(x)^{-1} = \frac{1}{h} \lim_{B \rightarrow \infty} B^{-1} \sum_{x \in G_k \setminus X_d(\alpha, B)} o(x)^{-1}.$$

Proof. If $T_1 \subset \mathfrak{M}(k)$ is a finite set containing T_0 then the T_0 -type α extends in various ways to a T_1 -type. Suppose that we can obtain the claim with T_1 in place of T_0 . Then by summing over all of the extensions of α to a T_1 -type we may obtain the original claim. Thus, we may suppose without loss of generality that T_0 contains the infinite places. Note that we could also assume that T_0 contains all the dyadic and triadic places, so that the full strength of Lemmas 5.3 and 5.4 is not really needed.

Henceforth, T_0 will be a finite set of places containing $\mathfrak{M}_\infty(k)$, α will be a T_0 -type, T will be a finite set of places containing T_0 and β will be a T -type extending α . If $d = 2$ then we assume, in addition, that β is a quadratic T -type. For $s \in \mathbb{C}$, $\omega \in \Omega(k)$ and $x \in V'_k$ we define

$$L_T(s) = \prod_{v \notin T} L_T(s)$$

and

$$\Xi_{x,T}(\omega) = \prod_{v \notin T} \Xi_{x,v}(\omega_v, \Phi_{0,v}).$$

Let χ be a character of $\text{Cl}(k)$ and regard χ also as a character of \mathbb{A}_k^1 trivial on J_k^1 . Note that $\tilde{\chi}$ is a character of \mathbb{A}_k^\times trivial on J_k . Both $L_T(s)$ and $\Xi_{x,T}(\omega_s \tilde{\chi})$ are Dirichlet series in s and, by Proposition 4.2 and Lemma 5.4, these series are absolutely convergent in the region $\text{re}(s) > 1$. Let us write

$$L_T(s) = \sum_{m=1}^\infty l_{T,m}^* m^{-s}$$

and

$$\Xi_{x,T}(\omega_s \tilde{\chi}) = \sum_{m=1}^\infty a_{x,T,m}^*(\chi) m^{-s}.$$

From Lemmas 5.3 and 5.4 and the usual rule for calculating the coefficients of a product of Dirichlet series, we see that

$$l_{T,m}^* = \sum \prod_{v \notin T} l_{v,n_v}$$

and

$$a_{x,T,m}^*(\chi) = \sum \prod_{v \notin T} a_{i_v(x), v, n_v} \tilde{\chi}_v(\varpi_v)^{n_v},$$

where the sum is over all factorizations $m = \prod_{v \notin T} q_v^{n_v}$. It follows that $a_{x,T,1}^*(\chi) = 1$, $l_{T,1}^* = 1$, $l_{T,m}^* \geq 0$ and $|a_{x,T,m}^*(\chi)| \leq l_{T,m}^*$ for all x, T, m and χ .

After these preliminaries, we introduce the Dirichlet series that are at the center of the our method. They are

$$\xi_{\alpha,T}(s) = \sum_{x \in G_k \setminus X_d(\alpha)} o(x)^{-1} \Xi_{x,T}(\omega_{2s}) \mathcal{N}(\Delta_{A_x/k})^{-s}$$

and

$$\xi_{\alpha,T,\mathcal{C}}(s) = \frac{1}{h} \sum_{\chi} \sum_{x \in G_k \setminus X_d(\alpha)} o(x)^{-1} \chi(S(\mathcal{O}_{A_x}) \mathcal{C}^{-1}) \Xi_{x,T}(\omega_{2s} \tilde{\chi}) \mathcal{N}(\Delta_{A_x/k})^{-s}.$$

In the second equation, the outer sum is over all characters χ of $\text{Cl}(k)$; we always understand such sums in this way. We must determine the analytic properties of these series. In order to do so, it is convenient to introduce the series

$$\xi_{\beta,T}(s) = \sum_{x \in G_k \backslash X_d(\beta)} o(x)^{-1} \Xi_{x,T}(\omega_{2s}) \mathcal{N}(\Delta_{A_x/k})^{-s}$$

and

$$\xi_{\beta,T,c}(s) = \frac{1}{h} \sum_{\chi} \sum_{x \in G_k \backslash X_d(\beta)} o(x)^{-1} \chi(S(\mathcal{O}_{A_x})\mathfrak{C}^{-1}) \Xi_{x,T}(\omega_{2s}\tilde{\chi}) \mathcal{N}(\Delta_{A_x/k})^{-s},$$

which differ from $\xi_{\alpha,T}(s)$ and $\xi_{\alpha,T,c}(s)$ only in that the sum includes fewer x . Note that we have

$$\xi_{\alpha,T,c}(s) = \sum_{\beta|T_0=\alpha} \xi_{\beta,T,c}(s) \tag{3}$$

and similarly with $\xi_{\alpha,T}(s)$. Thus, it suffices to determine the analytic properties of $\xi_{\beta,T}(s)$ and $\xi_{\beta,T,c}(s)$.

The argument we are about to make is the same but for notation whether $d = 2$ or $d = 3$. However, for definiteness, we assume that $d = 3$. Also we give the argument only for $\xi_{\beta,T,c}(s)$, as the other is simpler. Consider the function

$$I(s, \Phi) = \frac{1}{h} \sum_{\chi} \chi(\mathfrak{C}^{-1}) Z_3(2s, \chi, \Phi),$$

where $\Phi \in \mathfrak{S}_0(V_{\mathbb{A}})$ is yet to be chosen. By Proposition 4.1, $I(s, \Phi)$ is a holomorphic function in the region $\text{re}(s) > 1$, which has a meromorphic continuation to the region $\text{re}(s) > 5/6$, and whose only possible singularity in this region is a simple pole at $s = 1$ with residue $(C_3/h)\Sigma_3(\Phi)$, where C_3 is a positive constant depending only on k . Let us suppose that $\text{re}(s) > 1$. By Proposition 4.3, there is a positive constant C_0 , depending only on k , such that

$$I(s, \Phi) = \frac{C_0}{h} \sum_{\chi} \sum_{x \in G_k \backslash X_3} o(x)^{-1} \chi(S(\mathcal{O}_{A_x})\mathfrak{C}^{-1}) \Xi_x(\omega_{2s}\tilde{\chi}, \Phi) \mathcal{N}(\Delta_{A_x/k})^{-s}. \tag{4}$$

We now make a particular choice of Φ . For $v \in T \cap \mathfrak{M}_f(k)$, let $\Phi_{1,v}$ denote the characteristic function of the set $U_v w_{\beta(v),v}$ and put

$$\Phi = \bigotimes_{v \in \mathfrak{M}_{\infty}(k)} \Phi_v \otimes \bigotimes_{v \in T \cap \mathfrak{M}_f(k)} \Phi_{1,v} \otimes \bigotimes_{v \notin T} \Phi_{0,v},$$

where Φ_v for $v \in \mathfrak{M}_{\infty}(k)$ is chosen so that Φ_v is U_v -invariant, the support of Φ_v lies in $G_{k_v} w_{\beta(v),v}$, $Z_{\beta(v),v}(| \cdot |_v^s, \Phi_v)$ is entire and $Z_{\beta(v),v}(| \cdot |_v^2, \Phi_v) \neq 0$. According to Lemma 5.1, these requirements at infinite places can be met. The restrictions on the supports of the various factors in Φ imply that $\Xi_x(\omega_{2s}\tilde{\chi}, \Phi) = 0$ unless $x \in X_3(\beta)$. Now suppose that $x \in X_3(\beta)$. If $v \in \mathfrak{M}_{\infty}(k)$ then $\tilde{\chi}_v$ is trivial and so

$$\Xi_{x,v}(| \cdot |_v^{2s} \tilde{\chi}_v, \Phi_v) = Z_{\beta(v),v}(| \cdot |_v^{2s}, \Phi_v).$$

If $v \in T \cap \mathfrak{M}_f(k)$ then, by the choice of $\Phi_{1,v}$ and the fact that $\tilde{\chi}_v$ is unramified,

$$\Xi_{x,v}(| \cdot |_v^{2s} \tilde{\chi}_v, \Phi_{1,v}) = Z_{\beta(v),v}(| \cdot |_v^{2s} \tilde{\chi}_v, \Phi_{1,v})$$

is equal to the volume of the set $U_v H_{\beta(v),v}$, where $H_{\beta(v),v}$ denotes the isotropy group of $w_{\beta(v),v}$ in G_{k_v} , regardless of the values of s and $\tilde{\chi}_v$. (Note that $H_{\beta(v),v}$ is a finite group. It is, in fact, always contained in U_v , but we do not require this.) In particular, we may write

$$\Xi_{x,v}(| \cdot |_v^{2s} \tilde{\chi}_v, \Phi_{1,v}) = Z_{\beta(v),v}(| \cdot |_v^2, \Phi_{1,v})$$

for all $v \in T \cap \mathfrak{M}_f(k)$. On substituting these observations into (4), we obtain

$$\xi_{\beta,T,c}(s) = C_0^{-1} \prod_{v \in \mathfrak{M}_\infty(k)} Z_{\beta(v),v}(| \cdot |_{v}^{2s}, \Phi_v)^{-1} \prod_{v \in T \cap \mathfrak{M}_f(k)} Z_{\beta(v),v}(| \cdot |_{v}^2, \Phi_{1,v})^{-1} I(s, \Phi).$$

To avoid needless repetition, we call a Dirichlet series *standard* if it converges absolutely in the region $\text{re}(s) > 1$ and extends meromorphically to the region $\text{re}(s) > 5/6$ with its only possible singularity a simple pole at $s = 1$. We conclude from the previous equation that $\xi_{\beta,T,c}(s)$ is standard. By using Lemma 5.2, we see that the residue of $\xi_{\beta,T,c}(s)$ at $s = 1$ is

$$(C_3 C'_3 / h C_0) \prod_{v \in T} b_{\beta(v),v}^{-1} \prod_{v \notin T} \Sigma_{3,v}(\Phi_{0,v}).$$

By the choice of measures, $\Sigma_{3,v}(\Phi_{0,v}) = 1$ for all $v \notin T$ and so this residue is actually $(C_3 C'_3 / h C_0) \prod_{v \in T} b_{\beta(v),v}^{-1}$. Equation (3) then implies that $\xi_{\alpha,T,c}(s)$ is standard and that its residue at $s = 1$ is $(C_3 C'_3 / h C_0) \prod_{v \in T_0} b_{\alpha(v),v}^{-1} \prod_{v \in T \setminus T_0} E_{3,v}$. A simpler version of this argument shows that $\xi_{\alpha,T}(s)$ is standard and that its residue at $s = 1$ is $(C_3 C'_3 / C_0) \prod_{v \in T_0} b_{\alpha(v),v}^{-1} \prod_{v \in T \setminus T_0} E_{3,v}$. When $d = 2$ we obtain the same conclusions and the residues of $\xi_{\alpha,T}(s)$ and $\xi_{\alpha,T,c}(s)$ at $s = 1$ in this case turn out to be $(C_2 C'_2 / C_0) \prod_{v \in T_0} c_{\alpha(v),v}^{-1} \prod_{v \in T \setminus T_0} E_{2,v} \prod_{v \notin T} (1 - q_v^{-2})^{-1}$ and $1/h$ times this quantity, respectively. The key fact, valid in both cases, is that

$$\text{res}_{s=1}(\xi_{\alpha,T,c}(s)) = \frac{1}{h} \text{res}_{s=1}(\xi_{\alpha,T}(s)). \tag{5}$$

Now that (5) has been established, we may again treat the cases $d = 2$ and $d = 3$ simultaneously. Let us set

$$\xi_{\alpha,T}(s) = \sum_{m=1}^{\infty} c_{\alpha,T,m} m^{-s}$$

and

$$\xi_{\alpha,T,c}(s) = \sum_{m=1}^{\infty} c_{\alpha,T,m}(\mathcal{C}) m^{-s}.$$

Observe that

$$c_{\alpha,T,m} = \sum_{\substack{x \in G_k \setminus X_d(\alpha) \\ n^2 \mathcal{N}(\Delta_{A_x/k}) = m}} o(x)^{-1} a_{x,T,n}^*(\chi_0)$$

and

$$c_{\alpha,T,m}(\mathcal{C}) = \frac{1}{h} \sum_{\substack{x \in G_k \setminus X_d(\alpha) \\ n^2 \mathcal{N}(\Delta_{A_x/k}) = m}} \sum_{\chi} o(x)^{-1} \chi(S(\mathcal{O}_{A_x}) \mathcal{C}^{-1}) a_{x,T,n}^*(\chi).$$

From these equations and the properties of the $a_{x,T,n}^*(\chi)$ we obtain $c_{\alpha,T,m} \geq 0$ and $|c_{\alpha,T,m}(\mathcal{C})| \leq c_{\alpha,T,m}$ for all $m \geq 1$. We now have all the hypotheses necessary to apply a standard Tauberian theorem ([Lan86, Theorem 1, § 3, ch. 15], for example) to these standard Dirichlet series. As a consequence, we obtain

$$\lim_{B \rightarrow \infty} B^{-1} \sum_{m \leq B} \left(c_{\alpha,T,m}(\mathcal{C}) - \frac{1}{h} c_{\alpha,T,m} \right) = 0 \tag{6}$$

for each T .

Let us write $c_{\alpha,T,m} = P_m + Q_m$ and $c_{\alpha,T,m}(\mathcal{C}) = P_m(\mathcal{C}) + Q_m(\mathcal{C})$, where P_m and $P_m(\mathcal{C})$ are the parts of the above sums with $n = 1$ and Q_m and $Q_m(\mathcal{C})$ are the parts with $n \geq 2$. From the property $a_{x,T,1}^*(\chi) = 1$ for all χ and the orthogonality relation for the characters of $\text{Cl}(k)$, we obtain

the evaluations

$$P_m = \sum_{\substack{x \in G_k \setminus X_d(\alpha) \\ \mathcal{N}(\Delta_{Ax/k})=m}} o(x)^{-1}$$

and

$$P_m(\mathcal{C}) = \sum_{\substack{x \in G_k \setminus X_d(\alpha, \mathcal{C}) \\ \mathcal{N}(\Delta_{Ax/k})=m}} o(x)^{-1}.$$

We wish to estimate the terms Q_m and $Q_m(\mathcal{C})$ on average. To this end, recall that both the set of quadratic fields and the set of cubic fields with prescribed splitting at a finite number of places are known to have a non-zero density. That is, there are non-zero constants $\mathcal{D}_2(\alpha)$ and $\mathcal{D}_3(\alpha)$ such that

$$\lim_{B \rightarrow \infty} B^{-1} \sum_{x \in G_k \setminus X_d(\alpha, B)} o(x)^{-1} = \mathcal{D}_d(\alpha). \tag{7}$$

For $d = 3$ this is the main result of [DW88]. For $d = 2$ it is well known; it is stated as [DW88, Theorem 4.2], where possible references for a proof are given. In fact, a simpler version of the current proof would also serve to prove this fact. In any case, let us choose a constant \mathcal{D} so large that

$$\sum_{x \in G_k \setminus X_d(\alpha, B)} o(x)^{-1} \leq \mathcal{D}B$$

for all $B > 0$ (note that the left-hand side is 0 when $B < 1$). From the fact that $|a_{x, T, n}^*(\chi)| \leq l_{T, n}^*$ we obtain

$$\begin{aligned} \left| B^{-1} \sum_{m \leq B} Q_m(\mathcal{C}) \right| &\leq B^{-1} \sum_{\substack{x \in G_k \setminus X_d(\alpha) \\ n^2 \mathcal{N}(\Delta_{Ax/k}) \leq B, n \geq 2}} o(x)^{-1} l_{T, n}^* \\ &= B^{-1} \sum_{n=2}^{\infty} l_{T, n}^* \sum_{x \in G_k \setminus X_d(\alpha, Bn^{-2})} o(x)^{-1} \\ &\leq \mathcal{D} \sum_{n=2}^{\infty} l_{T, n}^* n^{-2} \\ &= \mathcal{D}(L_T(2) - 1) \end{aligned}$$

for $B > 0$. A similar argument gives

$$\left| B^{-1} \sum_{m \leq B} Q_m \right| \leq \mathcal{D}(L_T(2) - 1)$$

for $B > 0$.

Let $\epsilon > 0$. By Lemma 5.4, the product $\prod_{v \in \mathfrak{m}_f(k)} L_v(2)$ is absolutely convergent and so it is possible to choose a finite set $T \supset T_0$ such that $\mathcal{D}(L_T(2) - 1) < \epsilon/4$. For this set T we may, by (6), choose $B_0 > 0$ such that

$$\left| B^{-1} \sum_{m \leq B} \left(c_{\alpha, T, m}(\mathcal{C}) - \frac{1}{h} c_{\alpha, T, m} \right) \right| < \frac{\epsilon}{2}$$

for all $B \geq B_0$. Together with the evaluations of P_m and $P_m(\mathcal{C})$, these inequalities imply that

$$\left| B^{-1} \sum_{x \in G_k \setminus X_d(\alpha, \mathcal{C}, B)} o(x)^{-1} - B^{-1} \frac{1}{h} \sum_{x \in G_k \setminus X_d(\alpha, B)} o(x)^{-1} \right| < \epsilon$$

for all $B \geq B_0$. In combination with (7), this gives the required result. □

COROLLARY 5.6. *Let k be a number field, h the class number of k , $\mathcal{C} \in \text{Cl}(k)$ and $d \in \{2, 3\}$. Choose a finite set $T_0 \subset \mathfrak{M}(k)$, possibly empty, and for each $v \in T_0$ choose a separable k_v -algebra \mathcal{A}_v of degree d over k_v . For $B \geq 1$ let $F_d(B)$ denote the set of number fields K such that $K \supset k$, K/k has degree d , $\mathcal{N}(\Delta_{K/k}) \leq B$, and $k_v \otimes_k K \cong \mathcal{A}_v$ for every $v \in T_0$. If $d = 3$ then we add the condition that K/k be a non-normal extension. Let $F_d(\mathcal{C}, B) \subset F_d(B)$ be the set of those $K \in F_d(B)$ that also satisfy $S(\mathcal{O}_K) = \mathcal{C}$. Then we have*

$$\lim_{B \rightarrow \infty} \frac{|F_d(\mathcal{C}, B)|}{|F_d(B)|} = \frac{1}{h},$$

where the vertical bars denote cardinality.

Proof. We first observe that the number of cyclic cubic extensions K of k with $\mathcal{N}(\Delta_{K/k}) \leq B$ is $o(B)$. It is a simple consequence of [Wri89, Theorem I.2] that this number is bounded by a constant times $B^{1/2} \log(B)$, which is more than enough. In light of (7), our claim follows immediately from Theorem 5.5 and the fact that $o(x) = |\text{Aut}_k(K)|$ is 2 for all quadratic extensions and 1 for all non-normal cubic extensions of k . We have simply translated from the language of binary cubic forms to the language of number fields, using the Delone–Fadeev correspondence. \square

ACKNOWLEDGEMENTS

The authors would like to thank Akihiko Yukie for useful comments on a first draft of this paper. The first-named author realized the significance of the invariant S for the space of binary cubic forms whilst returning from a talk at the HAAR seminar at The Ohio State University. He would like to thank Eitan Sayag and Steve Rallis for stimulating discussions during his visit.

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